Vertical Velocity, Vertical Wind Shear, and Pressure Tendency in a Hydrostatic Atmosphere

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Motivations

- The barometer is one of the earliest instruments invented with a direct applicability to weather forecasting.
- The causes of atmospheric pressure change are age-old topics of conjecture, study, and debate, approached from various viewpoints, and still a matter of controversy. (Saucier, 1955)

Fundamental equations

Hydrostatic equation	$\frac{\partial p}{\partial z} = -\rho g$	$\frac{\partial}{\partial z}\left(\frac{\partial p}{\partial t}\right) = -g\frac{\partial \rho}{\partial t}$
First law of thermodynamics	$c_p\frac{dT}{dt} - \frac{1}{\rho}\frac{dp}{dt} = \dot{Q}$	$\frac{1}{\rho}\frac{d\rho}{dt} = \frac{c_v}{c_p}\frac{1}{p}\frac{dp}{dt} - \frac{\dot{Q}}{c_p T}$
Continuity equation	$\frac{1}{\rho}\frac{d\rho}{dt} = -\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} - \frac{\partial w}{\partial z} = -\nabla_z \cdot \vec{v} - \frac{\partial w}{\partial z}$	
Equations of motion	$\frac{du}{dt} \cong -\frac{1}{\rho}\frac{\partial p}{\partial x} + fv$	$= -fv_g + fv = fv_a$
	$\frac{dv}{dt} \cong -\frac{1}{\rho}\frac{\partial p}{\partial y} - fu$	$= fu_g - fu = -fu_a$
Ideal gas law	$p = \rho R T$	

The surface pressure tendency equation—a brief history

1. Margules (1904): combined the integrated form of the hydrostatic equation with the continuity equation, yielding a well-accepted and physically sound relationship between mass divergence and surface pressure tendency.

$$\frac{\partial p_s}{\partial t} = -g\int_{z_s}^{\infty} \nabla_z \cdot (\rho\vec{v})\, dz + \cancel{g(\rho w)_{z_s}}$$

2. Godson (1948) showed alternative methods of obtaining the surface pressure tendency by combining the hydrostatic equation with the thermodynamic equation instead of the continuity equation.

$$\frac{\partial p_s}{\partial t} = -\rho_s g\int_{z_s}^{\infty}\left[-\vec{v}\cdot\nabla_p \ln T + \left(\frac{d\ln T}{dp} - \frac{\partial \ln T}{\partial p}\right)\frac{dp}{dt} + \left.\frac{d\ln T}{dp}\right|_{\mathrm{db}}\right] dz$$

3. Saucier (1955): "We know that pressure tendency is the integral of mass divergence above the point, but the relative importance of the three factors—(i) horizontal velocity convergence, (ii) horizontal density advection, and (iii) vertical transport—is the point of doubt."

4. Hess (1959): "Those quantities which we can determine adequately are not ordinarily important (i.e. the density advection term), and those which are significant we cannot measure properly."

$$\frac{\partial p}{\partial t} = -g\int_{h}^{\infty}(\rho\nabla_z\cdot\vec{v} + \vec{v}\cdot\nabla_z\rho)\, dz + g(\rho w)_h$$

5. Kong (2006): Obtained essentially the same equation as in Godson but applied it to understand the genesis of Hurricane Vince.

$$\frac{\partial p_s}{\partial t} = -\rho_s g\int_{z_s}^{\infty}\left[-\vec{v}\cdot\nabla_p \ln T + \frac{R}{g}\left(\frac{g}{c_p} + \frac{\partial T}{\partial z}\right)\frac{\omega}{p} + \frac{\dot{Q}}{c_p T}\right] dz$$

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Derivations: Instead of making use of a single fundamental equation, attempt is made to make use of all fundamental equations in the derivations.

Derivation of the vertical velocity equation

We begin with the time-derivative form of the hydrostatic equation

$$\frac{\partial}{\partial z}\left(\frac{\partial p}{\partial t}\right) = -g\frac{\partial \rho}{\partial t}$$

The hydrostatic equation may also be combined with the thermodynamic energy equation to obtain the following relationship.

$$\frac{\partial p}{\partial t} = p\frac{c_p}{c_v}\left(\frac{1}{\rho}\frac{d\rho}{dt} + \frac{\dot{Q}}{c_p T}\right) - \vec{v}\cdot\nabla_z p + \rho g w$$

The continuity equation may now be substituted in place of the rate of change of density. If we differentiate the resulting equation with respect to z and substitute it into the time-derivative form of the hydrostatic equation above, we would get

$$\frac{\partial}{\partial z}\left[p\frac{c_p}{c_v}\left(-\nabla_z\cdot\vec{v} - \frac{\partial w}{\partial z} + \frac{\dot{Q}}{c_p T}\right) - \vec{v}\cdot\nabla_z p + \rho g w\right] = -g\frac{\partial \rho}{\partial t}$$

The continuity equation may be substituted again to eliminate the $\partial\rho/\partial t$ term on the R.H.S., resulting in the following second order differential equation with w being the independent variable.

$$\frac{\partial^2 w}{\partial z^2} - \frac{\rho g}{p}\frac{\partial w}{\partial z} = -\nabla_z\cdot\frac{\partial\vec{v}}{\partial z} + \frac{\rho g}{p}\left(\frac{R}{c_p}\nabla_z\cdot\vec{v} + \frac{c_v}{c_p}\frac{\partial\vec{v}}{\partial p}\cdot\nabla_z p - \frac{\dot{Q}}{c_p T}\right) + \frac{\partial}{\partial z}\left(\frac{\dot{Q}}{c_p T}\right)$$

If we neglect the higher order term $\frac{\partial^2 w}{\partial z^2}$ and express the divergence in isobaric coordinates, we find that

$$\frac{\partial w}{\partial z} = -\left(\frac{R}{c_p}\nabla_p\cdot\vec{v} - \frac{\partial\vec{v}}{\partial z}\cdot\nabla_p z - \frac{\dot{Q}}{c_p T}\right) + \frac{RT}{g}\frac{\partial}{\partial z}\left(\nabla_z\cdot\vec{v} - \frac{\dot{Q}}{c_p T}\right)$$

Integrating yields

$$\boxed{w_2 = w_1 - \int_{z_1}^{z_2}\left(\frac{R}{c_p}\nabla_p\cdot\vec{v} - \frac{\partial\vec{v}}{\partial z}\cdot\nabla_p z - \frac{\dot{Q}}{c_p T}\right) dz + \frac{R\bar{T}}{g}\left[\nabla_z\cdot\vec{v} - \frac{\dot{Q}}{c_p T}\right]_{z_1}^{z_2}}$$

This formulation of the vertical velocity satisfies the **hydrostatic**, **thermodynamic**, and **continuity** equations.

Physical interpretations

1. Low-level convergence leads to ascent aloft.
2. The wind shear term may be split into geostrophic and ageostrophic components, i.e.

   $$\frac{\partial\vec{v}}{\partial z}\cdot\nabla_p z = \left(\frac{\partial\vec{v}_g}{\partial z} + \frac{\partial\vec{v}_a}{\partial z}\right)\cdot\nabla_p z.$$

   Given the thermal wind relation $\frac{\partial\vec{v}_g}{\partial z} = \frac{g}{f}\hat{k}\times\nabla_p \ln T$ and the definition of the geostrophic wind $\vec{v}_g = \frac{g}{f}\hat{k}\times\nabla_p z$, we find that

   $$\frac{\partial\vec{v}_g}{\partial z}\cdot\nabla_p z = -\vec{v}_g\cdot\nabla_p \ln T$$

   which implies warm advection would lead to ascent aloft. As for the ageostrophic portion, given $\vec{v}_a = \frac{1}{f}\hat{k}\times\frac{d\vec{v}}{dt} = -\frac{1}{f^2}\nabla_p\left(\frac{\partial\Phi}{\partial t}\right) + \cdots$ and $\frac{\partial\Phi}{\partial p} = -\frac{RT}{p}$

   $$\frac{\partial\vec{v}_a}{\partial z}\cdot\nabla_p z = \underbrace{-\frac{g}{f^2 RT}\nabla_p\left(\frac{\partial T}{\partial t}\right)}_{\text{isallobaric}}\cdot\nabla_p z + \cdots$$

   which implies ascent aloft if local warming increases toward lower geopotential heights.
3. Diabatic heating leads to ascent aloft.
4. Diabatic heating and convergence at the lower boundary lead to ascent aloft.

Derivation of the pressure tendency equation

Starting from the time-derivative form of the hydrostatic equation, we substitute the continuity equation in place of $\partial\rho/\partial t$ to obtain

$$\frac{\partial}{\partial p}\left(\frac{\partial p}{\partial t}\right) = -\nabla_z\cdot\vec{v} - \frac{\partial w}{\partial z} - \vec{v}\cdot\nabla_z \ln\rho - \frac{\partial \ln\rho}{\partial z}w$$

Next we will attempt to eliminate w from the equation. We first solve for w from the second equation on the left and substitute the result into the above equation. After rearrangement, we obtain the following equation.

$$\frac{\partial}{\partial z}\left(\frac{1}{\rho}\frac{\partial p}{\partial t}\right) = -g\vec{v}\cdot\nabla_p \ln T - R\frac{c_p}{c_v}\left(\frac{g}{c_p} + \frac{\partial T}{\partial z}\right)\left(\nabla_z\cdot\vec{v} + \frac{\partial w}{\partial z}\right) + R\left(\frac{g}{R} + \frac{\partial T}{\partial z}\right)\frac{\dot{Q}}{c_v T}$$

where temperature advection is on a constant p surface, and $\frac{g}{c_p} + \frac{\partial T}{\partial z} = \frac{T}{\theta}\frac{\partial\theta}{\partial z}$.

This equation may be integrated from z_s to ∞, assuming $w_s = w_\infty = 0$ to get

$$\boxed{\frac{\partial p_s}{\partial t} = -\rho_s\int_{z_s}^{\infty}\left[-g\vec{v}\cdot\nabla_p \ln T - R\frac{c_p}{c_v}\left(\frac{g}{c_p} + \frac{\partial T}{\partial z}\right)\nabla_z\cdot\vec{v} + R\left(\frac{g}{R} + \frac{\partial T}{\partial z}\right)\frac{\dot{Q}}{c_v T}\right] dz}$$

Physical interpretations

1. Warm air advection leads to falling surface pressure.
2. Divergence leads to falling surface pressure, but **only if** $\frac{g}{c_p} + \frac{\partial T}{\partial z} < 0$ which means ***static instability*** (i.e. ***super-adiabatic)***. This implies that convergence in a stable atmosphere would also lead to falling surface pressure.
3. So far we have not considered the equations of motion in x and y coordinates. The term that links the pressure tendency equation to the (x, y) equations of motion is the divergence term via the vorticity equation.

Vorticity equation

$$\frac{d(\zeta + f)}{dt} = -(\zeta + f)\boldsymbol{\nabla}_z\cdot\vec{\boldsymbol{v}} - \hat{\boldsymbol{k}}\cdot\left(\boldsymbol{\nabla}_z w\times\frac{\partial\vec{\boldsymbol{v}}}{\partial z}\right) + \frac{1}{\rho^2}\hat{\boldsymbol{k}}\cdot(\boldsymbol{\nabla}_z\rho\times\boldsymbol{\nabla}_z p)$$

The vorticity equation contains a divergence term which can be solved to obtain

$$\boldsymbol{\nabla}_z\cdot\vec{\boldsymbol{v}} = \frac{1}{\zeta + f}\left[-\frac{\partial\zeta}{\partial t} - \underbrace{\vec{\boldsymbol{v}}\cdot\boldsymbol{\nabla}_z\zeta}_{\substack{\text{vorticity}\\ \text{advection}}} - w\frac{\partial\zeta}{\partial z} + \underbrace{\hat{\boldsymbol{k}}\times\frac{\partial\vec{\boldsymbol{v}}}{\partial z}\cdot\boldsymbol{\nabla}_z w}_{\text{tilting/twisting}} + \underbrace{f\vec{\boldsymbol{v}}_g\cdot\boldsymbol{\nabla}_z \ln T}_{\text{solenoid}} - \underbrace{\frac{2\Omega}{\mathrm{R_e}}(v\cos\phi)}_{\beta\text{ effect}}\right]$$

Substituting this into the pressure tendency equation further reveals the following dynamical effects on surface pressure tendency:

4. $\left(\frac{g}{c_p} + \frac{\partial T}{\partial z}\right)\frac{d(\zeta+f)}{dt}$ is the potential vorticity given a constant lapse rate. It is a conservative quantity following isentropic and adiabatic air flow. $\frac{d\zeta}{dt}$ may be split into local vorticity tendency and advection components.
5. Divergence due to tilting of vertical wind shear: Given the thermal wind relation, $\frac{\partial\vec{\boldsymbol{v}}_g}{\partial z} = \frac{g}{f}\hat{\boldsymbol{k}}\times\boldsymbol{\nabla}_p \ln T$, the tilting term can be written as

   $$-\frac{g}{f}\boldsymbol{\nabla}_p \ln T\cdot\boldsymbol{\nabla}_z w + \hat{\boldsymbol{k}}\times\frac{\partial\vec{\boldsymbol{v}}_a}{\partial z}\cdot\boldsymbol{\nabla}_z w$$

6. The solenoid term is geostrophic temperature advection in disguise. It partially cancels the temperature advection term in the pressure tendency equation.
7. The β effect: $-\frac{1}{\zeta+f}\frac{2\Omega}{\mathrm{R_e}}(v\cos\phi)$