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Toward relaxing the symmetric neutrality assumption of the potential intensity theory of tropical cyclones

Chau-Lam (Chris) Yu

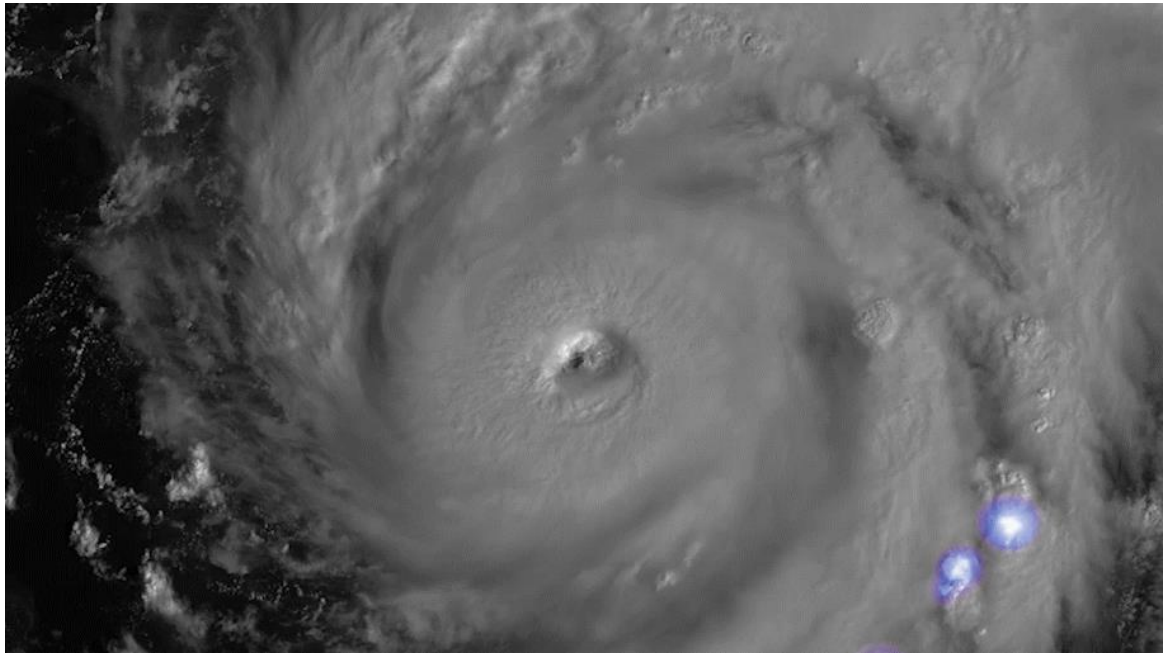
01/13/2025

105th AMS Annual Meeting

Background

- Rapid intensification (RI) of hurricane is challenging to forecast, particularly in terms of its onset timing and intensification rate.
- Many studies showed that storm asymmetries and environmental forcings can strongly influence the details of RI.
- However, even in axisymmetric framework, the RI onset condition and spin-up mechanism are not fully understood.

Hurricane Ian (2022)



credit: CIRA satellite library

Background

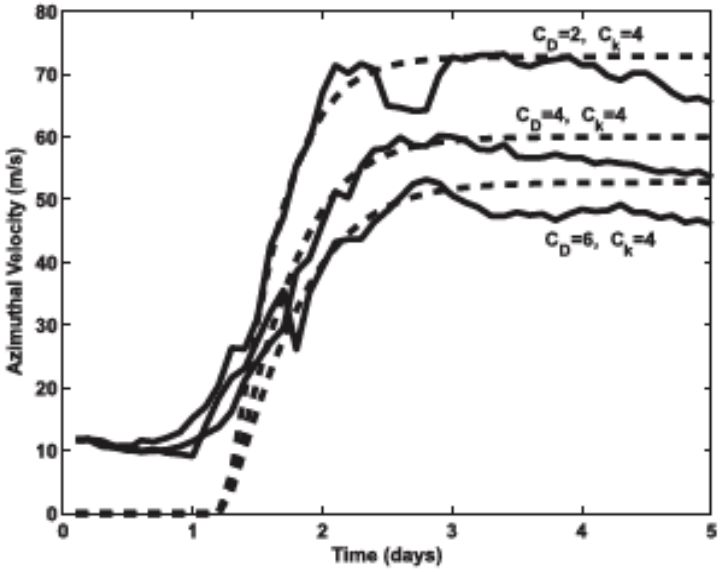
Time-dependent theory of TC intensification:

A number of time-dependent intensification models had been developed in the past decade:

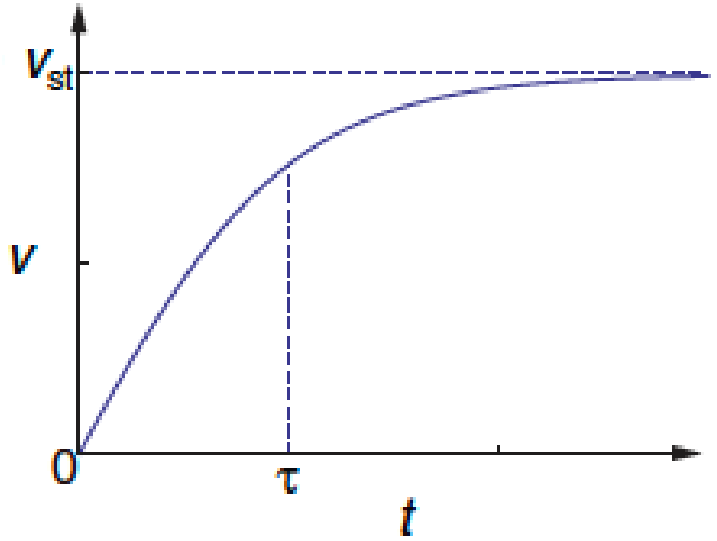
- Emanuel 2012 (steady-state Balanced dynamics)
- Ozawa and Shimokawa 2015 (cyclic return of parcels)
- Wang et al. 2020, 2021 (ad hoc parameters)

Some of these models rely on expressions from traditional PI-theory based on **S**ymmetric **N**eutrality (**SN**) assumption.

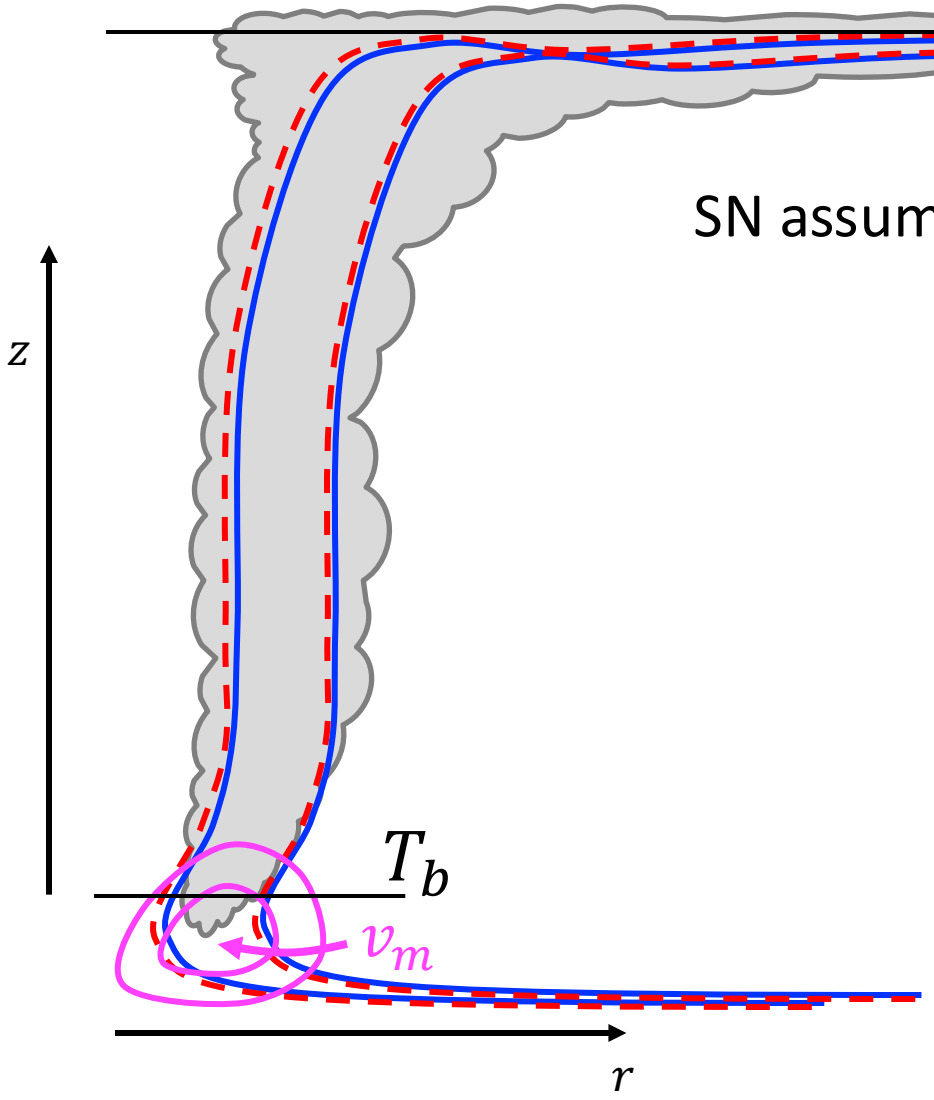
Emanuel 2012



Ozawa and Shimokawa 2015

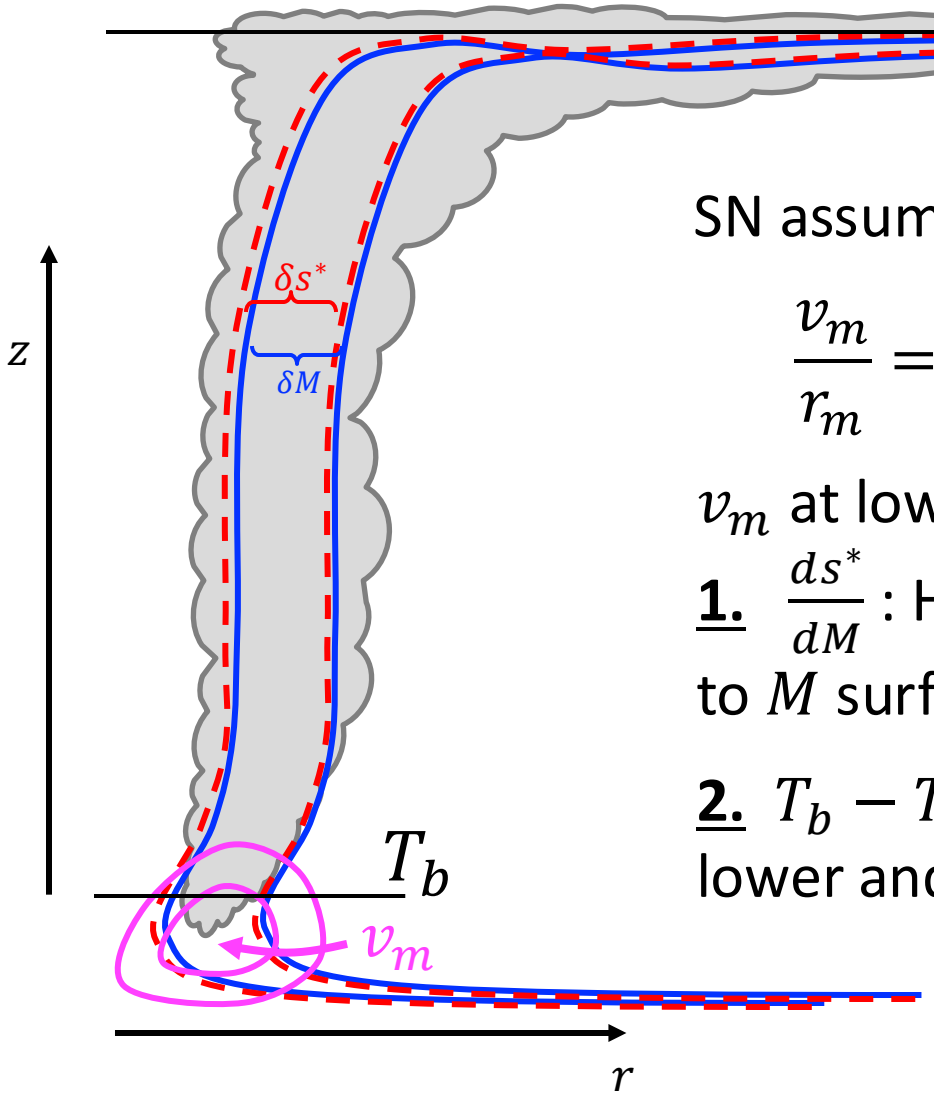


Symmetric Neutrality (SN) Assumption



SN assumption: $s^*(M)$, i.e., s^* and M are congruent

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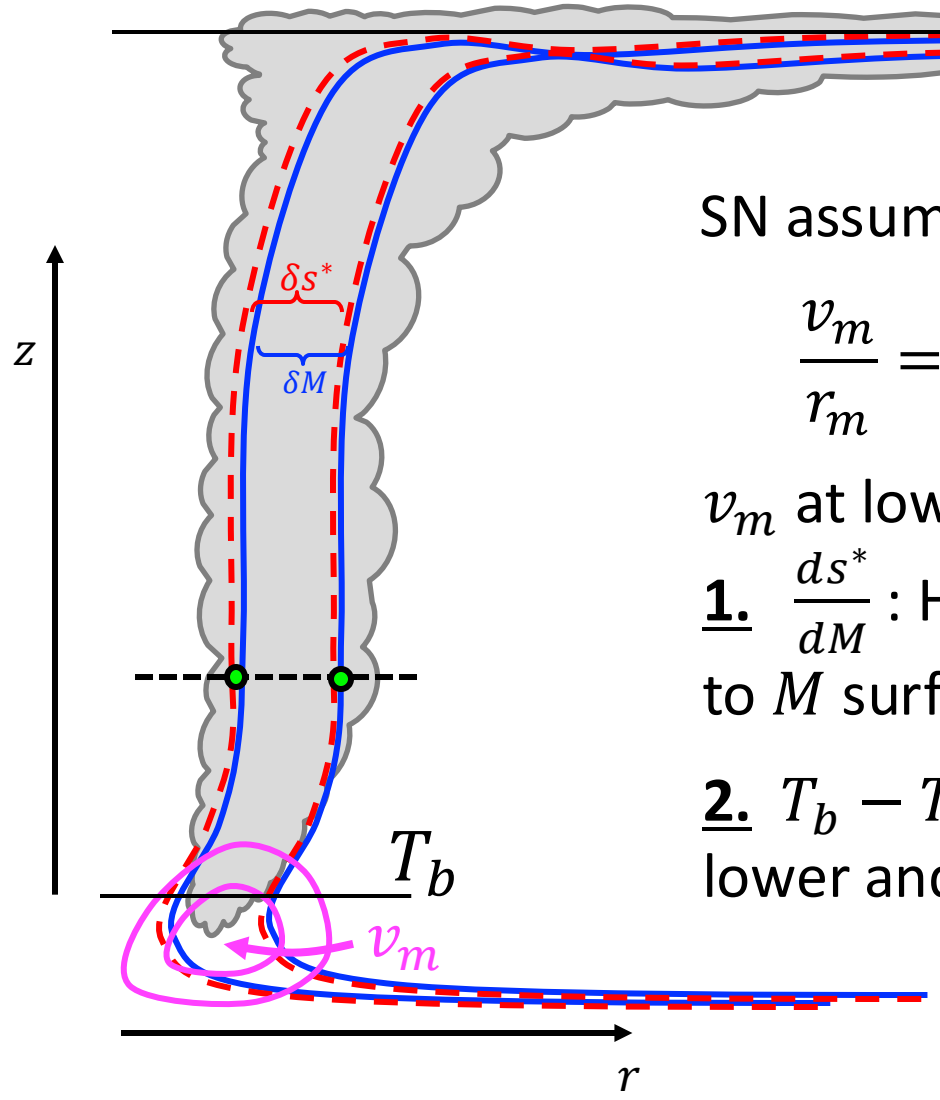
$$\frac{v_m}{r_m} = -\frac{ds^*}{dM}(T_b - T_o)$$ **Emanuel (1986)**

v_m at low level is determined by:

1. $\frac{ds^*}{dM}$: How closely spaced are the s^* surfaces relative to M surfaces

2. $T_b - T_o$: The temperature difference between the lower and upper levels

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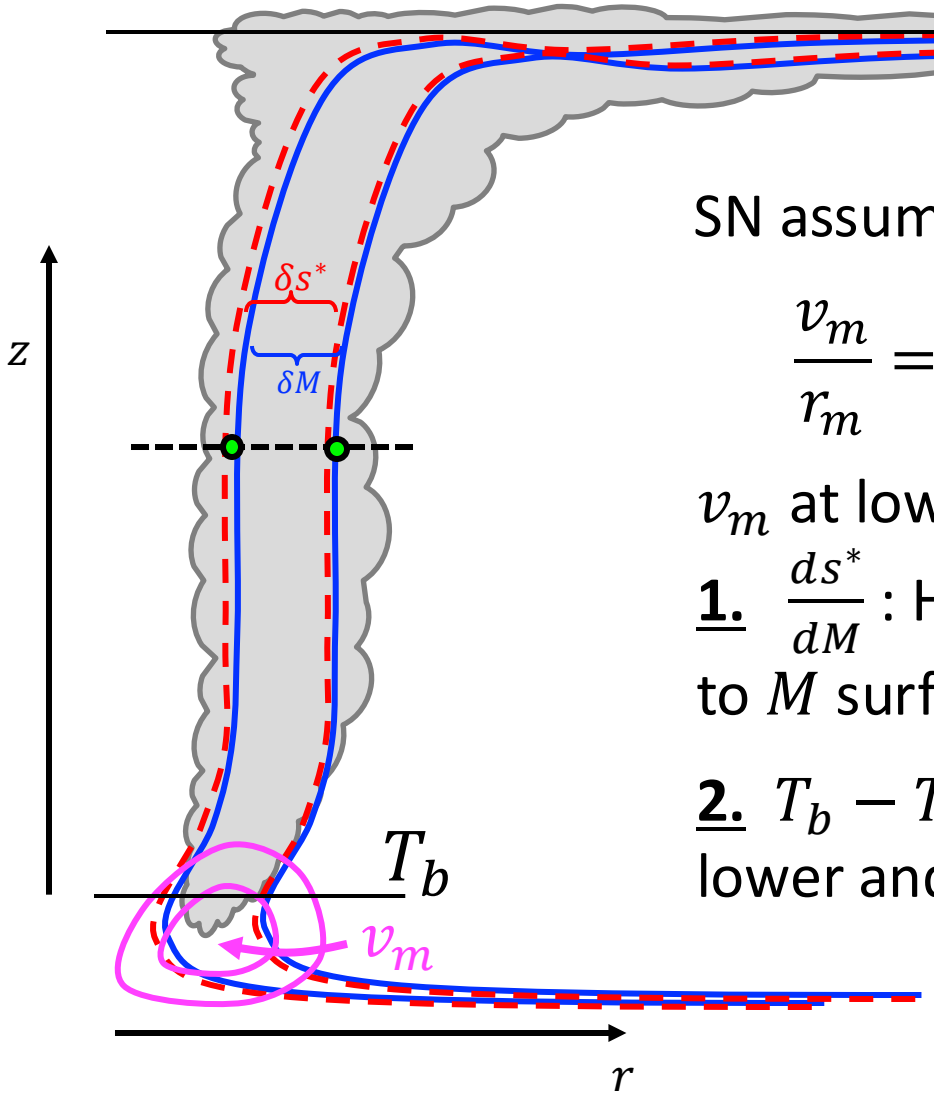
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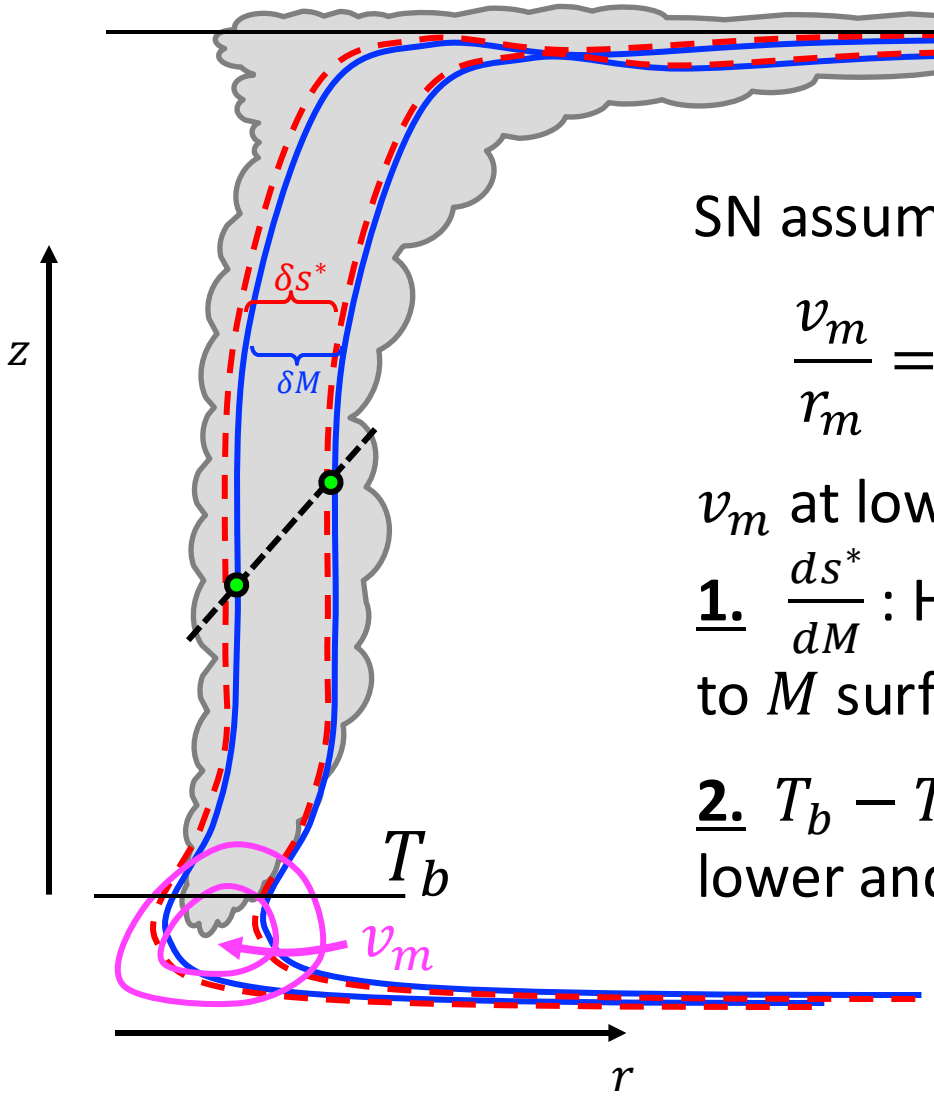
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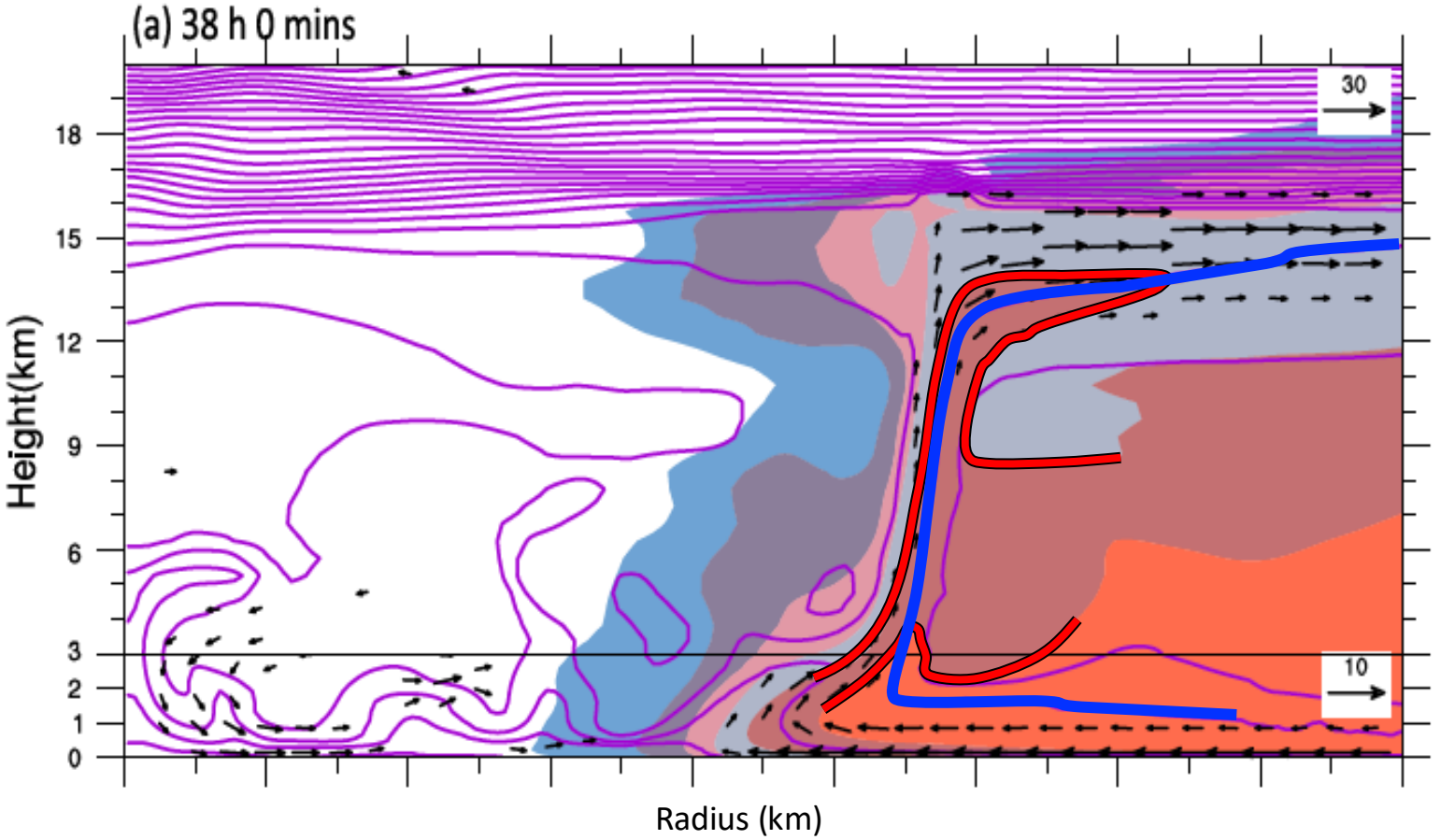
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Background: Non-Symmetric Neutrality (Non-SN)

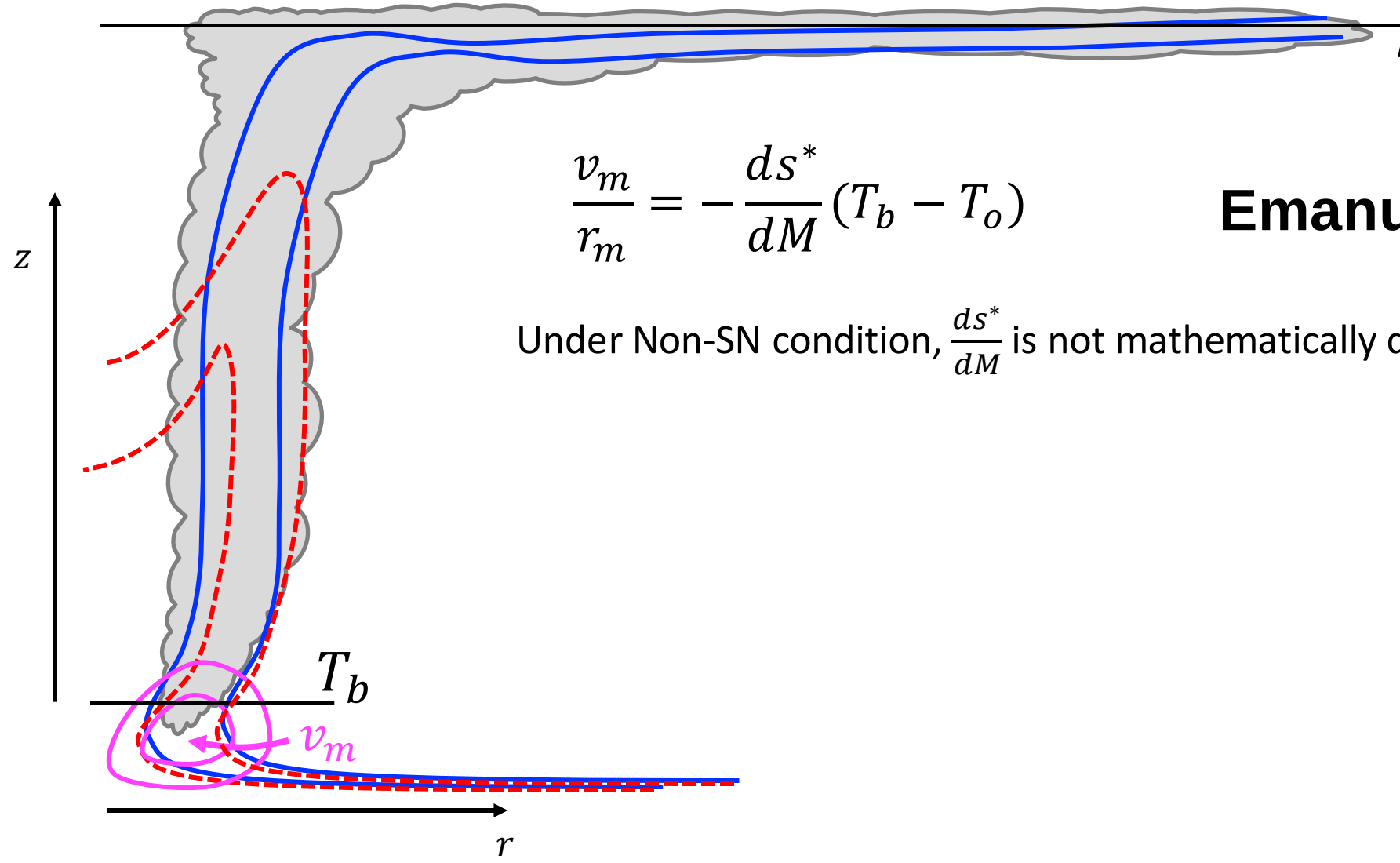
- Recent studies showed that SN condition is not met during the **early stage of intensification**

- M surfaces (**blue**) clearly cross s^* surfaces (**green**) during the early intensification period

Early intensification (Peng et al. 2019)



Non-Symmetrically Neutral Vortex

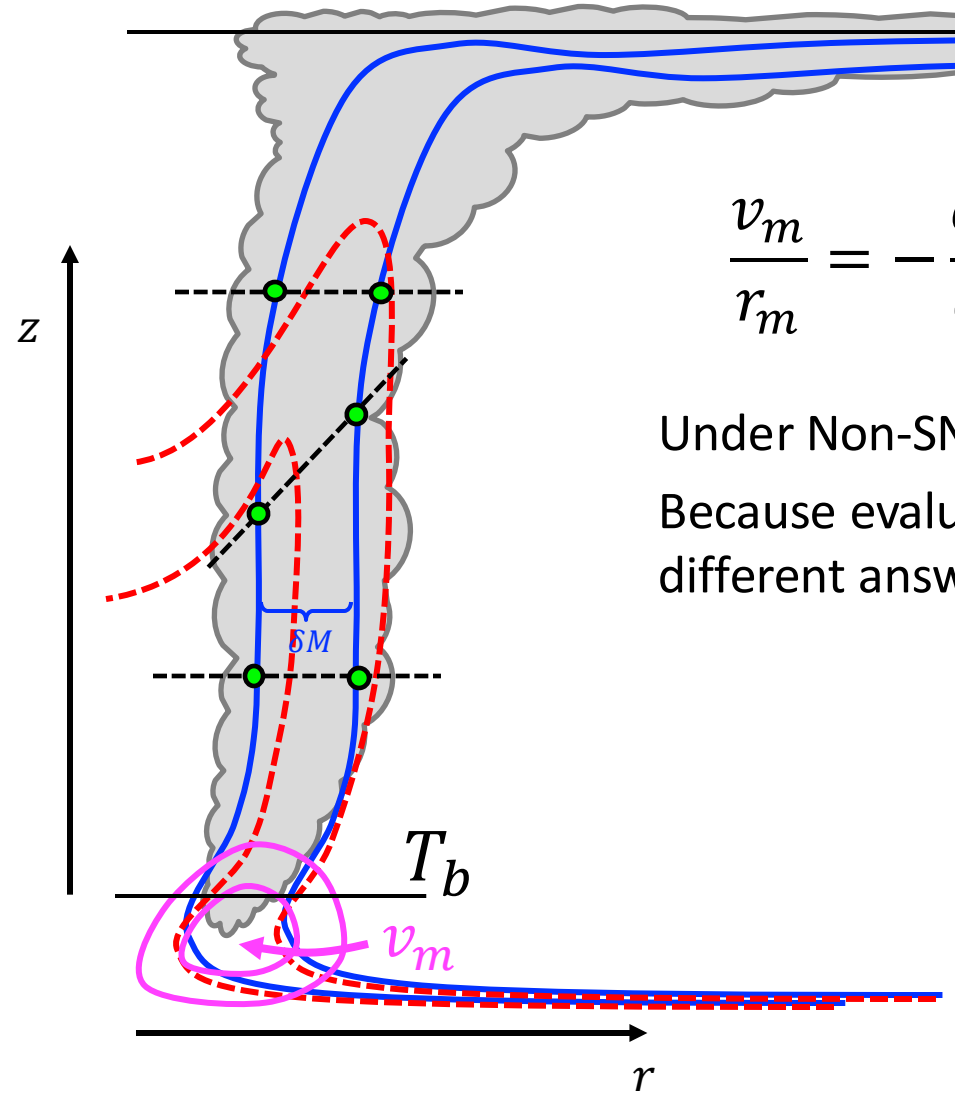


$$\frac{v_m}{r_m} = -\frac{ds^*}{dM}(T_b - T_o)$$

Emanuel (1986)

Under Non-SN condition, $\frac{ds^*}{dM}$ is not mathematically defined.

Non-Symmetrically Neutral Vortex



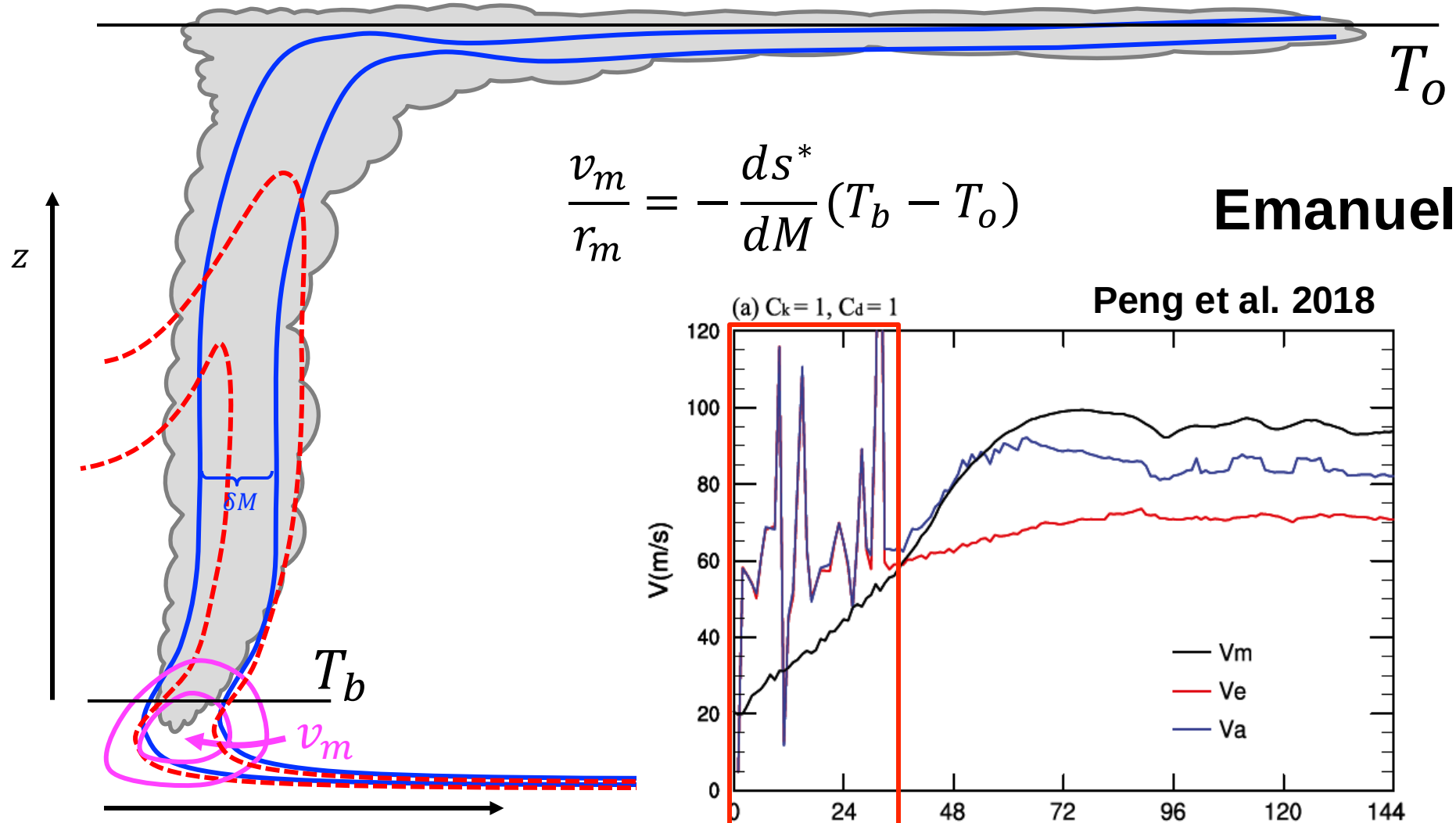
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Because evaluating it at different height, different direction gives you different answer.

Non-Symmetrically Neutral Vortex



$$\frac{v_m}{r_m} = -\frac{ds^*}{dM}(T_b - T_o)$$

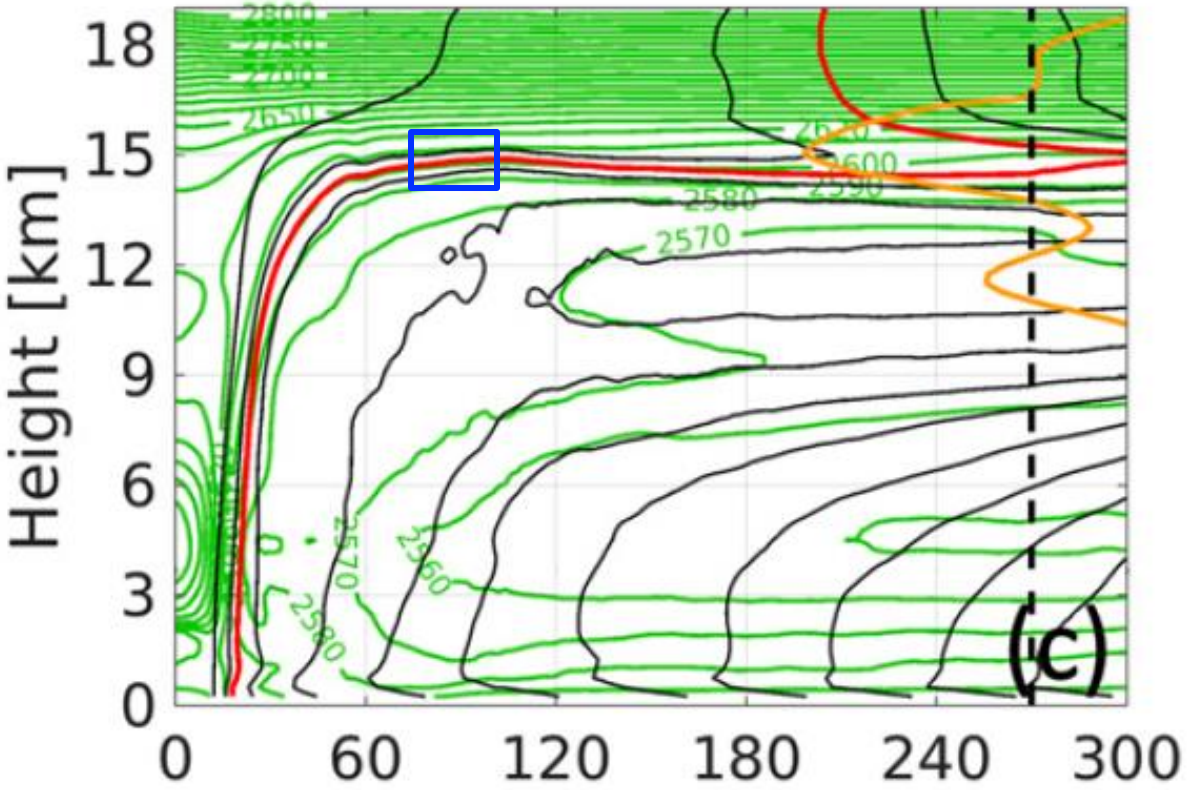
Emanuel (1986)

Peng et al. 2018

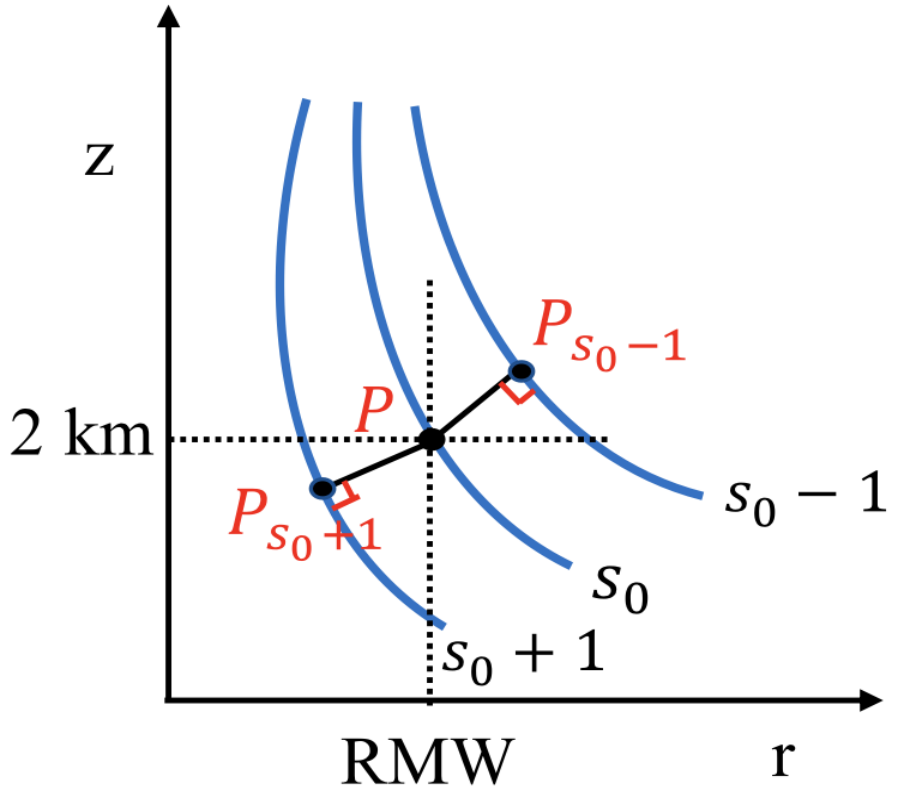
Non-Symmetrically Neutral Vortex

Currently, there is **no standard procedure** to compute $\frac{ds^*}{dM}$ (or $\frac{\partial s^*}{\partial M}$)

Tao et al. 2019



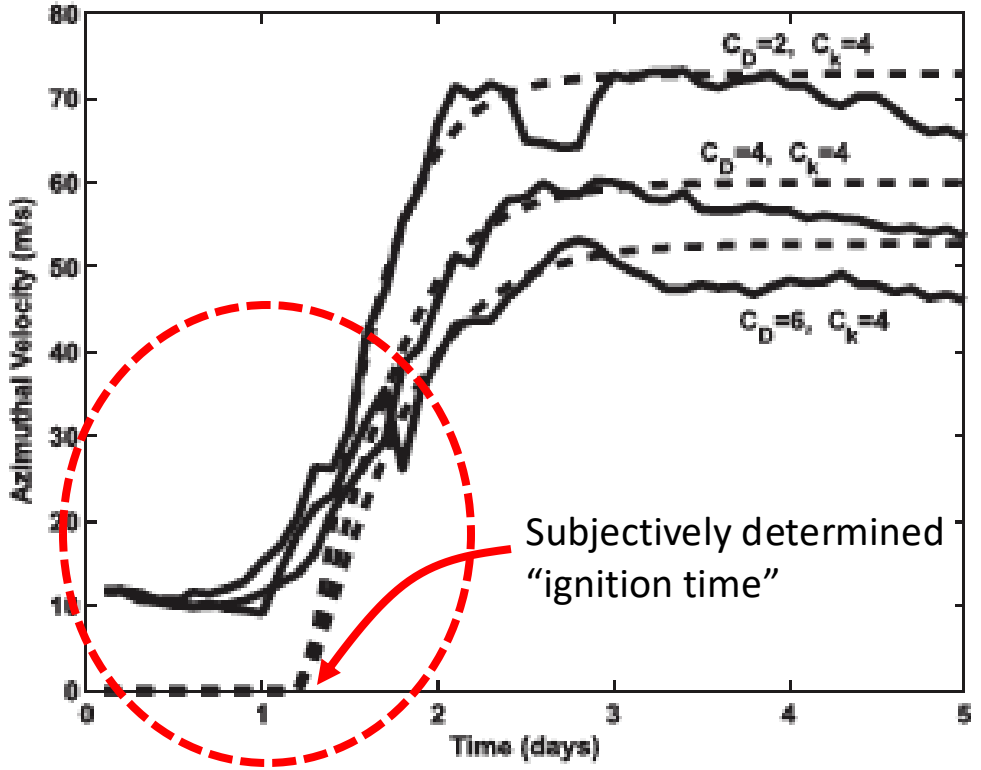
Wang et al. 2022; Li et al. 2024



Motivation

Problem:

The **breakdown of SN assumption** is the **biggest obstacle** that prevents us from developing a more complete understanding of RI.



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Goals:

Revisit the derivation of PI theory without making the SN assumption to uncover the moist vortex dynamics under Non-SN condition.

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The **breakdown of SN assumption** is the **biggest obstacle** that prevents us from developing a more complete understanding of RI.

Goals:

Revisit the derivation of PI theory without making the SN assumption to uncover the moist vortex dynamics under Non-SN condition.

Research questions:

1. How should we define $\frac{\partial s^*}{\partial M}$ such that it is dynamically linked with v_m at the low level? At what specific level and direction should we evaluate $\frac{\partial s^*}{\partial M}$?

2. Does the temperature difference $T_b - T_o$ play any role in determining v_m under Non-SN condition?

Revisiting PI theory without SN assumption

Emanuel (1986)

Current study

Revisiting PI theory without SN assumption

Emanuel (1986)

$$\frac{2M}{r^3}\left(\frac{\partial r}{\partial p}\right)_M = \left(\frac{\partial T}{\partial p}\right)_{s^*}\left(\frac{\partial s^*}{\partial M}\right)_p$$

thermal wind
+
saturation

Current study

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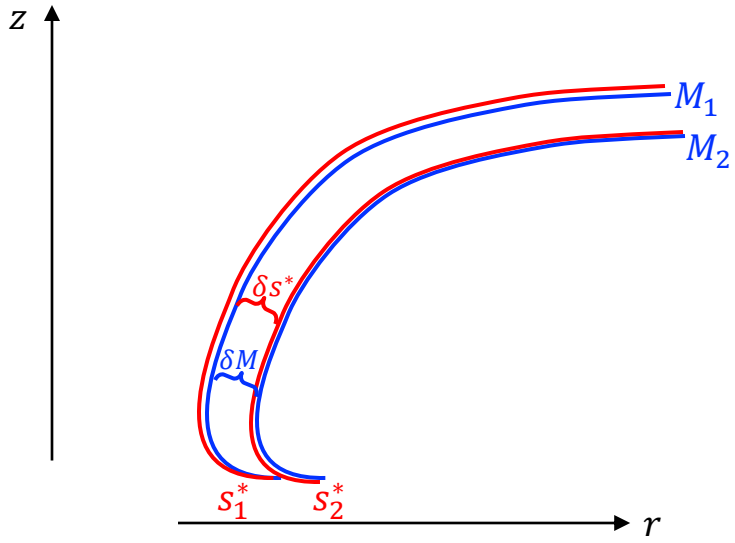
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Making SN assumption, $s^* = s^*(M)$

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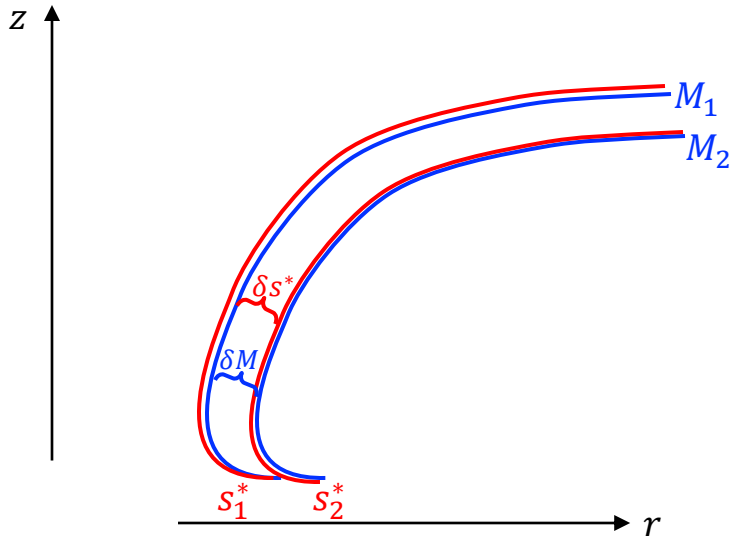
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$$\frac{2M}{r^3}\left(\frac{\partial r}{\partial p}\right)_M = \left(\frac{\partial T}{\partial p}\right)_{s^*}\left(\frac{\partial s^*}{\partial M}\right)_p = \begin{vmatrix} \frac{\partial s^*}{\partial M} & \frac{\partial s^*}{\partial p} \\ \frac{\partial T}{\partial M} & \frac{\partial T}{\partial p} \end{vmatrix}$$

Realizing that the R.H.S is a Jacobian Determinant

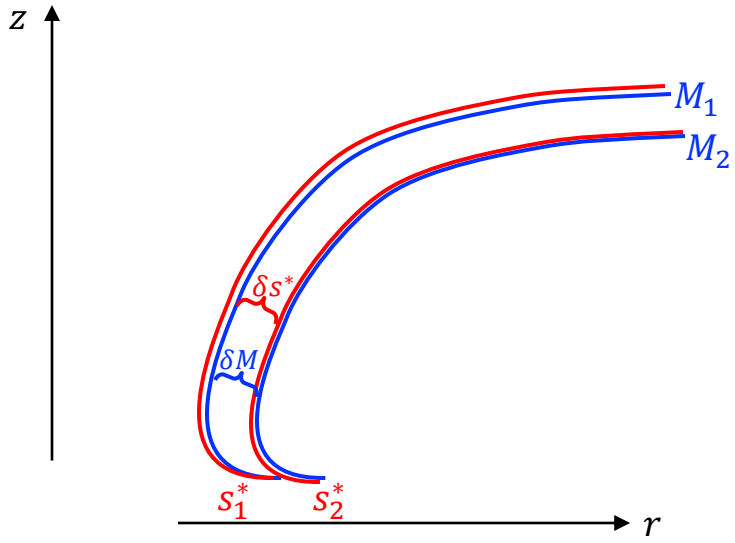


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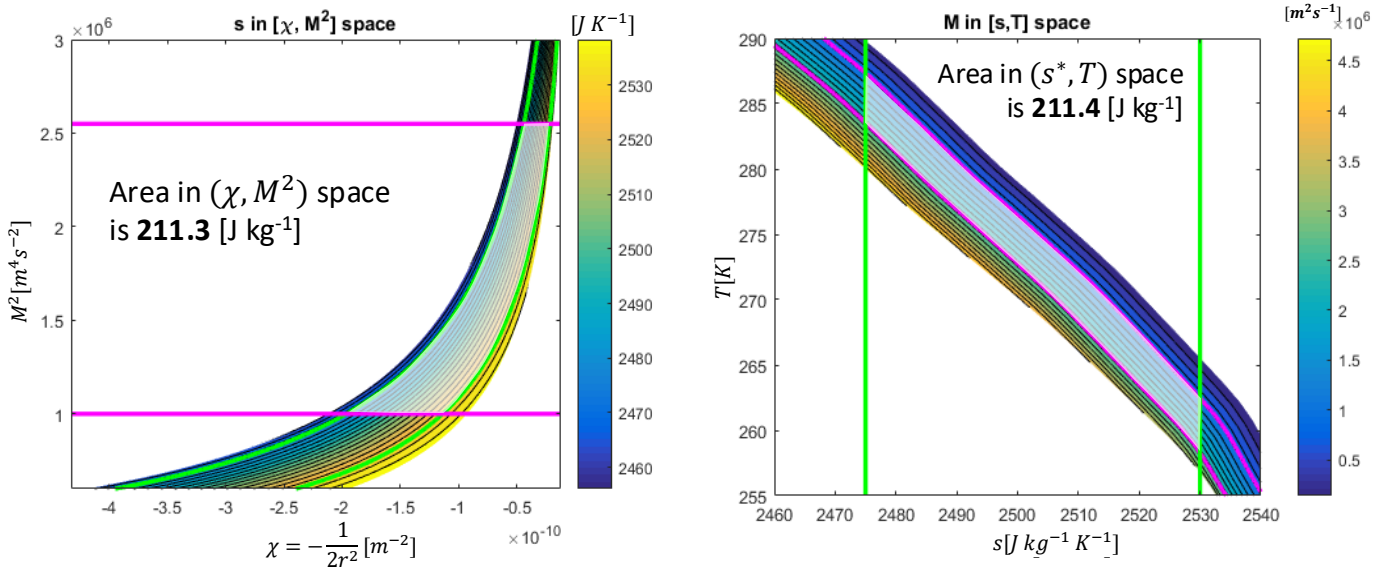


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Realizing that the R.H.S is a Jacobian Determinant

$$dM^2 d\left(-\frac{1}{2r^2}\right) = ds^* dT$$



Please see supplementary material for details if interested

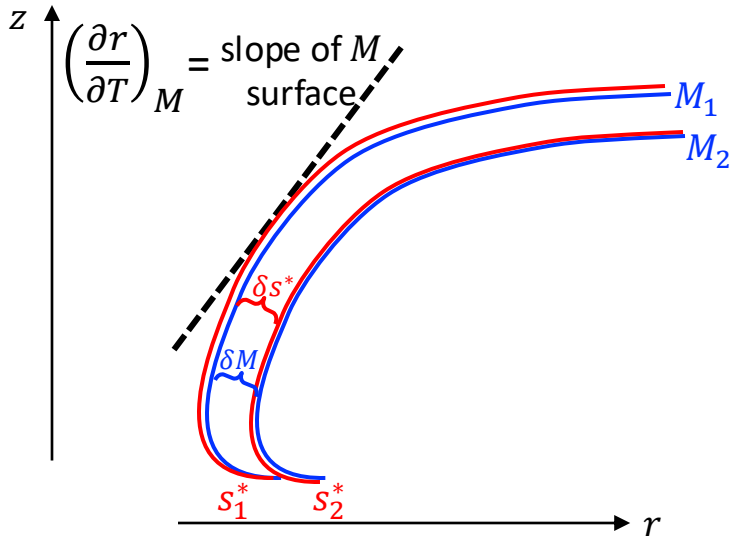
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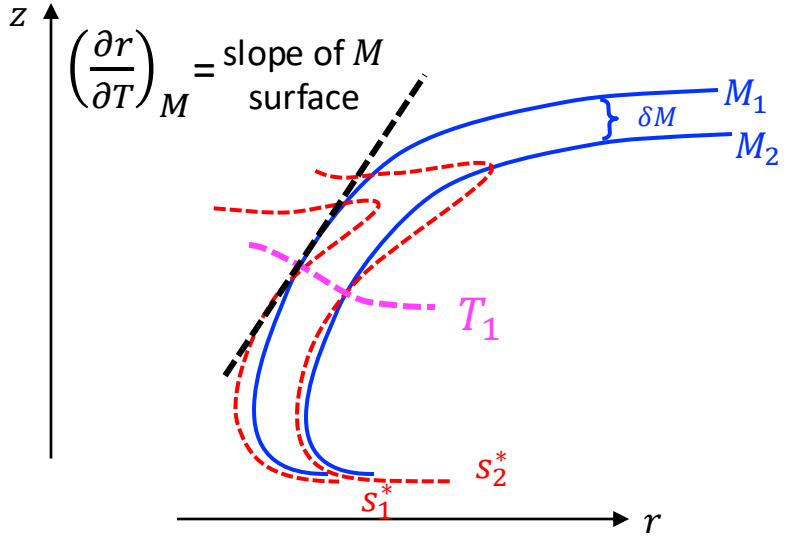


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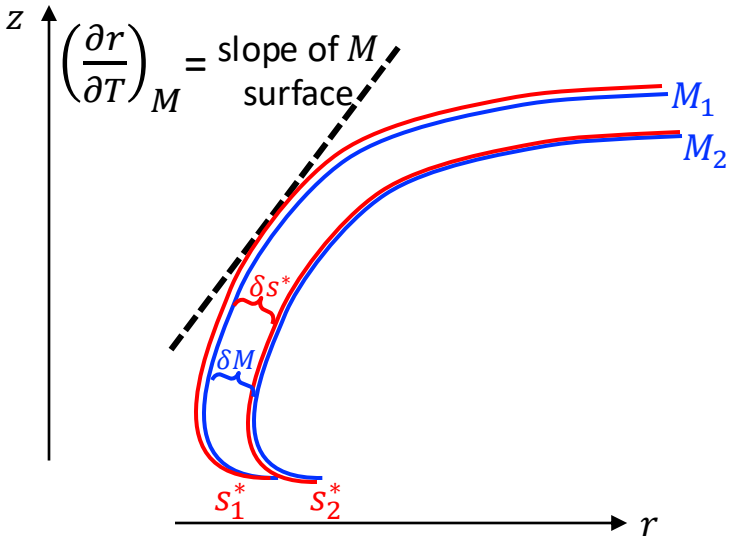
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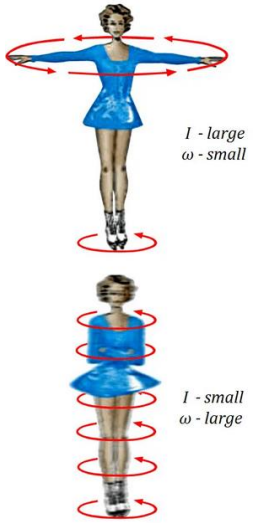
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Making SN assumption, $s^* = s^*(M)$

$$\frac{2M}{r^3}\left(\frac{\partial r}{\partial T}\right)_M = \frac{ds^*}{dM}$$





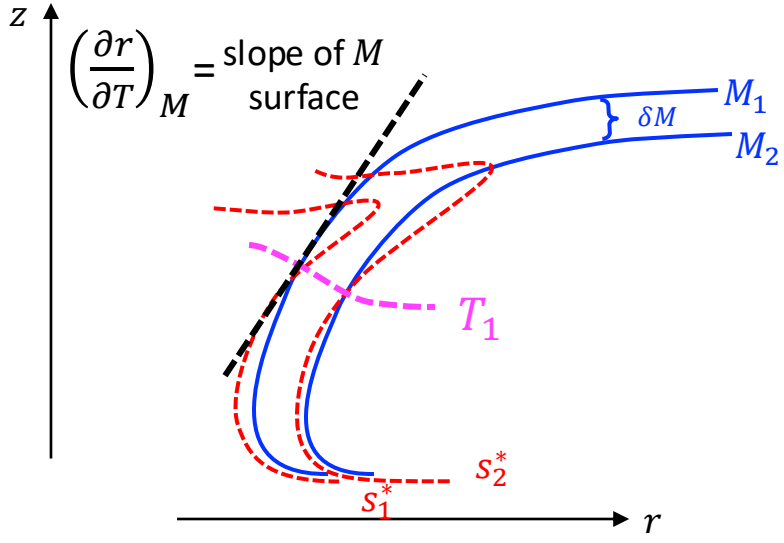
$\left(\frac{\partial s^*}{\partial M}\right)_T$ determines how much your arms can be pulled towards your body to generate spin

Current study

$$\frac{2M}{r^3}\left(\frac{\partial r}{\partial p}\right)_M = \left(\frac{\partial T}{\partial p}\right)_{s^*}\left(\frac{\partial s^*}{\partial M}\right)_p$$

Realizing that the R.H.S is a Jacobian Determinant

$$\frac{2M}{r^3}\left(\frac{\partial r}{\partial T}\right)_M = \left(\frac{\partial s^*}{\partial M}\right)_T$$



Revisiting PI theory without SN assumption

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$$\frac{2M}{r^3}\left(\frac{\partial r}{\partial T}\right)_M = \left(\frac{\partial s^*}{\partial M}\right)_T$$

$$\frac{v_m}{r_m} = \int_{T_b}^{T_o}\left(\frac{\partial s^*}{\partial M}\right)_T dT = \frac{d}{dM}\left(\int_{T_b}^{T_o} s^* dT\right)$$

Generalized findings:

Balanced v_{max} **at the low level** is controlled by:

1. Accumulation of inward bending effect of $\left(\frac{\partial s^*}{\partial M}\right)_T$ throughout the atmospheric from T_b to T_o
2. Alternatively, it is determined by the gradient of integrated thermal energy across the M surfaces.

→ Even under Non-SN condition, a TC eyewall is a **circular frontal zone** of **thermal energy**.

A generalized v_{max} equation: Final form

Balanced component
Converge to E86:

$$-\frac{ds^*}{dM}(T_b - T_o)$$

$$\frac{v_b}{r_b} = \int_{T_b}^{T_o} \left(\frac{\partial s^*}{\partial M}\right)_T dT$$

$$= \frac{d}{dM}\left(\int_{T_b}^{T_o} s^* dT\right)$$

Distribution of thermal energy

A generalized v_{max} equation: Final form

Balanced component
Converge to E86:

$$-\frac{ds^*}{dM}(T_b - T_o)$$

Unbalanced term
Converge to BR09:

$$\frac{d\psi}{dM}\left(\frac{\eta_o}{r_o\rho_{do}} - \frac{\eta_b}{r_b\rho_{db}}\right)$$

$$\frac{v_b}{r_b} = \int_{T_b}^{T_o}\left(\frac{\partial s^*}{\partial M}\right)_T dT + \int_{\psi_b}^{\psi_o}\left(\frac{\partial}{\partial M}\left(\frac{\eta}{r\rho_d}\right)\right)_\psi d\psi$$

$$= \frac{d}{dM}\left(\int_{T_b}^{T_o} s^* dT\right) + \frac{d}{dM}\left(\int_{\psi_b}^{\psi_o}\frac{\eta}{r\rho_d}d\psi\right)$$

Distribution of thermal energy

Distribution of azimuthal vorticity η

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New term due to work done against friction

$$\frac{v_b}{r_b} = \int_{T_b}^{T_o}\left(\frac{\partial s^*}{\partial M}\right)_T dT + \int_{\psi_b}^{\psi_o}\left(\frac{\partial}{\partial M}\left(\frac{\eta}{r\rho_d}\right)\right)_\psi d\psi - \int_{r_b}^{r_o}\left(\frac{\partial F_r}{\partial M}\right)_r dr - \int_{z_b}^{z_o}\left(\frac{\partial F_z}{\partial M}\right)_z dz$$

$$= \frac{d}{dM}\left(\int_{T_b}^{T_o} s^* dT\right) + \frac{d}{dM}\left(\int_{\psi_b}^{\psi_o}\frac{\eta}{r\rho_d}d\psi\right) - \frac{d}{dM}\left(\int_{r_b}^{r_o} F_r dr + \int_{z_b}^{z_o} F_z dz\right)$$

Distribution of thermal energy

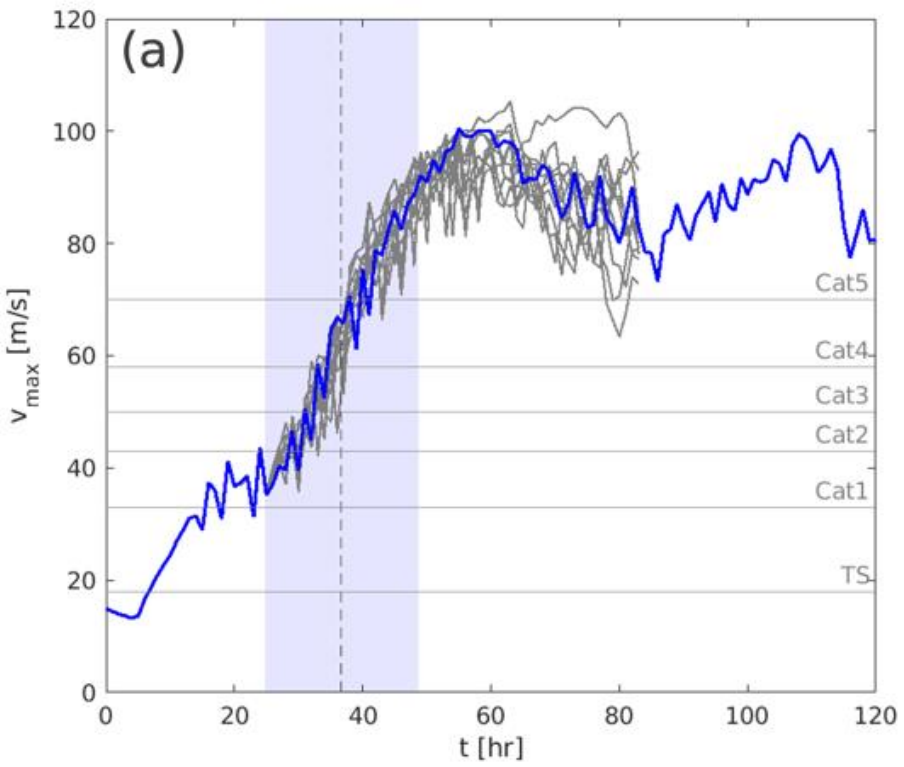
Distribution of azimuthal vorticity η

Distribution of work done against friction

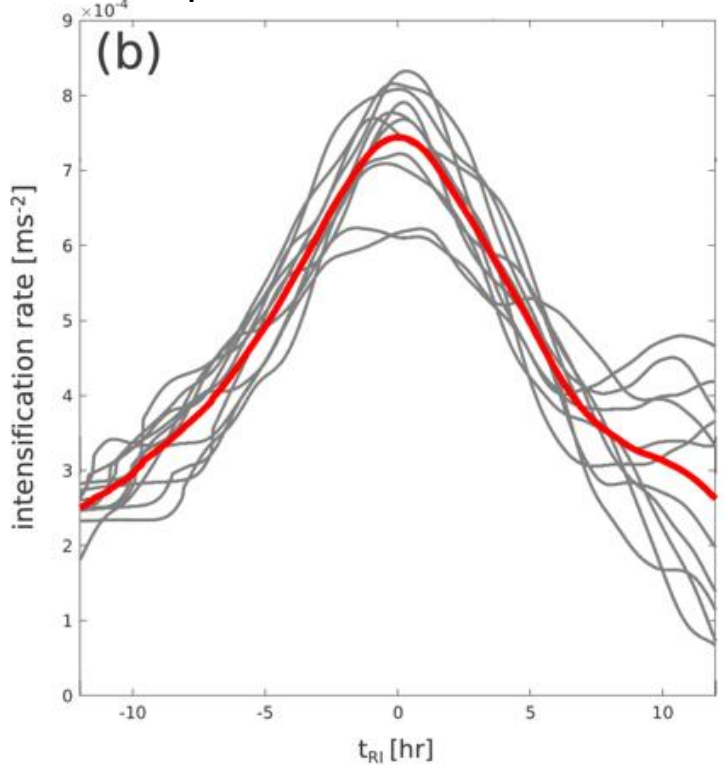
Verification against CM1 simulations

CM1 setting:	
Hori. resol	500m
# of Vert. level	141 levels
Microphysics	ER87 water-only scheme
PBL	BR09 Louis-type
Hori. mixing len	l_h=1500m

Vmax time series



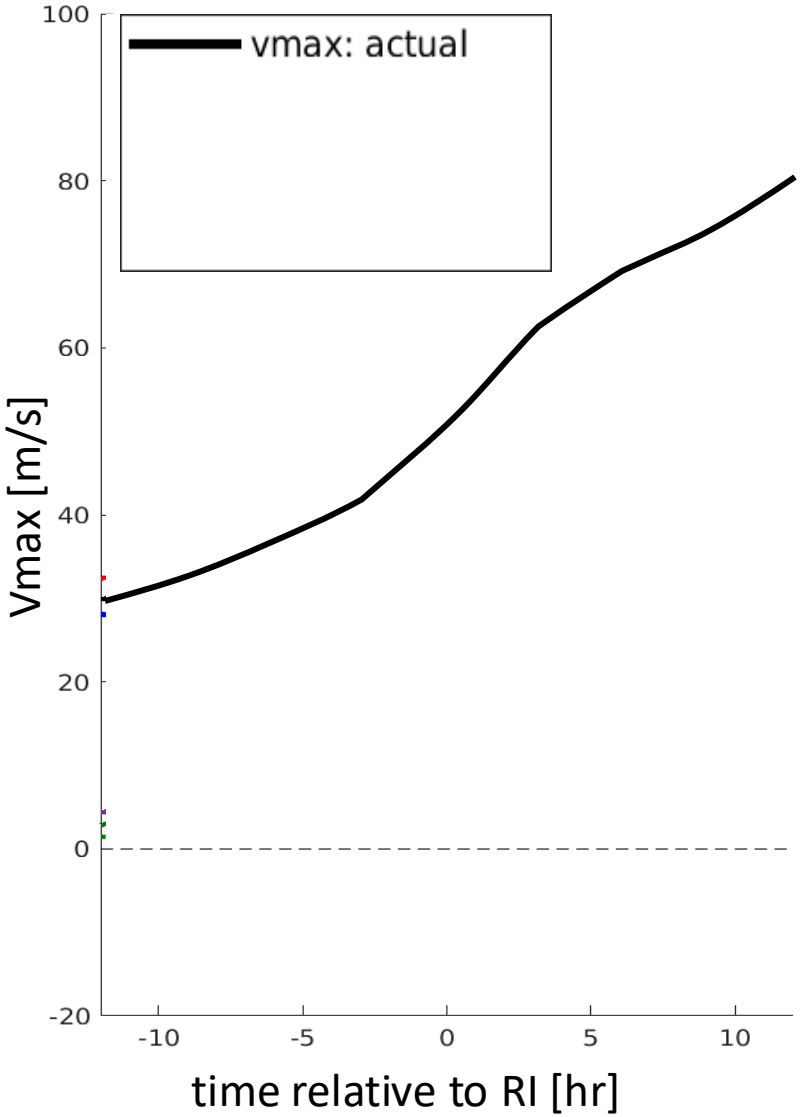
Composite: intensification rate



A generalized v_{max} equation: Final form

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$$- \int_{r_b}^{r_o} \left(\frac{\partial F_r}{\partial M}\right)_r dr - \int_{z_b}^{z_o} \left(\frac{\partial F_z}{\partial M}\right)_z dz$$

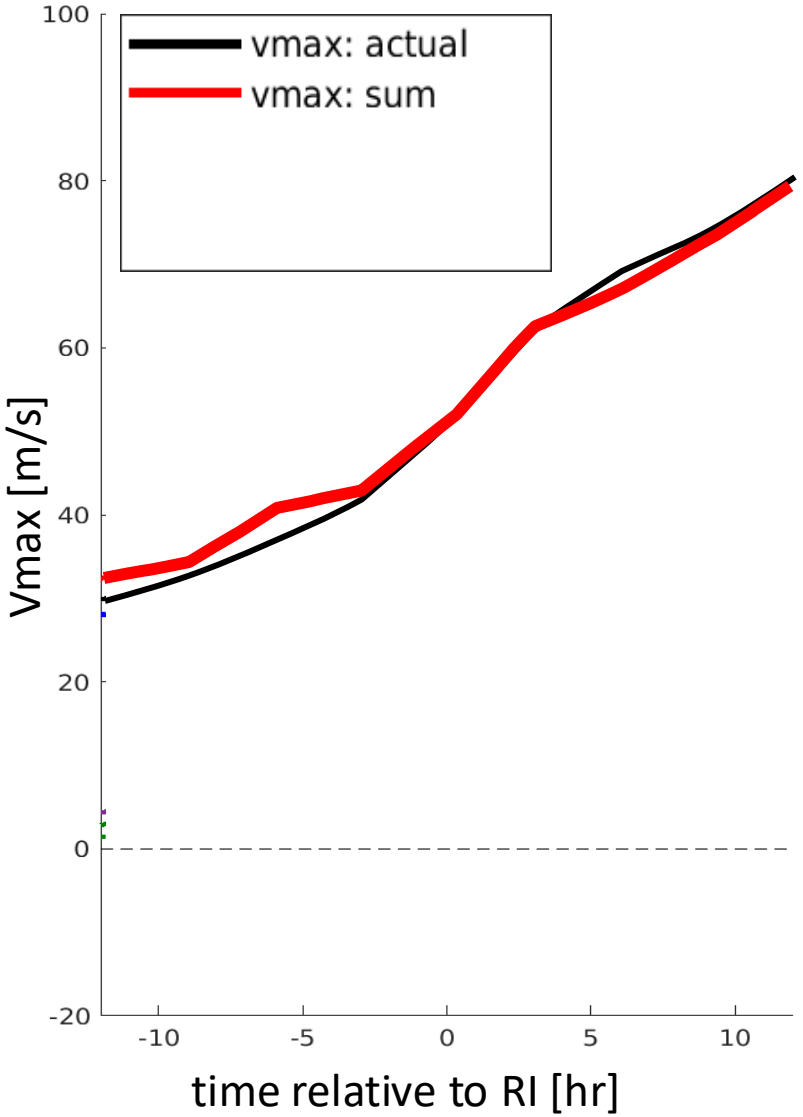
Composite mean of a 11-member ensemble simulation



A generalized v_{max} equation: Final form

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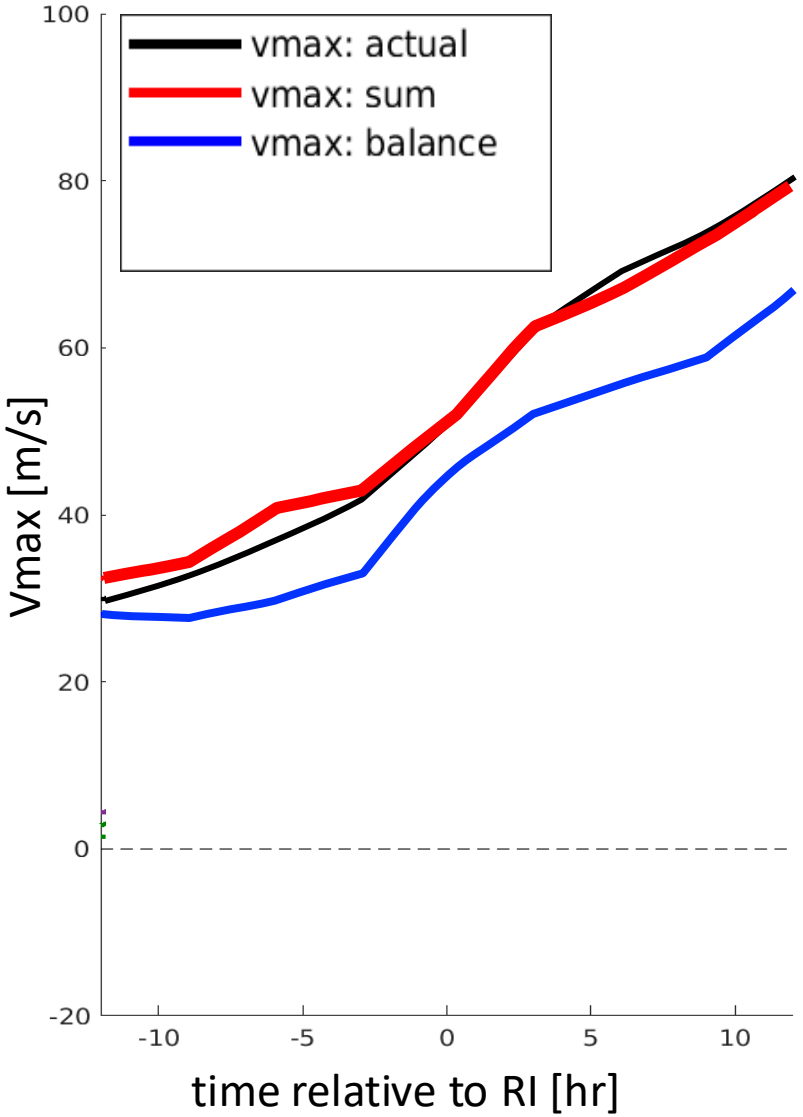
Composite mean of a 11-member ensemble simulation



A generalized v_{max} equation: Final form

$$\frac{v_b}{r_b} = \int_{T_b}^{T_o} \left(\frac{\partial s^*}{\partial M}\right)_T dT + \int_{\psi_b}^{\psi_o} \left(\frac{\partial}{\partial M}\left(\frac{\eta}{r\rho_d}\right)\right)_\psi d\psi - \int_{r_b}^{r_o} \left(\frac{\partial F_r}{\partial M}\right)_r dr - \int_{z_b}^{z_o} \left(\frac{\partial F_z}{\partial M}\right)_z dz$$

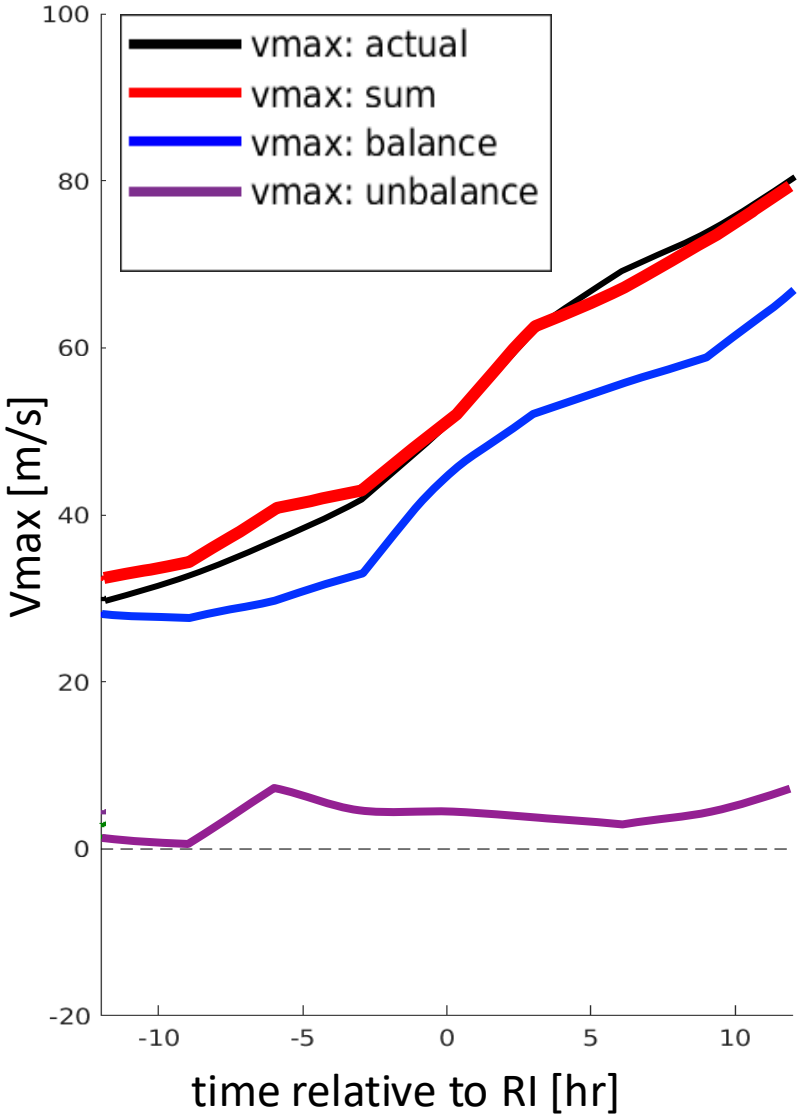
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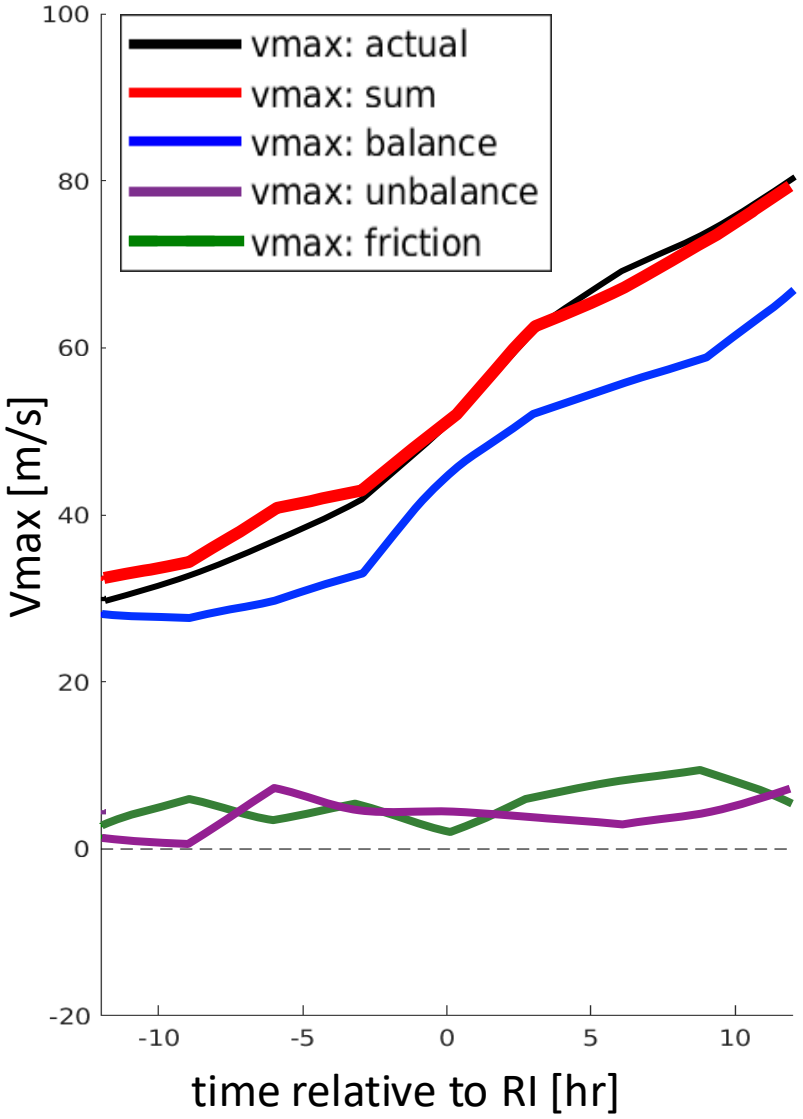
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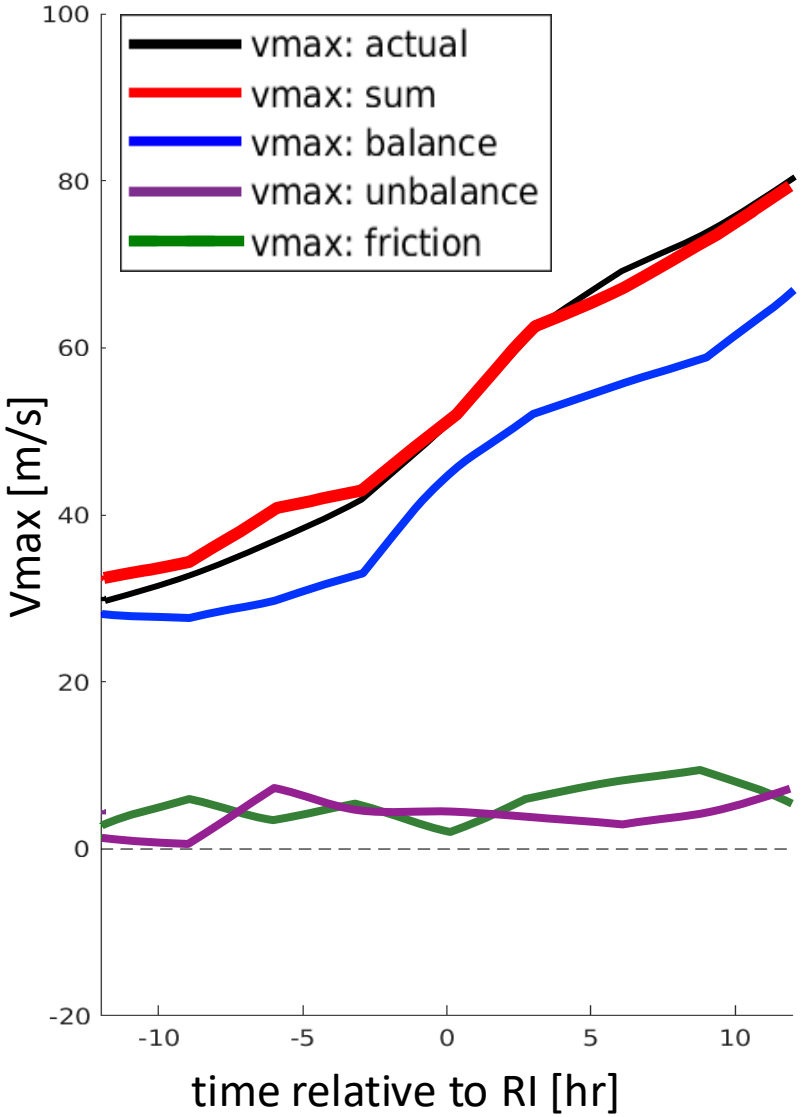
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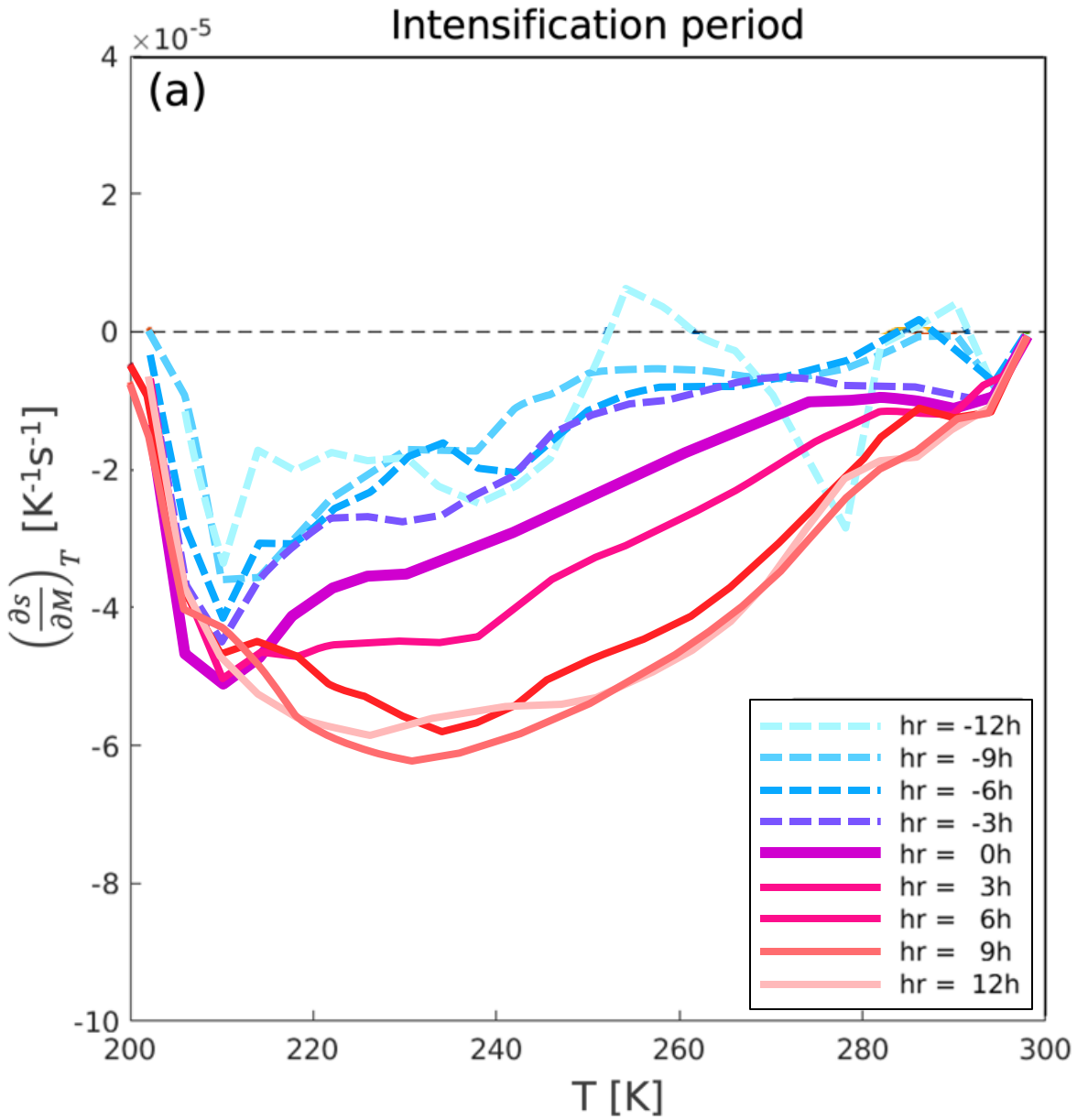
Intensification is mainly due to spin up of the balanced vortex with horizontal mixing length $l_h = 1500\,m$.

Composite mean of a 11-member ensemble simulation



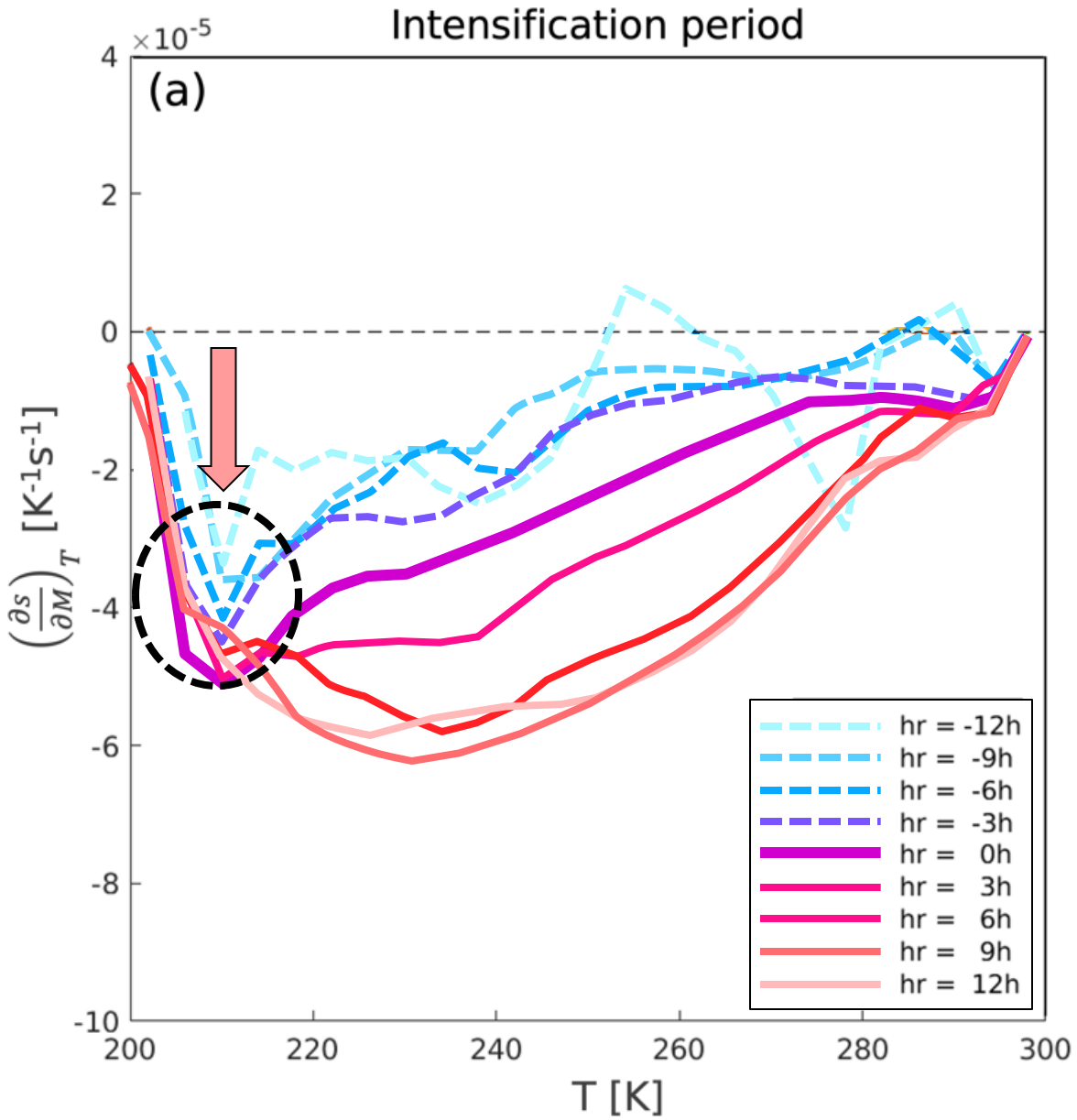
Structure of the balanced term

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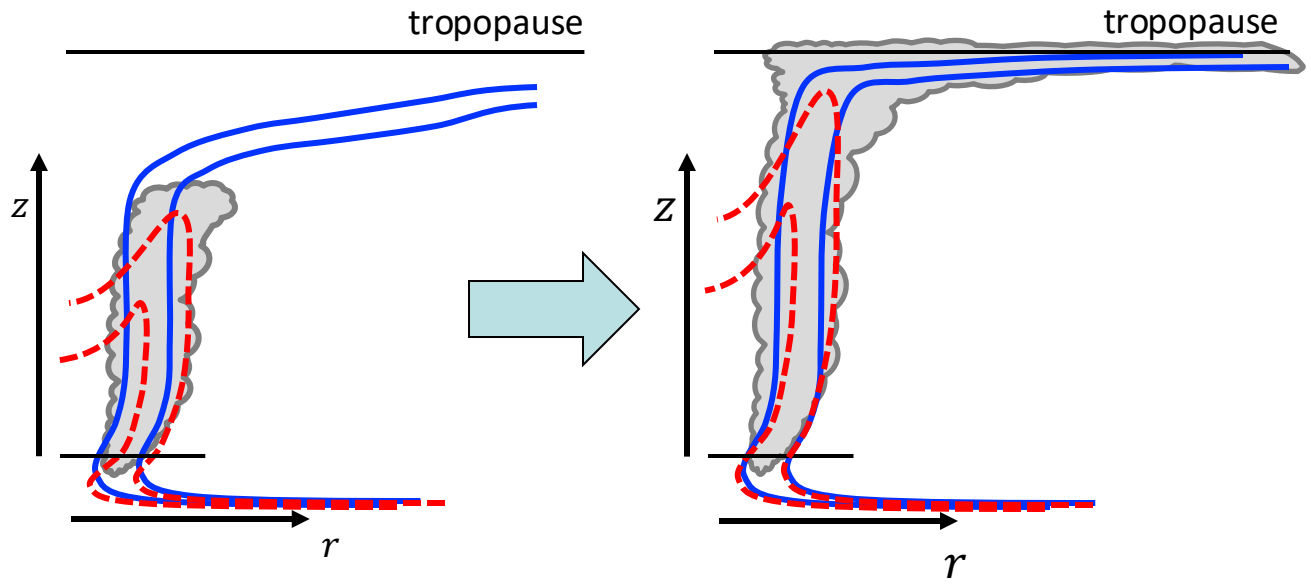
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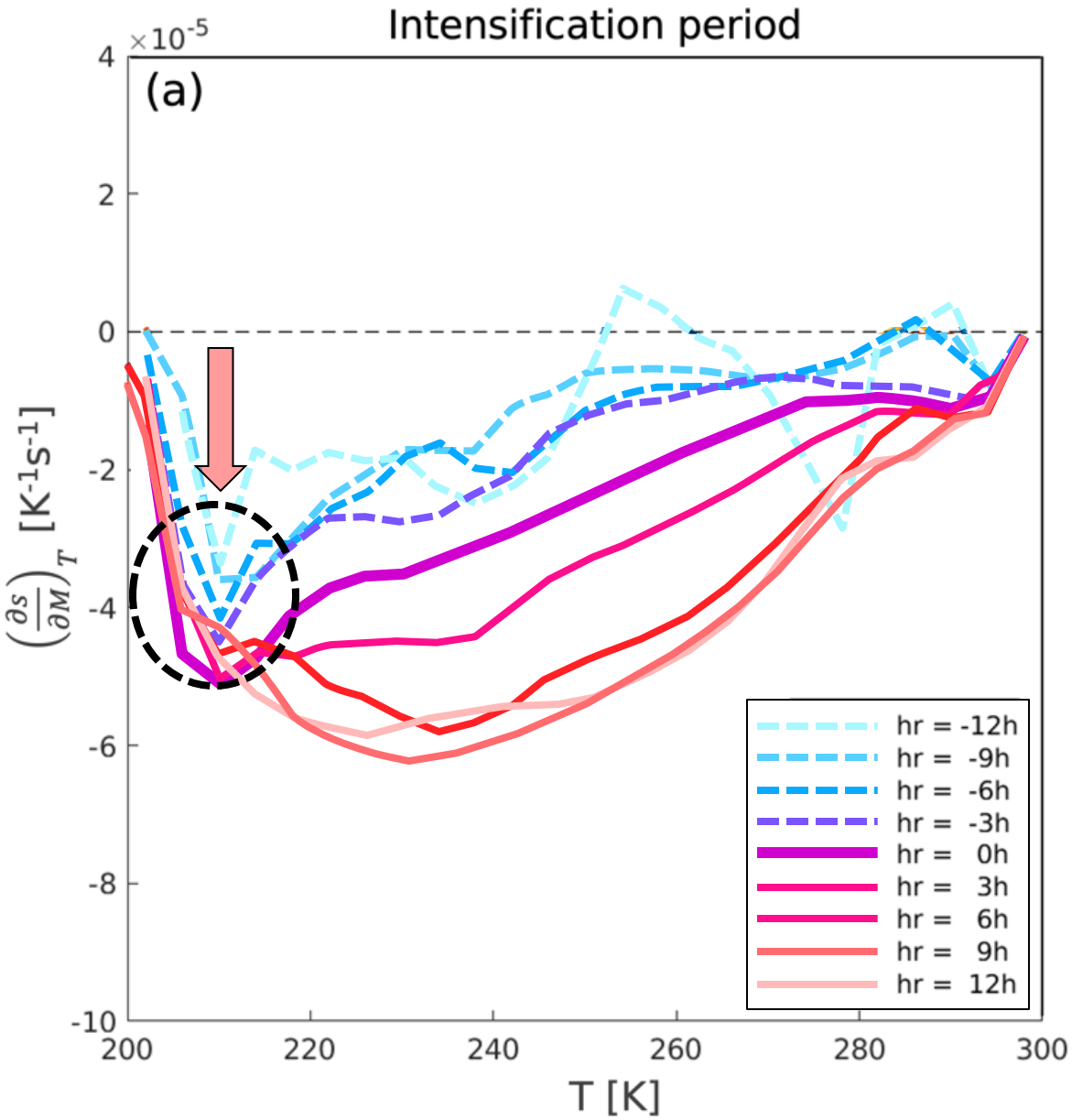


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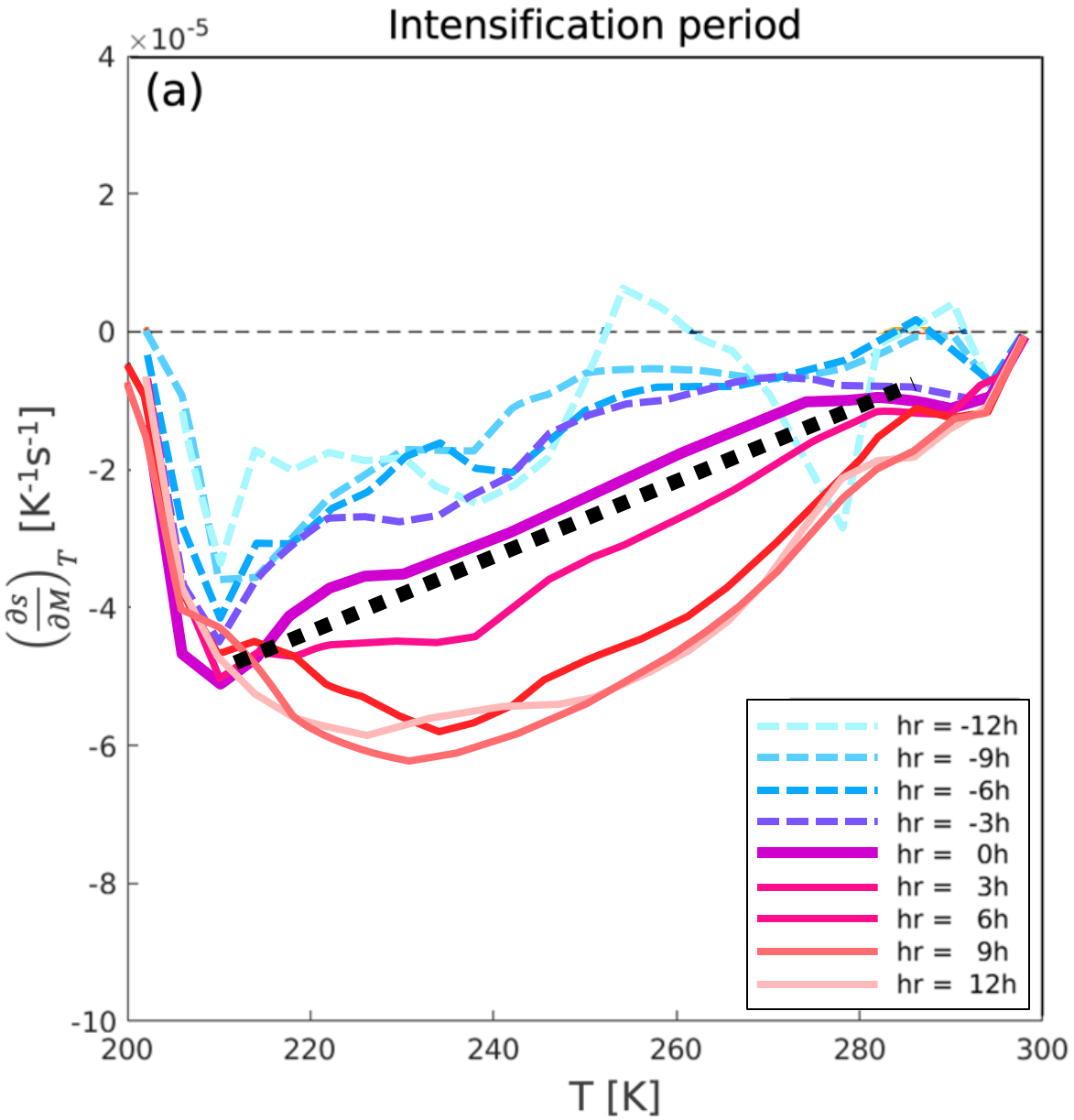


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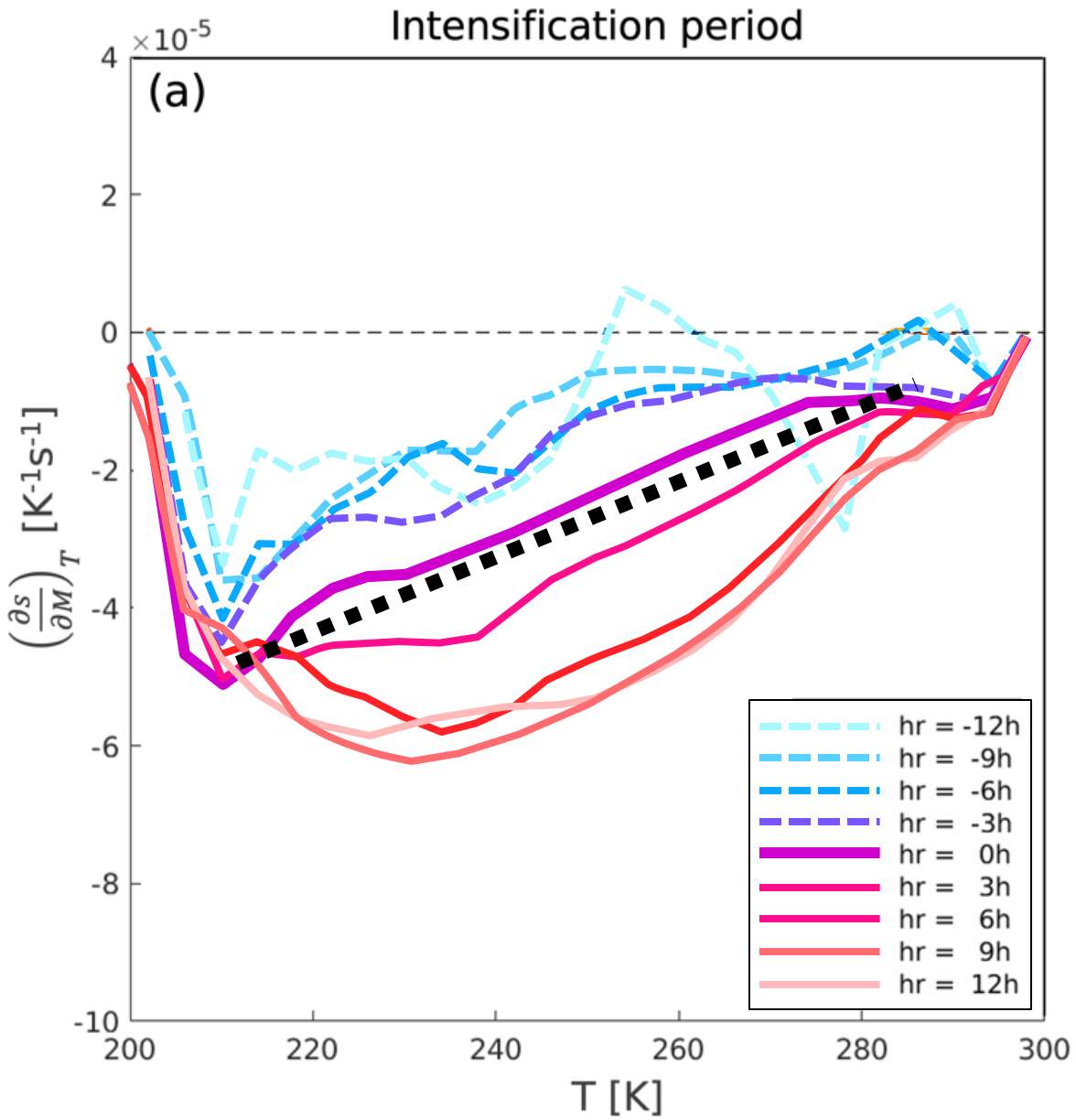
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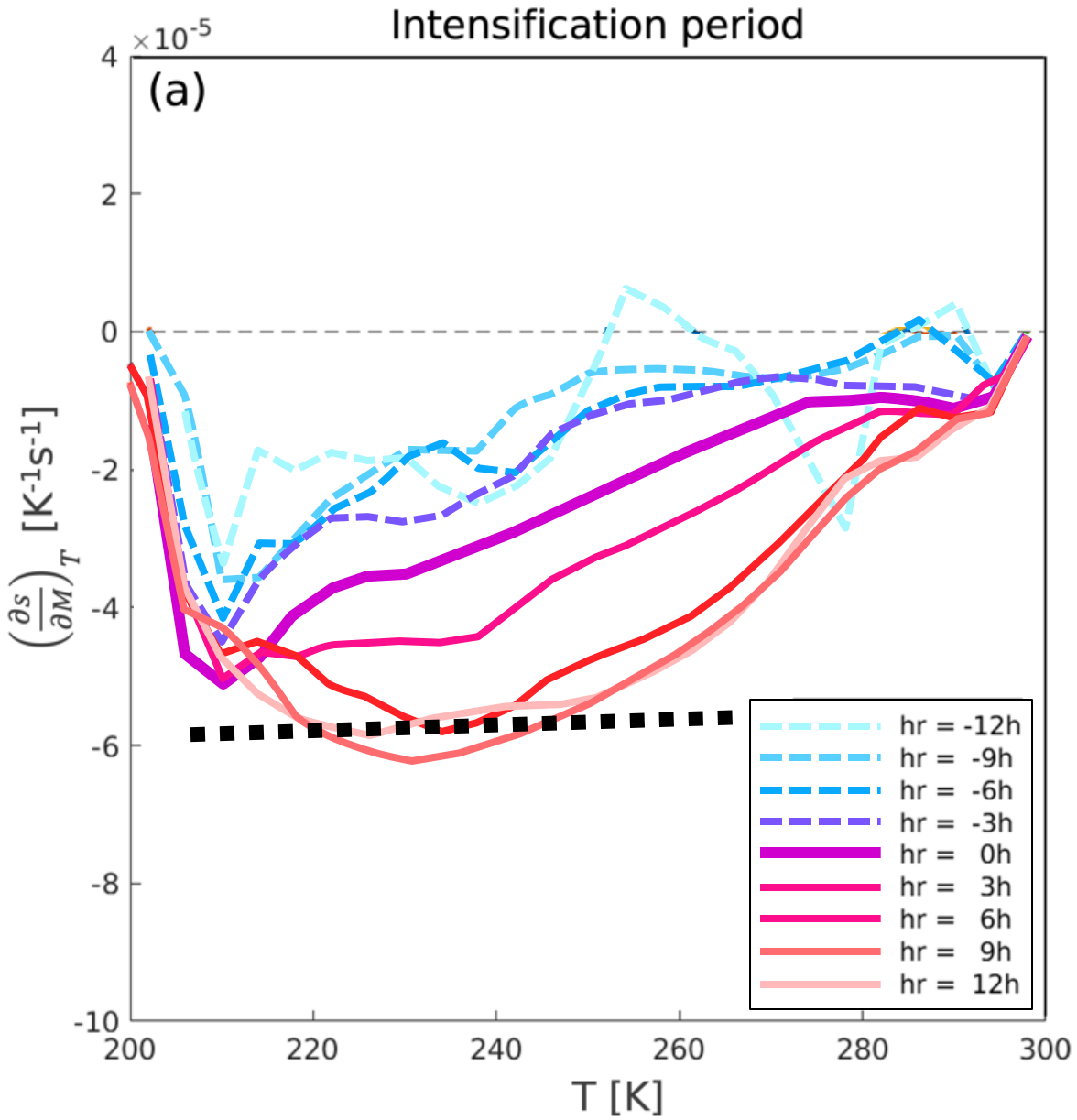
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→ The vortex is close to symmetrically neutral



Summary and implications

- *Under Non-SN condition*, we show that $\frac{\partial s^*}{\partial M}$ should be evaluated along ***constant T direction***, i.e., $\left(\frac{\partial s^*}{\partial M}\right)_T$, which controls the inward bending slope of an M surface.
- The balanced v_m precisely equals the accumulation of $\left(\frac{\partial s^*}{\partial M}\right)_T$ from T_b to T_o.
- Eyewall is a frontal zone of integrated thermal energy across M surfaces.
- Using the same technique, it is possible to derive a fully generalized expression of v_{max} that also capture the supergradient and frictional contributions under Non-SN condition.

Unique vortex structure during RI:

1. $\left(\frac{\partial s^*}{\partial M}\right)_T$ develops large value at tropopause before t_{RI}, indicating the development of a deep tall vortex that reaches the tropopause preconditions the onset of RI.

2. $\left(\frac{\partial s^*}{\partial M}\right)_T$ achieves a linear structure w.r.t T at t_{RI}. This vertical gradient indicates vertical advection may be important in TC intensification; an effect neglected in previous studies

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Thank you for your attention! My email: **cyu7@albany.edu**

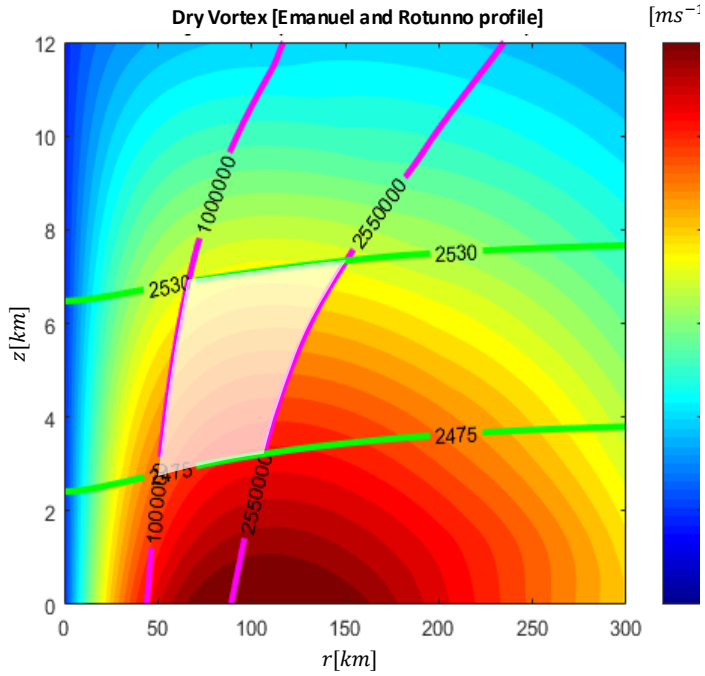
Energetic principle of balanced vortex

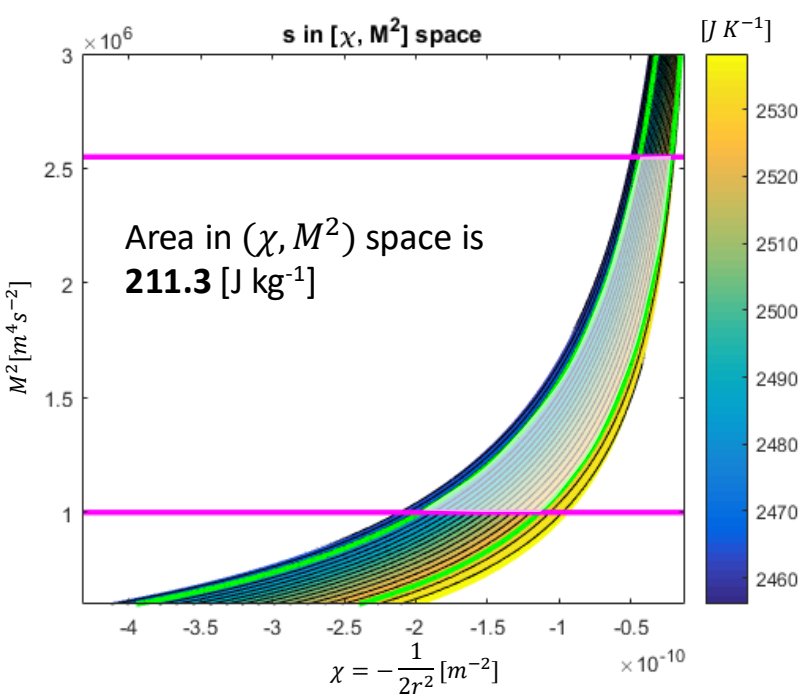
$\frac{2M}{r^3}\left(\frac{\partial r}{\partial p}\right)_M = \left(\frac{\partial T}{\partial p}\right)_{s^*}\left(\frac{\partial s^*}{\partial M}\right)_p$ represents torque balance between centrifugal force and baroclinicity

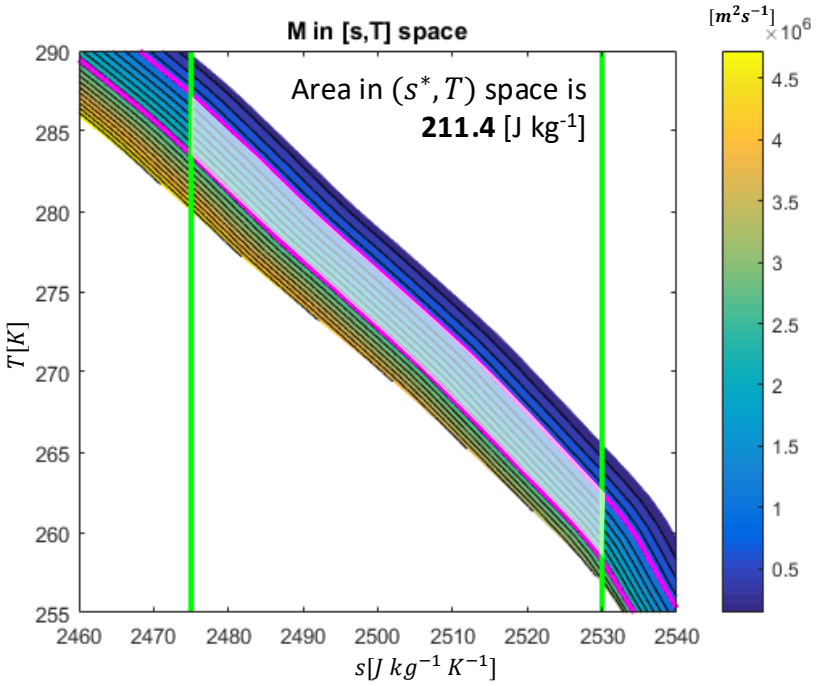
Such balance results in equipartition of virtual rotational work done by these two torques

$$\frac{2M}{r^3}\left(\frac{\partial r}{\partial p}\right)_M dMdp = \left(\frac{\partial T}{\partial p}\right)_{s^*}\left(\frac{\partial s^*}{\partial M}\right)_p dMdp \quad \Rightarrow \quad dM^2 d\left(-\frac{1}{2r^2}\right) = ds^* dT \quad \Rightarrow$$

Area in $(M^2, -1/2r^2)$ space =
Area in (T, s^*) space







For any region of a balanced vortex, the virtual rotational work associated with effective gravity must equal that of the baroclinicity. In this paper, I show that the equipartition of virtual rotational work constrains the structure of the vortex:

$$\frac{2M}{r^3}\left(\frac{\partial r}{\partial T}\right)_M = \left(\frac{\partial s^*}{\partial M}\right)_T \quad \Rightarrow \quad \frac{v_m}{r_m} = \int_{T_b}^{T_o}\left(\frac{\partial s^*}{\partial M}\right)_T dT$$