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Generalized Radiative Transfer Theory: Accounting for Unresolved Spatial Variability in Predominantly Scattering Regimes

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Motivation

Classic radiative transfer equations (RTEs) in integro-differential form are derived assuming the Beer-Lambert (BL) law accounts for light propagation in the medium, and various numerical solutions have been developed within this framework. However, natural media are to some extent variable in space, thus the BL law of exponential extinction does not apply *on average* due to its nonlinearity. What is the average value of direct transmission $T(s) = e^{-\sigma s}$, where s is a fixed distance and σ is random mean extinction coefficient over distance s?

To this problem, some approaches seek an effective material property σ_{eff} to use in the solution of a transport problem for a uniform medium [1-3], and some linearly combine solutions from uniform media and approximate solutions for spatially correlated medium [4-5]. Yet others introduce new RTEs to solve [6-10].

In the framework of developing new RTEs for spatially-correlated stochastic media, we derive and use a power-law† model for direct light transmission and solve the RTE numerically by a Monte Carlo method and by a Markov chain method (2-stream case only, for the moment) [11].

†Conley & Collins [12] propose the same power-law transmission law to model *spectral* variability, beyond correlated-*k*.

Generalized radiative transfer equation

In a uniform plane-parallel optical medium along the direction $\mathbf{\Omega}$ on the d-dimensional sphere

Integral form

$$I(\tau,\mathbf{\Omega}) = \int_{\Xi_d}\int_0^{\tau^*} K(\tau,\mathbf{\Omega};\tau',\mathbf{\Omega}')I(\tau',\mathbf{\Omega}')\,d\tau\,d\mathbf{\Omega}' + I_0(\tau,\mathbf{\Omega})$$

$I = I_0 + I_{dif}$, where I_0 is the direct light contribution and I_{dif} is the diffuse light contribution

$$I_0(\tau,\mathbf{\Omega}) = T(|\tau-\tau'|/\mu)\delta(\mathbf{\Omega}-\mathbf{\Omega}_0)$$

$$K(\tau,\mathbf{\Omega};\tau',\mathbf{\Omega}') = \omega p(\mathbf{\Omega}\cdot\mathbf{\Omega}')\Theta\left(\frac{\tau-\tau'}{\mu}\right)\frac{\dot{T}(|\tau-\tau'|/\mu)}{|\mu|}$$

where $\Theta(x)$ is the Heaviside step function

Power law transmission model

Assuming Gamma distribution of σ,

$$\Pr\{\sigma,d\sigma\} = \frac{1}{\Gamma(a)}\sigma^{a-1}\exp\left(-\frac{a\sigma}{\bar{\sigma}}\right)d\sigma$$

with mean extinction $\bar{\sigma}$ and variability parameter

$$a = \left(\overline{\sigma^2}/\bar{\sigma}^2 - 1\right)^{-1}$$

$$T_a(s) = \int_0^\infty \exp(-\sigma s)\Pr\{\sigma,d\sigma\} = \frac{1}{(1+\bar{\sigma}s/a)^a}$$

(a) Uniform medium σ

Beer-Lambert law: $T = T_{BL} = \exp(-x)$

$$\dot{T} = \dot{T}_{BL} = -T_{BL}$$

Reciprocity law follows.

Integro-differential RTE based on Beer's law

$$\left[\Omega_z\frac{d}{dz}+\sigma\right]I_{dif}(z,\mathbf{\Omega}) =$$

$$\sigma_s\int_{\Xi_d} p(\mathbf{\Omega}'\cdot\mathbf{\Omega})I_{dif}(z,\mathbf{\Omega}')\,d\mathbf{\Omega}' + F_0\exp\left(-\frac{\sigma z}{\mu_0}\right)\sigma_s p(\mathbf{\Omega}_0\cdot\mathbf{\Omega})$$

(b) Spatially correlated medium with $(\bar{\sigma},a)$

Power law model: $T = T_a = \left(1+x/a\right)^{-a}$

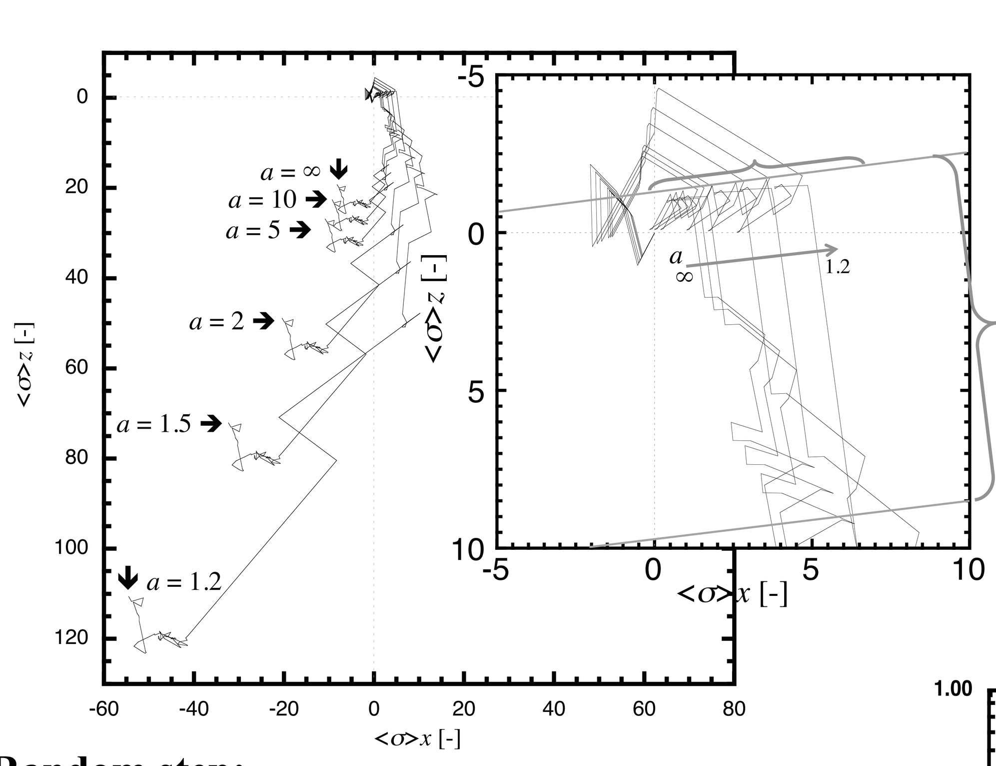
$$\dot{T} = \dot{T}_a = -\left(1+x/a\right)^{-(a+1)} \neq -T_a$$

Reciprocity law breaks !!!

Integro-differential RTE based on power law

Not yet known !!!
Cf. [10]

Monte Carlo solution in $d = 2$



Six traces of random walks with 100 isotropic scatterings and step sequences that follow power-law cumulative probabilities and PDFs. Both scatterings and steps use the same sequences of uniform random variables. Values of a are ∞ (exponential law), 10, 5, 2, 1.5 and 1.2.

Note how longer jumps occur as a decreases, and (inset) how shorter paths are required to cross a plane-parallel slab of a given thickness.

Random step:

$s = -\log\xi/\sigma$ becomes $a\times\left(\xi^{-1/a}-1\right)/\bar{\sigma}$

Random scattering (2D Henyey-Greenstein PF):

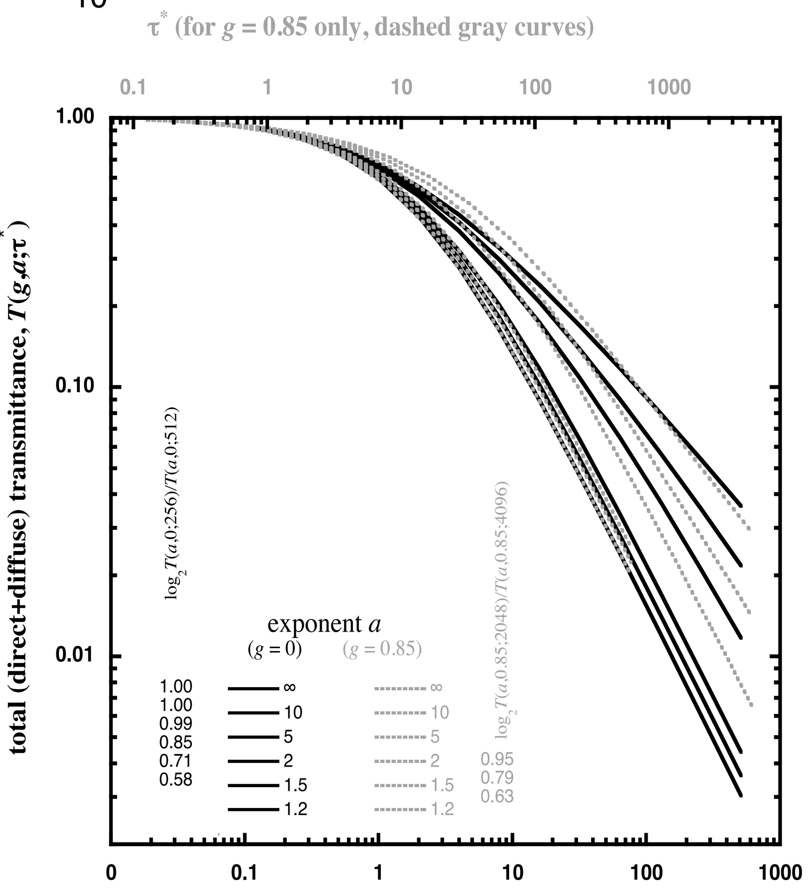
$$p(\theta_s) = \left(\frac{1}{2\pi}\right)\frac{1-g^2}{1+g^2-2g\cos\theta_s},$$

$$\text{hence } \theta_s = 2\frac{1-g}{1+g}\tan^{-1}\left[(\xi-1/2)\pi\right]$$

Monte Carlo evaluations of diffuse transmission $T(g,a;\tau^*)$ versus $(1-g)\tau^*$ in log-log axes for $g = 0$ and $g = 0.85$, and for $a = 1.2$, 1.5, 2, 5, 10, and ∞ in the absence of absorption.

Note how $T(g,a;\tau^*)$ increases when a decreases, as expected in spatially variable media. Asymptotic scaling is $\tau^{*-\min\{1,a/2\}}$, as predicted in [13]. ➔

➔ τ^* (for g = 0.85 only, dashed gray curves)



Markov chain solution in $d = 1$

a) General formalism

1D phase function: $p(0) = (1+g)/2$, $p(\pi) = (1-g)/2$

$$\mathbf{\Pi} = \mathbf{\Pi}_s + \mathbf{Q}\times\mathbf{\Pi}_s + \mathbf{Q}\times\mathbf{Q}\times\mathbf{\Pi}_s + \ldots = (\mathbf{E}-\mathbf{Q})^{-1}\times\mathbf{\Pi}_s$$

$\mathbf{\Pi}_s$: initial distribution of 1st order scattered photons

$\mathbf{Q}$: matrix of transition of photon from status $(\mathbf{x}_n, \mathbf{\Omega}_j)$ to $(\mathbf{x}_{n'}, \mathbf{\Omega}_{j'})$

b) Literal 1D, a.k.a. 2-stream model

Photon transient status (n, j): node $n = 1,\ldots,N$, direction j = + (up) or − (down)

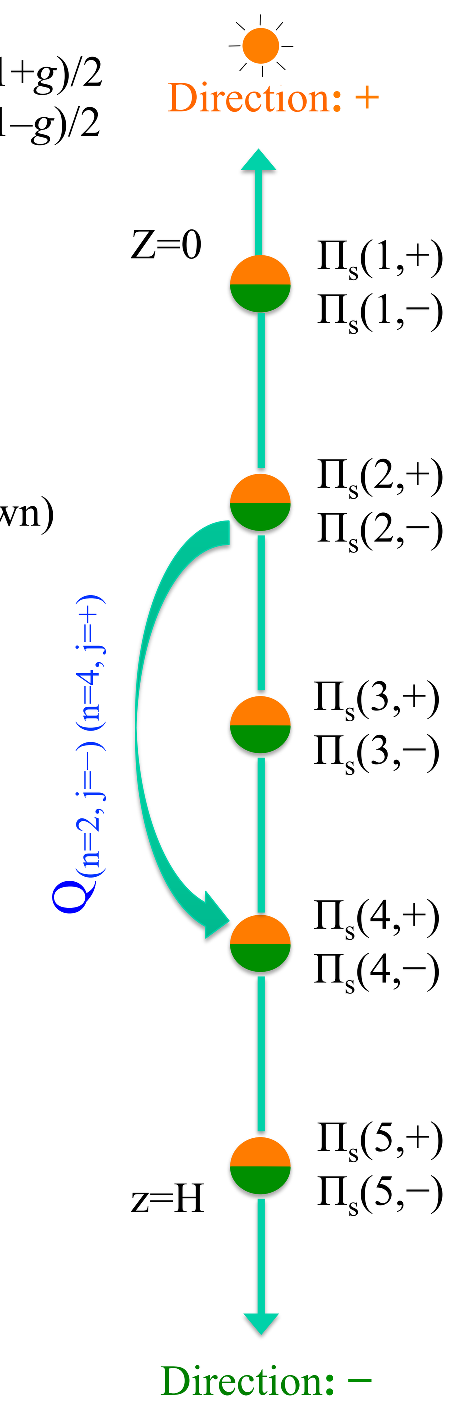
$$Q_{(n',j),(n,i)} = \begin{cases} -\dfrac{1}{\Delta\tau_n}\displaystyle\int_{\tau_{n-1}}^{\tau_n}\int_{\tau_{n'-1}}^{\tau_{n'}}\dot{T}_a(\tau-x)\omega p(\Delta\theta_{ji})\,dx\,d\tau, & n>n' \\ (1-P_{esc})\omega p(\Delta\theta_{ji}), & n=n' \\ -\dfrac{1}{\Delta\tau_n}\displaystyle\int_{\tau_{n-1}}^{\tau_n}\int_{\tau_{n'-1}}^{\tau_{n'}}\dot{T}_a(x-\tau)\omega p(\Delta\theta_{ji})\,dx\,d\tau, & n<n' \end{cases}$$

$$\Pi_{s,(n,i)} = -F_0\omega p(\Delta\theta_{i0})\int_{\tau_{n-1}}^{\tau_n}\dot{T}_a(\tau)\,d\tau$$

Diffuse field

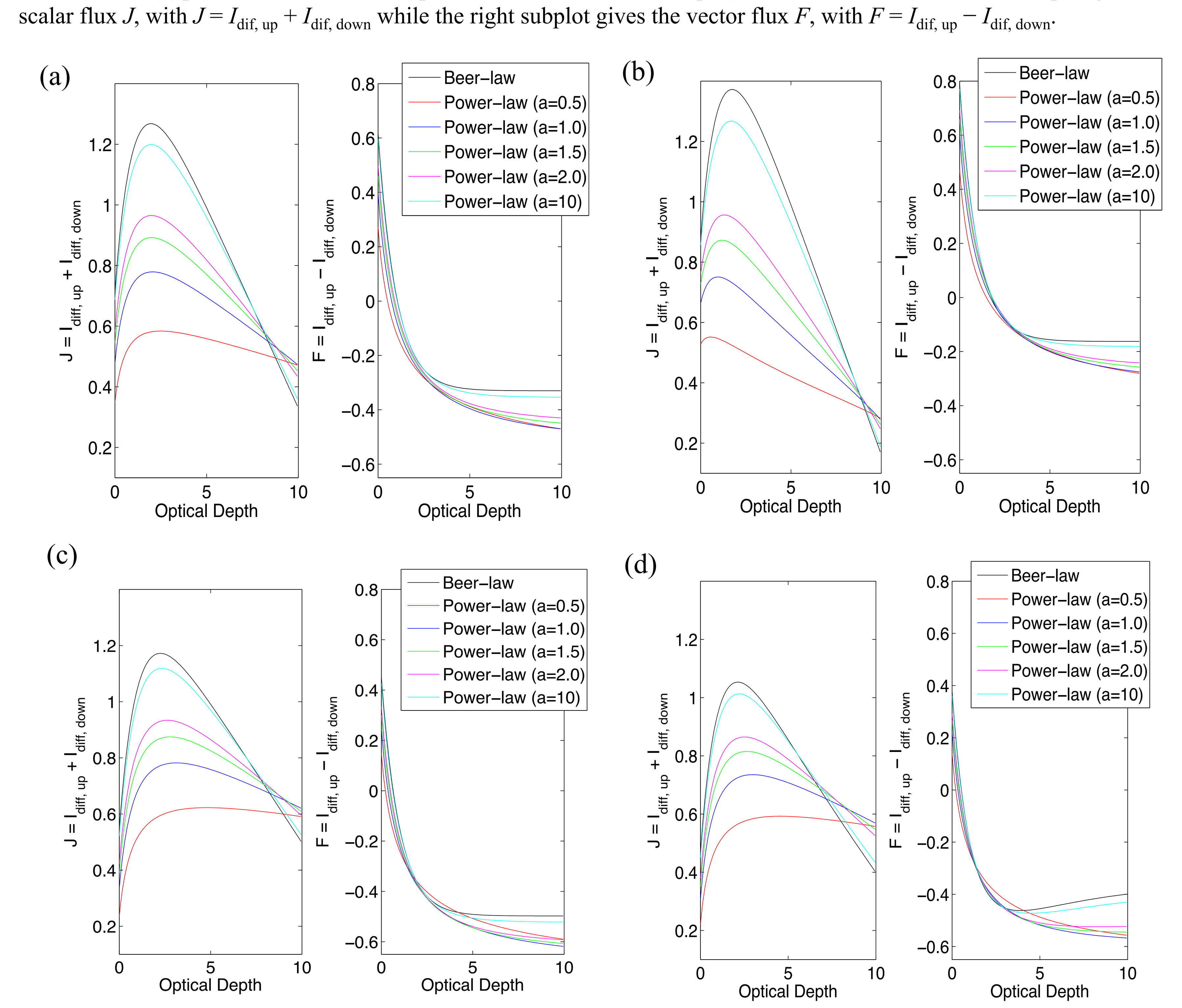
$$I_{diff,up}^{(n)} = \sum_{k=n}^{N}\Pi_{(k,1)}\overline{T}_a(k,\tau_{n-1}) \qquad I_{diff,dn}^{(n)} = \sum_{k=1}^{n}\Pi_{(k,0)}\overline{T}_a(k,\tau_n)$$

$$\overline{T}_a(n,x) = \begin{cases} \dfrac{1}{\Delta\tau_n}\displaystyle\int_{\tau_{n-1}}^{\tau_n}T_a(\tau-x)\,d\tau, & x\le\tau_{n-1} \\ \dfrac{1}{\Delta\tau_n}\displaystyle\int_{\tau_{n-1}}^{\tau_n}T_a(x-\tau)\,d\tau, & x\ge\tau_n. \end{cases}$$



Numerical investigation in $d = 1$

Radiation fields throughout the 1D medium with $\tau = 10$ and (a) forward phase function $p(0) = 0.8$ and single scattering albedo $\omega = 1$; (b) $p(0) = 0.5$ and $\omega = 1$; (c) $p(0) = 0.9$ and $\omega = 1$, and (d) $p(0) = 0.9$ and $\omega = 0.98$. The left subplot gives scalar flux J, with $J = I_{dif,up} + I_{dif,down}$ while the right subplot gives the vector flux F, with $F = I_{dif,up} - I_{dif,down}$.



References:

[1] A. Davis, et al., *J. Geophys. Res.* **95**, 11729 (1990).
[2] R.F. Cahalan, *Nonlinear Proc. Geophys.* **1**, 156 (1994).
[3] B. Cairns et al., *J. Atmos. Sci.* **57**, 700 (2000).
[4] R.F. Cahalan et al., *J. Atmos. Sci.* **51**, 3776 (1994).
[5] H.W. Barker, *J. Atmos. Sci.* **53**, 2289 (1996).
[6] O.A. Avaste and G.M. Vainikko, *Acad. Sci. USSR, Atmos. Oceanic Phys.* **10**, 1054 (1973).
[7] C.D. Levermore, G.C. Pomraning, D.L. Sanzo, and J. Wong, *J. Math. Phys.* **27**, 2526 (1986).
[8] G.L. Stephens, *J. Atmos. Sci.* **45**, 1837 (1988).
[9] A.B. Davis, *Notes in Computational Science and Engineering* **84**, Springer, New York (NY), 2006.
[10] E.W. Larsen and R. Vasques, *J. Quant. Spectrosc. Radiat. Transfer* **112**, 619 (2011).
[11] A.B. Davis and F. Xu, *Transport Theory and Statistical Physics* (2014, under revision).
[12] A.J. Conley and W.D. Collins, *J. Quant. Spectrosc. Radiat. Transfer* **112**, 1525 (2011).
[13] A. Davis and A. Marshak, in *Fractal Frontiers*, Eds. M.M. Novak and T.G. Dewey, World Scientific (Singapore), pp. 63–72 (1997).

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