A Quasi-Geostrophic System of Equations in Terrain-Following Coordinates

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1. Introduction

The quasi-geostrophic (QG) system of equations has been very useful for studying and understanding the large-scale and synoptic-scale dynamic processes (Hoskins et al. 1985). The QG system is also useful for studying the interactions between the synoptic-scale and mesoscale processes as the secondary circulation forced by the primary geostrophically balanced flow in the QG system plays a key role in the development of some subsynoptic-scale flow phenomena such as fronts. However, the classic QG system was formulated and has been widely used in the pressure coordinate with the lower boundary set to the mean sea level pressure. When the secondary circulation was diagnosed from this system, the effects of surface pressure variations and their interactions with the terrain were neglected. To solve this and other related problems, a QG system is derived in terrain-following coordinates. This QG system preserves the potential vorticity (PV) conservation and contains the effects of terrain and surface pressure variation. A complete set of diagnostic equations is derived for the secondary circulation in the new QG system, and this set of diagnostic equations also contains the effects of terrain and surface pressure variation.

2. PE systems in η coordinates

The PE system in η-coordinates is given by

$$d_t\mathbf{v} + f\mathbf{k}\times\mathbf{v} + \nabla\phi + \eta\alpha\nabla\mu = 0, \tag{1a}$$
$$d_t\theta = 0, \tag{1b}$$
$$\partial_\eta\phi = -\mu\alpha, \tag{1c}$$
$$d_t\mu + \mu\nabla_3\cdot\mathbf{v}_3 = 0, \tag{1d}$$
$$\alpha = \theta R(\mu\eta + p_t)^{k-1}/p_0^{\,k}, \tag{1e}$$

where $d_t \equiv \partial_t + \mathbf{v}_3\cdot\nabla_3$, $\nabla_3 \equiv (\nabla, \partial_\eta)$, $\nabla \equiv (\partial_x, \partial_y)$, $\mathbf{v_3} \equiv (\mathbf{v}, \omega)$, $\mathbf{v} \equiv (u, v)$ is the horizontal velocity along the η-surface, $\eta \equiv (p - p_t)/\mu$, $\mu = p_s - p_t$, $\omega \equiv d_t\eta$ is the vertical velocity in η coordinates, α is the specific volume, θ is the potential temperature, $k = R/c_p$, R is the gas constant, c_p is the specific heat under constant pressure, and p_0 is the mean sea level pressure. The PV equation is $d_tq = 0$, where $q = (f\mathbf{k} + \nabla_3\times\mathbf{v}_3)\cdot\nabla_3\theta/\mu$ is the PV in this PE system.

3. Geostrophy in η coordinates

The geostrophic wind $\mathbf{v_g}$ is constrained by

$$f_o\mathbf{k}\times\mathbf{v_g} + \nabla\phi' + \eta\alpha_o\nabla\mu' = 0, \tag{2}$$

where $(\ )' \equiv (\ ) - (\ )_o$ and $(\ )_o$ denotes the averaged value of ( ) over each η-surface. Substituting $\mathbf{v_g} \equiv \mathbf{k}\times\nabla\psi_g$ into (2) gives

$$f_o\psi_g = \phi' + \eta\alpha_o\mu'. \tag{3}$$

The divergence and vertical vorticity of the geostrophic wind are given by

$$\nabla\cdot\mathbf{v_g} = \nabla\cdot(\mathbf{k}\times\nabla\psi_g) = 0$$
and $$\zeta_g \equiv \nabla\times\mathbf{v_g} = \nabla^2\psi_g,$$

respectively.

4. QG system in η coordinates

The QG-truncated (1a) is

$$d_g\mathbf{v}_g + f_o\mathbf{k}\times\mathbf{v}' + f\mathbf{k}\times\mathbf{v}_g + \nabla\phi' + \eta\alpha_o\nabla\mu' = 0, \tag{4}$$

where $d_g \equiv \partial_t + \mathbf{v_g}\cdot\nabla$ and $\mathbf{v}' \equiv \mathbf{v} - \mathbf{v_g}$ is the ageostrophic wind in this QG system. Subtracting (2) from (4) gives

$$d_g\mathbf{v_g} + f_o\mathbf{k}\times\mathbf{v}' + f'\mathbf{k}\times\mathbf{v_g} = 0. \tag{5a}$$

The QG-truncated (1b)-(1e) are given by

$$d_g\theta' + \omega\partial_\eta\theta_0 = 0, \tag{5b}$$
$$\partial_\eta\phi' = -\mu_0\alpha', \tag{5c}$$
$$d_g\mu' + \mu_0\nabla_3\cdot\mathbf{v}_3 = 0, \tag{5d}$$
$$r_0\alpha' = \theta', \tag{5e}$$

where $r_0 \equiv (\mu_0\eta + p_t)^{1-k}p_0^{\,k}/R$, and (1e) is truncated to $\alpha = \theta/r_0$ with $r_0\alpha_0 = \theta_0$.

The QG vorticity equation is given by $\mathbf{k}\cdot\nabla\times$(5a):

$$d_g\zeta_g + f_o\nabla\cdot\mathbf{v}' + \mathbf{v}_g\cdot\nabla f' = 0,$$

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or $\quad d_g Z_g + f_o\nabla\cdot\mathbf{v}' = 0,$ (6)

where $Z_g = f + \zeta_g$, $\mathbf{k}\cdot\nabla\times(f_o\mathbf{k}\times\mathbf{v}') = f_o\nabla\cdot\mathbf{v}'$ and $\mathbf{k}\cdot\nabla\times(f'\mathbf{k}\times\mathbf{v}_g) = \nabla\cdot(f'\mathbf{v}_g) = \mathbf{v}_g\cdot\nabla f' = \mathbf{v}_g\cdot\nabla f$ are used.

Combining (6) with $f_o\partial_\eta[(5\text{b})/\partial_\eta\theta_o]$ and using (5d) give the following PV conservation equation:

$$d_g q_{qg} = 0, \quad (7)$$

where $q_{qg} \equiv Z_g + f_o\partial_\eta(\theta'/\partial_\eta\theta_o) - f_o\mu'/\mu_o$ is the PV for this QG system. This PV can be considered as a linearized leading-order truncation of the semi-geostrophic (SG) PV, defined by $q_{sg} \equiv -(f\mathbf{k} + \nabla_3\times\mathbf{v_g})\cdot\nabla_3\theta/\mu$ in a SG system [derived similarly to the semibalance system (Xu and Cao 2012) except that the primary flow is $\mathbf{v_g}$ defined in (2) instead of the nonlinearly balanced flow $\mathbf{v_b}$]. Such a linearized leading-order truncation of q_{sg} is derived through the following steps:

$$\begin{aligned} q_{sg} &\approx -(f\mathbf{k} + \nabla\times\mathbf{v_g})\cdot\nabla_3\theta/\mu \\ &= -f_o(\partial_\eta\theta_o/\mu_o)[1 + (f' + \nabla^2\psi_g)/f_o](1 + \partial_\eta\theta'/\partial_\eta\theta_o)/(1 + \mu'/\mu_o) \\ &\approx -f_o(\partial_\eta\theta_o/\mu_o)[1 + (f' + \nabla^2\psi_g)/f_o + \partial_\eta\theta'/\partial_\eta\theta_o - \mu'/\mu_o] \\ &\approx -(\partial_\eta\theta_o/\mu_o)q_{qg}, \end{aligned}$$

where $\partial_\eta\theta'/\partial_\eta\theta_o \approx \partial_\eta(\theta'/\partial_\eta\theta_o)$ is used in the last step. Note that $-\partial_\eta\theta_o/\mu_o > 0$ and $d_g(\partial_\eta\theta_o/\mu_o) = 0$, so $-(\partial_\eta\theta_o/\mu_o)q_{qg}$ has the same sign as q_{qg} and is also conserved in the above QG system.

Note from (5e), (5c) and (3) that $\theta' = r_o\alpha' = r_o\partial_\eta\phi'/\mu_o = r_o f_o\partial_\eta\psi_g/\mu_o - \theta_o\mu'/\mu_o$. Substituting this into $q_{qg} \equiv Z_g + f_o\partial_\eta(\theta'/\partial_\eta\theta_o) - f_o\mu'/\mu_o$ gives

$$\begin{aligned} q_{qg} &= f + \nabla^2\psi_g + f_o^2 r_o\partial_\eta(\partial_\eta\psi_g/\partial_\eta\theta_o)/\mu_o \\ &\quad + f_o[\theta_o\partial_\eta^2\theta_o/(\partial_\eta\theta_o)^2 - 2]\mu'/\mu_o, \end{aligned} \quad (8)$$

so q_{qg} is invertible for given μ' and boundary values of ψ_g ($= \phi' + \eta\alpha_o\mu'$).

5. Secondary circulation – $\mathbf{v_3}' \equiv (\mathbf{v}', \omega)$

The secondary circulation consists of the ageostrophic wind $\mathbf{v}'$ and vertical velocity ω. To obtain a complete solution of the secondary circulation, it is necessary and convenient to partition $\mathbf{v}'$ into two parts: a barotropic part and a baroclinic part (Xu and Keyser 1993). The barotropic part of $\mathbf{v}'$ is denoted and defined by $\mathbf{v}'^b \equiv \int_0^1\mathbf{v}'d\eta$. As shown later in (13), this part can be solved in terms of ψ_3, defined by $\mathbf{k}\times\nabla\psi_g \equiv \mathbf{v}'^b$, from the vertical component of the C-vector equation (Xu 1992). The baroclinic part of $\mathbf{v}'$ is denoted and defined by $\mathbf{v}'^a \equiv \mathbf{v}' - \mathbf{v}'^b$, so it satisfies $\int_0^1\mathbf{v}'^a d\eta = \int_0^1\mathbf{v}'d\eta - \mathbf{v}'^b = 0$. As shown later in (18), this part can be solved in terms of $\mathbf{\Psi}$ from the horizontal component of the C-vector equation, where $\mathbf{\Psi} \equiv (\psi_1, \psi_2)^T$ is the vector psi-streamfunction defined by $-\partial_\eta\mathbf{\Psi} \equiv \mathbf{v}'^a$ with the homogeneous boundary conditions of $\mathbf{\Psi} = 0$ at $\eta = 0$ and 1.

From (5c) and (5e) we obtain

$$\partial_t\theta' = -r_o\partial_\eta\partial_t\phi'/\mu_o. \quad (9)$$

Substituting (9) into (5b) gives

$$-\omega\partial_\eta\theta_o = \mathbf{v_g}\cdot\nabla\theta' - r_o\partial_\eta\partial_t\phi'/\mu_o,$$

or $\quad -\mu_o\alpha_o\omega\partial_\eta\ln\theta_o = \mu_o\alpha_o\mathbf{v_g}\cdot\nabla\theta'/\theta_o - \partial_\eta\partial_t\phi'.$ (10)

Combining $-f_o\mathbf{k}\times(5\text{a}) + \mathbf{k}(10)$ gives

$$\mathbf{\Pi v_3}' = \mathbf{P} - \partial_t\nabla_3\phi' - \partial_t\nabla(f_o\psi_g - \phi'), \quad (11)$$

where

$$\begin{aligned} &\mathbf{P} \equiv f_o\mathbf{k}\times\mathbf{A} + \mathbf{k}B, \\ &\mathbf{A} \equiv \mathbf{v_g}\cdot\nabla\mathbf{v_g} - f'\nabla\psi_g, \\ &B = \mu_o\alpha_o\mathbf{v_g}\cdot\nabla\theta'/\theta_o, \\ &\mathbf{\Pi} = (f_o^2, f_o^2, -\mu_0\alpha_0\partial_\eta\ln\theta_0)^{\text{diag}}. \end{aligned}$$

The C-vector diagnostic equation for the secondary circulation can be derived from $\nabla_3\times(11)$; that is,

$$\nabla_3\times(\mathbf{\Pi v_3}') = 2\mathbf{C}, \quad (12)$$

where

$$\begin{aligned} 2\mathbf{C} &= \nabla_3\times\mathbf{P} - \mathbf{k}\times\nabla\partial_\eta\partial_t(f_o\psi_g - \phi') \\ &= \nabla_3\times(f_o\mathbf{k}\times\mathbf{A} + \mathbf{k}B) - \mathbf{k}\times\nabla[(\alpha_o + \eta\partial_\eta\alpha_o)\partial_t\mu'] \\ &= f_o\mathbf{k}(\nabla\cdot\mathbf{A}) - f_o\partial_\eta\mathbf{A} - \mathbf{k}\times\nabla[B + (\alpha_o + \eta\partial_\eta\alpha_o)\partial_t\mu'], \end{aligned}$$

where $\nabla_3\times\nabla_3\phi' = 0$, $\nabla_3\times\nabla(f\psi_g - \phi') = \mathbf{k}\times\nabla\partial_\eta(f\psi_g - \phi') = (\alpha_o + \eta\partial_\eta\alpha_o)\mathbf{k}\times\nabla\mu'$ [see (3)], $\nabla_3\times(\mathbf{k}\times\mathbf{A}) = \mathbf{k}(\nabla\cdot\mathbf{A}) - \partial_\eta\mathbf{A}$ for $\mathbf{A} = (A_1, A_2, 0)^T$ and $\nabla_3\times(\mathbf{k}B) = -\mathbf{k}\times\nabla B$ are used. Integrating the vertical component of (12) over the entire depth from $\eta = 0$ to 1 gives the following diagnostic equation for ψ_3:

$$f_o\nabla^2\psi_3 = \int_0^1[\nabla\cdot(f'\nabla\psi_g) - 2J_{xy}(\partial_x\psi_g, \partial_y\psi_g)]d\eta, \quad (13)$$

where $\mathbf{v}' = \mathbf{v}'^a + \mathbf{v}'^b$, $\int_0^1\mathbf{v}'^a d\eta = 0$ and $\mathbf{v}'^b = \mathbf{k}\times\nabla\psi_g$ are used.

Substituting $\mathbf{v}' = \mathbf{v}'^a + \mathbf{v}'^b = \mathbf{k}\times\nabla\psi_3 - \partial_\eta\mathbf{\Psi}$ and $\mathbf{v}_g = \mathbf{k}\times\nabla\psi_g$ into (5d) gives:

$$\partial_\eta \omega = -\nabla\cdot\mathbf{v} - \mathbf{v}_g\cdot\nabla\mu'/\mu_o - \partial_t\mu'/\mu_o$$

$$= \partial_\eta \nabla\cdot\mathbf{\Psi} - [J_{xy}(\psi_g, \mu') + \partial_t\mu']/\mu_o. \quad (14)$$

Integrating (14) over the entire depth from $\eta = 0$ to 1, with the homogeneous boundary conditions of $\omega = 0$ and $\mathbf{\Psi} = 0$ at $\eta = 0$ and 1, gives

$$\partial_t\mu' = Y(\mu', 1, \psi_g) \equiv J_{xy}(\mu', \textstyle\int_0^1 \psi_g d\eta). \quad (15)$$

Integrating (14) vertically from 0 to η with the homogeneous boundary conditions for ω and $\mathbf{\Psi}$ at $\eta = 0$ gives

$$\omega = \nabla\cdot\mathbf{\Psi} + E/\mu_o, \quad (16)$$

where $E = Y(\mu', \eta, \psi_g) - \eta Y(\mu', 1, \psi_g)$, $Y(\mu', \eta, \psi_g) \equiv J_{xy}(\mu', \int_0^\eta \psi_g d\eta)$, and (15) is used.

Combining $\mathbf{v}' = \mathbf{k}\times\nabla\psi_3 - \partial_\eta\mathbf{\Psi}$ with $\mathbf{k}$(16) gives

$$\mathbf{v}_3' = \mathbf{D\Psi} + \mathbf{k}\times\nabla\psi_3 + \mathbf{k}E/\mu_o, \quad (17)$$

where

$$\mathbf{D} \equiv \begin{pmatrix} -\partial_\eta & 0 \\ 0 & -\partial_\eta \\ \partial_x & \partial_y \end{pmatrix}.$$

Substituting (17) into $\mathbf{k}\times$(12) gives the following diagnostic equation for $\mathbf{\Psi}$:

$$\mathbf{D}^\mathbf{T}\mathbf{\Pi}\mathbf{D\Psi} = 2\mathbf{Q} - \mathbf{D}^\mathbf{T}\mathbf{\Pi}(\mathbf{k}\times\nabla\psi_3 + \mathbf{k}E/\mu_0), \quad (18)$$

where $\mathbf{Q} = \mathbf{k}\times\mathbf{C}$, $\mathbf{k}\times\nabla_3\times(\ ) = \mathbf{D}^\mathbf{T}(\ )$ and $\partial_\eta\mathbf{v}'^b = 0$ are used. As explained earlier, the top and bottom boundary conditions for $\mathbf{\Psi}$ are homogeneous; that is, $\mathbf{\Psi} = 0$ at $\eta = 0$ and 1. Admissible lateral boundary conditions can be derived in two (Dirichlet and Nuemann) types based on the variational formulations of (18) similarly to those proposed in Xu and Davies-Jones (1993).

A single diagnostic equation for $\omega_- \equiv \nabla\cdot\mathbf{\Psi} = \omega - E/\mu_o$ can be derived from $\nabla\cdot$(18), that is,

$$(f_o^2\partial_3^2 + N_o^2\nabla\cdot\nabla)\omega_- = \nabla\cdot[2\mathbf{Q} - \mathbf{D}^\mathbf{T}\mathbf{\Pi}(\mathbf{k}\times\nabla\psi_3 + \mathbf{k}E/\mu_o)],$$

where $N_o^2 = -\mu_o\alpha_o\partial_3\ln\theta_o$. Note $Y(\mu', 0, \psi_g) = 0$ and thus $E = 0$ at $\eta = 0$ and 1. This gives $\omega_- = \omega = 0$ at $\eta = 0$ and 1. Admissible lateral boundary conditions can be derived for ω in two (Dirichlet and Nuemann) types based on the variational formulations (omitted here).

6. Applications

The above QG system uses the same η coordinates as the WRF model (Skamarock et al. 2005), so it can be applied directly to WRF simulated fields to extract the geostrophic flow and its forced secondary circulation affected by terrain and surface pressure variation and to diagnose the roles of the QG-forced secondary circulation in severe weather initiation. Since q_{qg} is conserved and invertible, this QG system can be used to study the effects of terrain on geostrophic turbulence (Charney 1971) and related predictability problems.

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