Stable and symmetric barotropic flows over topography

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Quasi-geostrophic framework

The nondimensional equation

$$J(\psi,Q) \equiv \hat{\mathbf{k}}\cdot\nabla\psi\times\nabla\left[\nabla^2\psi + h(x,y)\right] = 0 \tag{2.1}$$

governs the steady advection of potential vorticity $Q=\nabla^2\psi + h(x,y)$ by the geostrophic current $\mathbf{u}=\hat{\mathbf{k}}\times\nabla\psi$ in the presence of the $O(1)$ bathymetry $z=h(x,y)$. The assumed fluid domain

$$D=[0\le x\le\pi]\times[0\le y\le\pi] \tag{2.2}$$

is included in a certain $f-plane$, and the usual no mass-flux boundary condition

$$\psi(x,y)=0\,\forall(x,y)\in\partial D \tag{2.3}$$

will be applied to single out the model solutions. If

$$\frac{dQ}{d\psi}=-2 \tag{2.5}$$

then the solutions of problem (2.1), (2.3) are stable in the norm $\|\phi\|=\left(\int_D \phi^2\,dxdy\right)^{1/2}$ where $\phi=\phi(x,y,t)$ is any perturbation superimposed to the basic state ψ and satisfying the time-dependent version of (2.1), (2.3).

A class of stable solutions

A special solution is Taylor's vortex

$$\Psi=\sin(x)\sin(y) \tag{2.7}$$

over a flat bottom, i.e., for $h(x,y)=0$; in fact, stream function (2.7) identically verifies (2.3) and (2.6). This solution suggests to derive further solutions in the presence of a modulated bathymetry, that is for $h(x,y)\neq 0$, setting

$$\psi(x,y)=\Psi(x,y)\;[1+\eta(x,y)] \tag{2.8}$$

where $\eta(x,y)$ is determined by the request

$$\nabla^2[\Psi(1+\eta)]+h=-2\Psi(1+\eta) \tag{2.9}$$

Note that, owing to (2.7), boundary condition (2.3) is satisfied by (2.8) provided that $\eta(x,y)$ takes finite values everywhere in the fluid domain (2.2). Criterion (2.5) is still satisfied by (2.9), so solution (2.8) is stable. Equation (2.9) yields the following link between $\eta(x,y)$ and $h(x,y)$

$$h(x,y)=-2\left[\cos(x)\sin(y)\frac{\partial\eta}{\partial x}+\sin(x)\cos(y)\frac{\partial\eta}{\partial y}\right]-\sin(x)\sin(y)\nabla^2\eta \tag{2.10}$$

In the present investigation only the class of functions

$$\eta(x,y)=-\frac{h_0}{2}(px+qy+rxy) \tag{2.11}$$

is taken into account, where h_0,p,q,r are $O(1)$ parameters. Then, substitution of (2.11) into (2.10) yields the bathymetric profiles

$$h(x,y)=h_0\left[\cos(x)\sin(y)\;(p+ry)+\sin(x)\cos(y)\;(q+rx)\right] \tag{2.12}$$

together with the stream functions

$$\psi(x,y)=\Psi(x,y)\left[1-\frac{h_0}{2}(px+qy+rxy)\right] \tag{2.13}$$

which are produced according to (2.8).

Equation $1-\frac{h_0}{2}(px+qy+rxy)=0$. This line separates two current systems; once p,q,r are fixed, the shape and the extension of these currents depend on h_0 in a way that will be clarified by means of some examples in what follows. Integration of (2.6) on D, results in the equation

$$\oint_{\partial D}\mathbf{u}\cdot\hat{\mathbf{t}}\;ds=-\int_D(h+2\psi)\;dxdy \tag{2.14}$$

Then, substitution of (2.12) and (2.13) into (2.14) gives

$$\oint_{\partial D}\mathbf{u}\cdot\hat{\mathbf{t}}\;ds=-8+2\pi h_0(p+q)+\pi^2h_0r \tag{2.15}$$

If $h_0=8\left[2\pi(p+q)+\pi^2r\right]^{-1}$, then the so obtained solution is constituted of two counter-rotating vortices, with opposite relative vorticity.

The fluid domain (2.2) is transformed into itself by the mirror reflections

$$R_1:(x,y)\to(x',y')=(y,x) \tag{3.1a}$$
$$R_2:(x,y)\to(x',y')=(\pi-y,\pi-x) \tag{3.1b}$$
$$R_3:(x,y)\to(x',y')=(\pi-x,y) \tag{3.1c}$$
$$R_4:(x,y)\to(x',y')=(x,\pi-y) \tag{3.1d}$$

If the bathymetry is invariant

$$h(x,y)=h(x',y') \tag{3.2}$$

under a certain reflection among (3.1a)-(3.1d) then the stream function is invariant under the same reflection for which (3.2) holds true, that is

$$\psi(x,y)=\psi(x',y') \tag{3.4}$$

In the following examples some bathymetric reliefs satisfying (3.2) will be considered.

Current fields over some special bathymetric reliefs

a) If $p=1,\; q=r=0$. Then

$$h=h_0\cos(x)\sin(y) \tag{4.1}$$

and

$$\psi=\sin(x)\sin(y)(1-h_0x/2) \tag{4.2}$$

according to (2.15) the two vortices contribute equally (and oppositely) to relative vorticity for $h_0=4/\pi$.
These results are illustrated in left panel.

b) If $p=q=1,\; r=0$. Then

$$h=h_0\sin(x+y) \tag{4.4}$$

while (2.13) becomes

$$\psi=\sin(x)\sin(y)\left[1-\frac{h_0}{2}(x+y)\right] \tag{4.5}$$

Equation (2.15) shows that the two vortices have opposite vorticities for $h_0=2/\pi$. Middle panel shows the isobaths and some snapshots of the model solution

c) If $p=q=0,\; r=1$ then

$$h=h_0\left[y\cos(x)\sin(y)+x\sin(x)\cos(y)\right] \tag{4.6}$$

and

$$\psi(x,y)=\sin(x)\sin(y)\left(1-\frac{h_0}{2}xy\right) \tag{4.7}$$

If $h_0>2/\pi^2$, the streamlines of (4.7) exhibit the formation of a couple of counter-rotating vortices. Right panel shows the isobaths and some snapshots of the model solution.