2.19 EXTENDED PRIMITIVE EQUATIONS IN ISENTROPIC COORDINATES

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1. INTRODUCTION

In the troposphere, water vapor as a gas constituent alters the air parcel's specific gas constant and specific heat capacity at constant pressure from their dry values R_d and c_{pd} to moist values R_m and c_{pm}. Despite the knowledge that the potential temperature θ of an air parcel has a dependence on its water vapor content, potential temperature is often still calculated ignoring the specific humidity dependence in the exponent, with a few notable exceptions, e.g. Emanuel (1994). We derive the form that the isentropic primitive equations take when the dependence of potential temperature θ on water vapor content is included, and show that the resultant forcing from moisture gradients are non-negligible in the tropical troposphere. The new moisture gradient terms fundamentally alter the form of the isentropic equations.

2. THEORY

Assuming that moist air behaves like an ideal gas, the potential temperature of a moist air parcel is

$$\theta = T\left(\frac{p_{ref}}{p}\right)^{\kappa_m} \tag{1}$$

where T is the temperature of the air parcel, p the pressure, p_{ref} the reference pressure, $\kappa_m = R_m/c_{pm}$. $\theta$is often approximated using $\kappa = R_d/c_{pd}$ in place of κ_m . When the moisture dependence of the exponent κ_m is retained, the isentropic primitive equations need to be re-derived. We shall call this new set the "extended isentropic equations".

2.1 DEFINITIONS

Define the constants

$$a_R \equiv \frac{R_v}{R_d} - 1 \approx 0.608 \tag{2}$$

$$a_c \equiv \frac{c_{pv}}{c_{pd}} - 1 \approx 0.859 \tag{3}$$

where R_v and c_{pv} are the specific gas constant and specific heat capacity at constant pressure of water vapour respectively, where the slight variation of c_{pd} and c_{pv} with temperature and pressure is ignored.

Thus, the moist specific gas constant and moist specific heat capacity at constant pressure are dependent on the specific humidity q as follows:

$$R_m = (1 + a_R q) R_d \tag{4}$$

$$c_{pm} = (1 + a_c q) c_{pd} \tag{5}$$

$$\kappa_m = \left(\frac{1 + a_R q}{1 + a_c q}\right)\kappa \tag{6}$$

In the extended isentropic theory, due to the dependence of θ on q, this expression appears repeatedly: for s = x, y, or θ ,

$$\begin{aligned}
&\frac{\partial}{\partial s}\log c_{pm} + \log\left(\tfrac{T}{\theta}\right)\frac{\partial}{\partial s}\log\kappa_m \\
&= \left[1 - \log\left(\tfrac{T}{\theta}\right)\right]\frac{\partial}{\partial s}\log c_{pm} + \log\left(\tfrac{T}{\theta}\right)\frac{\partial}{\partial s}\log R_m \\
&= \left[1 - \log\left(\tfrac{T}{\theta}\right)\right]\frac{a_c}{1 + a_c q}\frac{\partial q}{\partial s} + \log\left(\tfrac{T}{\theta}\right)\frac{a_R}{1 + a_R q}\frac{\partial q}{\partial s} \\
&= \varepsilon\frac{\partial q}{\partial s}
\end{aligned} \tag{7}$$

where with a little algebraic manipulation,

$$\begin{aligned}
\varepsilon &\equiv \frac{1}{1 + a_c q}\left[a_c + \frac{a_R - a_c}{1 + a_R q}\log\left(\tfrac{T}{\theta}\right)\right] \\
&= \tfrac{c_{pd}}{c_{pm}}\left[a_c + (a_R - a_c)\tfrac{R_d}{R_m}\log\left(\tfrac{T}{\theta}\right)\right]
\end{aligned} \tag{8}$$

Exner's function and Montgomery function can be generalized in a moist atmosphere as

$$\Pi_m \equiv c_{pm}\left(\frac{p}{p_{ref}}\right)^{\kappa_m} = c_{pm}\frac{T}{\theta} \tag{9}$$

$$M_m \equiv c_{pm}T + \Phi = \theta\Pi_m + \Phi \tag{10}$$

2.2 MOIST AND DRY PRIMITIVE EQUATIONS

The extended moist isentropic primitive equations are:

$$\rho = \frac{p_{ref}}{R_m\theta}\left(\frac{\Pi_m}{c_{pm}}\right)^{1/\kappa_m - 1} \tag{11}$$

$$\sigma = -\frac{\rho}{g}\left(\frac{\partial}{\partial\log\theta} - \varepsilon\frac{\partial q}{\partial\log\theta}\right)\Pi_m \tag{12}$$

$$\frac{\partial M_m}{\partial\log\theta} = \left(1 + \varepsilon\frac{\partial q}{\partial\log\theta}\right)\theta\Pi_m \tag{13}$$

$$\frac{Du}{Dt} = fv + F_x - \frac{\partial M_m}{\partial x} + \varepsilon\frac{\partial q}{\partial x}\theta\Pi_m \tag{14}$$

$$\frac{Dv}{Dt} = -fv + F_y - \frac{\partial M_m}{\partial y} + \varepsilon\frac{\partial q}{\partial y}\theta\Pi_m \tag{15}$$

$$\frac{\partial\sigma}{\partial t} = \frac{\partial}{\partial x}(\sigma u) + \frac{\partial}{\partial y}(\sigma v) + \frac{\partial}{\partial\theta}(\sigma Q) \tag{16}$$

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$$\frac{Dq}{Dt} = E - C \qquad (17)$$

$$\varepsilon = \frac{c_{pd}}{c_{pm}}\left[a_c + \left(a_R - a_c\right)\frac{R_d}{R_m}\log\left(\frac{\Pi_m}{c_{pm}}\right)\right] \qquad (18)$$

where $Q \equiv D\theta/Dt$ is the diabatic heating rate; E and C are the rates of evaporation and condensation per unit mass of (moist) air respectively. The extended equations can be compared to the set of dry isentropic primitive equations from Andrews et al.(1987) where the basic terms are analogous:

$$\rho = \frac{p_{ref}}{R_d\theta}\left(\frac{\Pi}{c_{pd}}\right)^{1/\kappa-1} \qquad (19)$$

$$\sigma = -\frac{\rho}{g}\frac{\partial\Pi}{\partial\log\theta} \qquad (20)$$

$$\frac{\partial M}{\partial\log\theta} = \theta\Pi \qquad (21)$$

$$\frac{Du}{Dt} = fv + F_x - \frac{\partial M}{\partial x} \qquad (22)$$

$$\frac{Dv}{Dt} = -fv + F_y - \frac{\partial M}{\partial y} \qquad (23)$$

$$\frac{\partial\sigma}{\partial t} = \frac{\partial}{\partial x}(\sigma u) + \frac{\partial}{\partial y}(\sigma v) + \frac{\partial}{\partial\theta}(\sigma Q) \qquad (24)$$

3. RESULTS

Table 1 lists the terms compared to evaluate the influence of the moisture gradient terms on the moist dynamics. The vertical gradient of moisture reduces the isentropic density by up to 10% (Figure 1c). Even more significant is its impact on the vertical gradient of the moist Montgomery function, reducing the latter by up to 50% (Figure 1a). Finally, the isentropic moisture gradient term is comparable to the moist Montgomery gradient force in contributing to the horizontal momentum tendency (Figure 2).

4. CONCLUSIONS

Vertical and horizontal moisture gradients exert dynamical influences. The driving of horizontal momentum by the isentropic moisture gradient is comparable to that by the moist Montgomery gradient force in the tropical troposphere. These findings reflect the weak horizontal or isobaric temperature gradients in the tropics. In such a setting, the inhomogeneous distribution of water vapor creates isobaric gradients in potential temperature comparable to those created by temperature variations and thus contributes to pressure forces on an isentrope. It is therefore not advisable to ignore the water vapor dependence of potential temperature when studying tropical dynamics in the isentropic framework.

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Equation	ε-term [A]	basic term [B]	ratio [A]:[B]
Isentropic density (12)	$-\varepsilon\frac{\partial q}{\partial\log\theta}$	$\frac{\partial\log\Pi_m}{\partial\log\theta}$	≤ 10%
Hydrostatic balance (13)	$\varepsilon\frac{\partial q}{\partial\log\theta}$	1	≤ 50%
Horizontal momentum tendency (14), (15)	$\frac{\varepsilon}{1+\varepsilon\frac{\partial q}{\partial\log\theta}}\left\lvert\nabla_\theta q\right\rvert$	$\left\lvert\nabla_{M_m}(\log\theta)\right\rvert$	about 1

Table 1: The ratios that determine the significance of the moisture gradient terms in the extended primitive equations. Equation (13) is used in the last row.

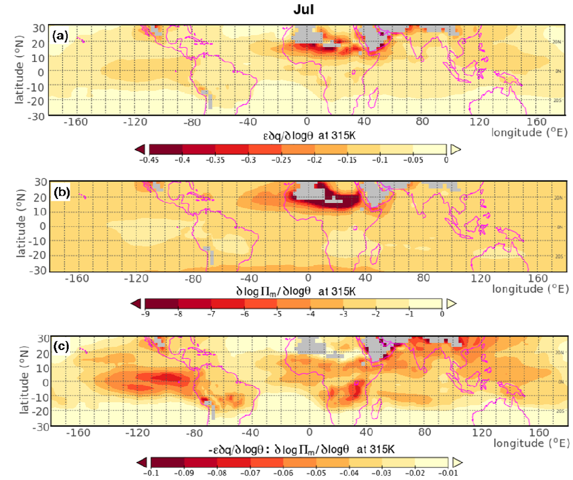

Figure 1. The 1981-2010 July monthly climatology at θ=315K. (a) $\varepsilon\frac{\partial q}{\partial\log\theta}$, (b) $\frac{\partial\log\Pi_m}{\partial\log\theta}$, (c) $-\varepsilon\frac{\partial q}{\partial\log\theta} : \frac{\partial\log\Pi_m}{\partial\log\theta}$ on different scales.

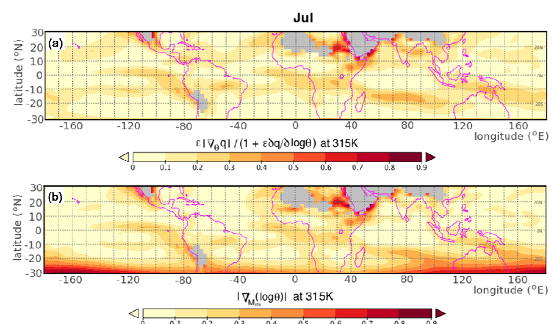

Figure 2. The 1981-2010 July monthly climatology at θ=315K. (a) $\left\lvert\nabla_\theta q\right\rvert\varepsilon/\left(1+\varepsilon\frac{\partial q}{\partial\log\theta}\right)$, (b) $\left\lvert\nabla_{M_m}(\log\theta)\right\rvert$, on the same scale in 10^{-3} km^{-1}.

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