6.115 The equality between the radial extension of the tropical cyclone and the Rossby radius

Florin Spineanu and Madalina Vlad
National Institute of Laser, Plasma and Radiation Physics
Bucharest, Romania
(spineanu@nipne.ro)

The generation of a large scale atmospheric vortex of high intensity (tropical cyclone) involves both theromodynamics and mechanical processes. After cyclogenesis a stationary state can be reached (assuming that the topographic perturbations are absent) and the main spatio-temporal characteristics of the vortex only have small changes while the whole structure is drifting. This is an ideal limit which corresponds to the state where the dynamical evolution is almost factorised into two distinct, cuasi-independent sub-systems: the thermodynamic processes and the mechanical balance. They are almost independent, with only a flow of energy from the thermic sub-system toward the mechanical processes needed for the latter to overcome the loss due to the friction. The energy involved in the purely thermic processes is estimated to be about 400 times greater that the mechanical energy. Further, the loss of the mechanical energy by friction in the vortical motion is a small fraction of the total mechanical energy. At stationarity this represents the only interaction between the two subsystems. A useful image of this state would be that of a matrix expressing the relative influences of thermodynamic and mechanical variables, factorized into two Jacobi blocks, with only slight interconnection remaining off the diagonal, to connect the two sub-systems.

Such ideal state suggests that the vorticity dynamics is the essential factor in establishing the spatio-temporal characteristics of the atmospheric vortex at stationarity. Then the Taylor - Proudman theorem and the typical dimensions of the observed cyclones allow a $2D$ approximation for the tropical vortex. The vorticity dynamics in $2D$ is radically different than in $3D$: there is inverse cascade and the asymptotic relaxed states of the fluid are highly organized, the vortical structures can be close to stability. This is true for the ideal incompressible (Euler) fluid where the asymptotic states are governed by the self-duality property and satisfy the equation sinh - Poisson equation (also known as elliptic sinh - Gordon equation) $\Delta\psi + \sinh(\psi) = 0$, where ψ is the streamfunction. The equation is exactly integrable and the *stable* solutions are necessarly doubly periodic, like - for example, the Mallier - Maslowe chain of vorticies. No other solution is stable, in particular a single monopolar vortex, but it can be very robust and may appear in reality as sufficiently long-lived to be considered stable.

In the case of the $2D$ approximation of the atmosphere the sinh - Poisson equation may only be an indicative approximation. The new physical element is the Rossby radius, that changes the physics and the mathematical possibility of relaxed states.

In previous papers we have described the asymptotic states of the $2D$ atmospheric vortex using the extremum of an action functional. The idea was to extend the approach that was based exclusively on conservation laws (density, momentum, energy, etc.) by considering the states that results from the variational formulation based on a Lagrangian density. The ideal (Euler) fluid is described in $2D$ by the equation $d\boldsymbol{\omega}/dt = 0$ where $\boldsymbol{\omega}$ is the vorticity, a vector directed along the perpendicular on the plane.

This equation is known to be equivalent with the equations of motion of a discrete set of point-like vortices interacting in plane by a mutual long-range potential (the ln function of the relative distance between point-like vortices). We have developed a field theoretical formulation of the continuous limit of this system [1]. It was then possible to derive, in purely analytical terms, the sinh-Poisson equation. We have extended the treatment to the $2D$ atmosphere [2]. Since now there is a short range interaction between the point-like vortices of the discrete model, the resulting equation is different and the self-duality property is lost

$$\Delta\psi + \frac{1}{2}\sinh(\psi)\left[\cosh(\psi) - 1\right] = 0 \tag{1}$$

Solving numerically the equation for a wide range of *nondimensional* parameters, we have derived two equations, connecting the three important characteristics of the atmospheric vortex: the maximum azimuthal wind speed $v_\theta^{\max}$, the radius where the azimuthal wind has this maximum, $r_{v_\theta^{\max}}$ and the radial extension of the tropical cyclone, $R_{\max}$. The equations are [3]

$$v_\theta^{\max} = \frac{e^2}{2}\left[\alpha \exp\left(\frac{\sqrt{2}}{R_{\max}}\right) - 1\right] \tag{2}$$

and

$$\frac{r_{v_\theta^{\max}}^{phys}}{R_{\max}^{phys}} = \frac{r_{v_\theta^{\max}}}{R_{\max}} = \frac{1}{4}\left[1 - \exp\left(-\frac{R_{\max}}{2}\right)\right] \tag{3}$$

It clear that the use of these equations requires an input in terms of physical length, for example $R_{\max}^{phys}$ and the ratio $r_{v_\theta^{\max}}^{phys}/R_{\max}^{phys}$ as observed from satellite. This is because the quantities appearing in the equations are normalized as follows: distances to the Rossby radius, R_{Rossby} and time to f_0^{-1}, the inverse of the Coriolis frequency.

We have checked these equations by starting from observations of tropical cyclones and determining first R_{Rossby}. We have noted that, systematically, there was a correlation between the maximum extension of the tropical cyclone $R_{\max}^{phys}$ (expressed in physical units) and the Rossby radius R_{Rossby}, aslo expressed in physical units. The connection is linear

$$R_{\max}^{phys} = \gamma R_{Rossby} \tag{4}$$

with the suggested value $\gamma \sim 1.1$.

This is a physical result and can be expressed as follows: the radius of Rossby for a tropical cyclone is approximately equal (up to a factor γ) with the maximal radial extension of the atmospheric vortex.

We find that this equality is actually a natural result in the field theoretical formulation that we have developed for the $2D$ atmosphere.

The field theoretical formalism defines two fields:

- the scalar ("matter") field ϕ; its particle is the Higgs scalar; the mass of the particle is m_H. The mass m_H is the inverse of the characteristic range of spatial decay of the field ψ solution, *i.e.* the inverse of the radius of characteristic decay of the vortex flow

$$m_H = \frac{1}{R_{max}} \tag{5}$$

- the gauge field A_μ whose particle is the "photon". The necessity of a *mass* for this field comes from the fact (shown by Morikawa [4]) that the system of point-like vortices in the case of the $2D$ atmosphere has short range interaction (equivalently, the "photon" has a mass) and the operator $\Delta\psi$ of the case of the Euler fluid is now replaced by

$$\Delta\psi - \frac{1}{R_{Rossby}^2}\psi \tag{6}$$

which leads to an interaction potential $K_0(x/R_{Rossby})$ instead of the long range $\ln(x/L)$. For atmosphere the photon has a mass, m_{photon}, which is the inverse of the characteristic range of spatial decay of the interaction. The latter is Rossby radius

$$m_{photon} = \frac{1}{R_{Rossby}} \tag{7}$$

The field theory finds: $m_H = m_{photon}$, which means $R_{max} \sim R_{Rossby}$. The fact that we introduce a factor γ is due to the nature of decay represented by the two "masses": for the spatial extension of the vortex we have to consider that the decay is at half-heigth, as in a Gaussian profile.

To study the scalings implied by the scatterplot of the pairs

$$\left(r_{v_\theta^{\max}}, v_\theta^{\max}\right) \tag{8}$$

Using the Eq.(3) it results, with $R_{\max} = \sqrt{2}L$, where L is the length of the side of the box of integration of the Equation

$$-\frac{L}{\sqrt{2}} = \ln\left(1 - 2\sqrt{2}\frac{r_{v_\theta^{\max}}}{L}\right)$$

An approximation is possible if

$$2\sqrt{2}\frac{r_{v_\theta^{\max}}}{L} \equiv \varepsilon \ll 1$$

which is a strong approximation, since $\varepsilon \sim 0.25...0.5$. We only adopt it for order - of magnitude verification. Then the equation becomes $-\frac{L}{\sqrt{2}} \simeq -2\sqrt{2}\frac{r_{v_\theta^{\max}}}{L}$ or

$$L \simeq 2\sqrt{r_{v_\theta^{\max}}}$$

(both distances are normalized to R_{Rossby}). In physical terms, since $L = \frac{R_{\max}^{phys}/\sqrt{2}}{R_{Rossby}}$ and $r_{v_\theta^{\max}} = \frac{r^{phys}_{v_{\theta\max}}}{R_{Rossby}}$, the equation becoms

$$R_{Rossby} \simeq 8\frac{\left(R_{\max}^{phys}\right)^2}{r^{phys}_{v_{\theta\max}}}$$

Then the Rossby radius for a particular atmospheric vortex can be estimated on the basis of the knowledge of the maximum radius of the vortex and of the radius of maximum tangential wind. Since very often the ratio of these two quantities is in the range

$$\frac{R_{\max}^{phys}}{r^{phys}_{v_{\theta\max}}} \sim \frac{1}{10}...\frac{1}{8}$$

we obtain that

$$R_{Rossby} \simeq (0.8...1) \times R_{\max}^{phys}$$

This very rough estimation is only useful to suggest that the adimensional L , the essential parameter of the problem, is of the order 1.

The sensitivity of any result obtained from the above equations for L in the range around $L \approx 1$ imposes more careful calculations, including more accurate expressions in Eqs.(2 - 3). The latter are obtained from a large number of numerical solutions of Eq.(1) and may need to be adapted to sub-ranges of L. However the field - theoretical result, that the masses of the photon and of the scalar partile are equal, naturally implies that the Rossby radius and the maximum extension of the tropical cyclone are equal, irrespective of the details of the calculations.

References

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