Is Weather Chaotic?

Coexistence of Chaos and Order within a Generalized Lorenz Model

by

Bo-Wen Shen[1,*], Roger A. Pielke Sr.[2], Xubin. Zeng[3], Jong-Jin Baik[4]
Tiffany A.L. Reyes[1], Sara Faghih-Naini[1,5], Robert Atlas[6], and Jialin Cui[1,7]

[1]Department of Mathematics and Statistics, San Diego State University, San Diego, CA, USA
[2]CIRES, University of Colorado at Boulder, Boulder, CO, USA
[3]Department of Hydrology and Atmospheric Science, University of Arizona, Tucson, AZ, USA
[4]School of Earth and Environmental Sciences, Seoul National University, Seoul, South Korea
[5]Chair for Applied Mathematics 3, Friedrich-Alexander-University Erlangen-Nuremberg, Erlangen, Germany
[6]AOML, National Oceanic and Atmospheric Administration, Miami, FL, USA
[7]Department of Computer Sciences, San Diego State University, San Diego, CA, USA

[*]bshen@sdsu.edu

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Abstract

The pioneering study of Lorenz in 1963 and a follow-up presentation in 1972 changed our view on the predictability of weather by revealing the so-called butterfly effect, also known as chaos. Over 50 years since Lorenz's 1963 study, the statement of ``weather is chaotic'' has been well accepted. Such a view turns our attention from regularity associated with Laplace's view of determinism to irregularity associated with chaos. Stated alternatively, while Lorenz (1993) documented that "*as with Poincare and Birkhoff, everything centers around periodic solutions,*" he himself and chaos advocates focused on the existence of non-periodic solutions and their complexities. Now, a refined statement is suggested based on recent advances in high-dimensional Lorenz models and real-world global models. In this study, we provide a report to: (1) Illustrate two kinds of attractor coexistence within Lorenz models. Each kind contains two of three attractors including point, chaotic, and periodic attractors corresponding to steady-state, chaotic, and limit cycle solutions, respectively. (2) Suggest that the entirety of weather possesses the dual nature of chaos and order associated with chaotic and non-chaotic processes, respectively. Specific weather systems may appear chaotic or non-chaotic within their finite lifetime. While chaotic systems contain a finite practical predictability, non-chaotic systems (e.g., dissipative processes) could have better predictability (e.g., up to their lifetime). The refined view on the nature of weather is neither too optimistic nor pessimistic as compared to the Laplacian view of deterministic unlimited predictability and the Lorenz view of deterministic chaos with finite predictability.

Capsule

By revealing two kinds of attractor coexistence within Lorenz models, we suggest that the entirety of weather possesses a dual nature of chaos and order. The refined view on the nature of weather is neither too optimistic nor pessimistic as compared to the Laplacian view of deterministic predictability and the Lorenz view of deterministic chaos.

1. Introduction

Is weather chaotic? A view that weather is chaotic was proposed and is recognized based on the pioneering work of Lorenz (1963) who first introduced the concept of deterministic chaos. Defined as aperiodic solutions that display sensitive dependence on initial conditions (ICs), chaos is also known as the butterfly effect. The appearance of deterministic chaos suggests finite predictability, in contrast to the Laplacian view of deterministic predictability that is unlimited. After a follow-up conference presentation in 1972 (Lorenz 1972), the butterfly effect has come to be known as a metaphor for indicating that a tiny perturbation that is as small as a butterfly's flap may generate a large impact that could create a tornado. The original Lorenz 1963 study and a 1972 presentation, as well as his 1969 study (Lorenz 1969), laid the foundation for chaos theory that is viewed as one of the three scientific achievements of the 20th century, inspiring numerous studies in multiple fields, including earth science, mathematics, philosophy, physics, etc. (Gleick 1987).

While the finding of a chaotic attractor has suggested a finite predictability for weather over the past fifty years, such chaotic solutions indeed occur over a finite interval of time-independent parameters within the Lorenz model. Therefore, other features of the original Lorenz model and generalized Lorenz models that were discovered in subsequent studies (Guckenheimer and Williams 1979; Sparrow 1982; Smale 1998; Tucker 2002; Musielak et al. 2005; Roy and Musielak, 2007; Yang and Chen 2008; Sprott et al. 2013; Moon et al. 2017, 2019; Felicio and Rech, 2018; Shen 2014-2017, 2019a) should be taken into consideration in order to reveal the true nature of weather. For example, in addition to chaotic solutions, other types of solutions indeed appear over different intervals of parameters within the Lorenz model (Sparrow 1982), but their role in weather has been overlooked. Furthermore, as emphasized by recent studies using a generalized high-dimensional Lorenz model (e.g., Shen 2019a; Shen et al. 2019; Reyes and Shen 2019), two types of solutions (e.g., chaotic and non-chaotic solutions) may coexist within the same model parameters but for different ICs (e.g., Sprott et al. 2005; Sprott and Xiong 2015). Such a coexistence indicates the possibility of a dual nature for chaos and order for weather. Thus, it is important to understand whether or not and how other types of solutions and their coexistence may help illustrate a more comprehensive view on the nature of weather, and improve our understanding of chaotic and non-chaotic processes within different types of solutions. Stated alternatively, we may ask whether the statement of ``weather is chaotic'' that exclusively considers chaotic solutions is realistic. In this study, a specific type of solution is referred to as an ``attractor'', defined as the smallest attracting point set that cannot be decomposed into two or more subsets with distinct regions of attraction (e.g., Sprott et al. 2013).

To address the above, here, we provide a review of major solutions using the Lorenz model (LM), including three types of solutions (i.e., three attractors) and one kind of attractor coexistence. We then summarize our recent findings for two kinds of attractor coexistence using a newly developed, generalized, high-dimensional LM (GLM) (e.g., Shen 2019a). Based on an analysis of the Lorenz model and the GLM, we suggest a refined view on the dual nature of weather. Concluding remarks are provided at the end. Using a realistic value for the Prandtl number (i.e., $\sigma = 1$) within the Lorenz model, Supplemental Materials are presented in order to support the findings for two kinds of attractor coexistence.

2. The Lorenz 1963 Model

In his 1963 study, Prof. Lorenz presented an elegant system of three ordinary differential equations (ODEs) using three parameters derived from the governing equations for the Rayleigh-Benard convection (e.g., Saltzman 1962; Lorenz 1963). The three ODEs describe the time evolution of three variables, X, Y, and Z, as follows:

$$\frac{dX}{d\tau} = \sigma Y - \sigma X, \quad (1)$$

$$\frac{dY}{d\tau} = -XZ + rX - Y, \quad (2)$$

$$\frac{dZ}{d\tau} = XY - bZ. \quad (3)$$

Here, τ is dimensionless time. The three, time-independent parameters are σ, r, and b. The first two parameters represent the Prandtl number and the normalized Rayleigh number (or the heating parameter), respectively. The third parameter is a function of the ratio between the vertical scale of the convection cell and its horizontal scale. (X, Y, Z) represent the amplitudes of the three Fourier modes for dynamic and thermodynamic variables (e.g., Table 1 of Shen 2014). Specifically, X represents the amplitude of the stream function, and Y and Z represent the amplitudes of the temperature deviation. Equations (1)-(3) contain three types of physical processes, including buoyancy/heating, dissipative, and nonlinear processes. The linear buoyancy force and the heating force are represented by σY in Eq. (1) and rX in Eq. (2), respectively. The three dissipative terms are $-\sigma X$, $-Y$, and $-bZ$ and are ignored under the dissipationless condition. The two nonlinear terms, $-XZ$ and XY, are derived from the nonlinear advection of the temperature term within the governing equation for the Rayleigh-Benard convection (e.g., Saltzman 1962). With the exception of the heating parameter (r), the following parameters are kept constant: $\sigma = 10$ and $b = 8/3$. A choice of $\sigma = 1$ and $b = 2/5$ is also discussed in the Supplemental Materials.

In additional to control runs, parallel runs with ICs that consist of control run ICs and tiny perturbations ($\in = 10^{-10}$) or finite perturbations ($\in = -0.9$) are performed in order to reveal the difference of two solutions between the control and parallel runs.

Using the state variables X, Y, and Z as coordinates, a phase space can be defined for the analysis of solutions. Therefore, the dimension[1] of the phase space is equal to the number of time-dependent variables or the number of ODEs. Equations (1)-(3) with three variables are referred to as a three-dimensional Lorenz model (3DLM). High-dimensional LMs contain more than three variables (e.g., Shen 2014). To display a solution effectively, its time varying components are plotted within the phase space, which is referred to as an orbit or a trajectory.

Lorenz's Chaotic and Non-Chaotic Attractors

Depending on the competitive or collective impact of nonlinear processes and linear heating and dissipative processes, measured by values of the three parameters, various types of solutions (i.e., different attractors) appear within the Lorenz model. Historically, the dependence of their appearance on the strength of heating measured by the normalized Rayleigh parameter (r) has been a focus. Steady-state, chaotic, and nonlinear oscillatory solutions have been shown to occur under conditions of weak, moderate, and strong heating, respectively (e.g., Sparrow 1982; Drazin 1992; Ott 2002)[2]. The three different types of solutions are shown using r = 20, 28, and 350, respectively,

[1] The term "dimension" is conventionally used for a system of ODEs (e.g., Hirsch et al. 2013; Thompson and Stewart 2002). In this study, the 5DLM and 7DLM are referred to as high-dimensional or high-order Lorenz models (e.g., Moon et al. 2017).

[2] Similar findings for the dependence of various solutions (i.e., chaotic and limit cycle solutions) on the strength of heating were also reported using a two-layer, quasi-geostrophic model that describes the finite-amplitude evolution of a single baroclinic wave by Pedlosky and Frenzen (1980).

in Fig. 1. The top panels display solutions for control runs within the *X*-*Y* space, while the bottom panels display the time evolution of the Y components for both control and parallel runs. For a steady-state solution, its orbit eventually approaches a single point, that is, a non-trivial equilibrium point within the *X*-*Y* space (Fig. 1a), appearing as a point attractor; and its amplitude remains constant over time after arriving at the equilibrium point. Mathematically, equilibrium points, also called critical points, are defined as solutions of the time-independent nonlinear system (e.g., no time derivatives in Eqs. (1)-(3), Guckenheimer and Holmes (1983))[3]. When the heating parameter exceeds the critical value of $r_c = 24.74$, the 3DLM with r = 28 displays the so-called chaotic solution or a chaotic attractor with irregular oscillations. The solution's boundary within the *X*-*Y* space appears as a tilted "8" pattern. Interestingly, when heating becomes larger (e.g., r = 350), the system produces a nonlinear periodic solution known as a limit cycle solution or a periodic attractor, as shown in Figs. 1c and 1f. Additional details on the characteristics of nonlinear oscillatory solutions may be found in earlier studies (e.g., Shimizu 1979; Sparrow 1982; Strogatz 2015) and/or recent studies (e.g., Fig. 9 of Reyes and Shen 2019). In summary, three types of attractors, including a point attractor, a chaotic attractor, and a periodic attractor (e.g., Sprott et al. 2013) appear and are associated with weak, moderate, and strong heating, respectively. Below, the impact of a tiny initial perturbation on these attractors is further discussed.

Parallel runs with a tiny initial perturbation ($\epsilon = 10^{-10}$) are compared to control runs in order to reveal the difference (also referred to as the divergence) of initial, nearby trajectories within the phase space of the 3DLM. For steady-state and nonlinear oscillatory solutions, control and parallel runs produce almost identical results, only appearing in red, for example, in Figs. 1d and 1f. The

[3] In our 5D-, 7D-, and 9D LMs, we can obtain closed form solutions of trivial and non-trivial equilibrium points and use them to verify the numerical solutions of equilibrium points.

runs indicate insignificant impacts by a tiny initial perturbation. In other words, steady-state and nonlinear oscillatory solutions are insensitive to a tiny change in ICs. In comparison, within the chaotic regime, two solution orbits whose starting points are very close to each other display very different time evolutions, as clearly shown in blue and red in Fig. 1e. The phenomenon is called the sensitive dependence of solutions on ICs. As further discussed below, such a feature only appears within a chaotic solution.

Boundedness and Divergence of Chaotic Trajectories

Within the chaotic regime of the 3DLM, a sensitive dependence of solutions on ICs is referred to as the butterfly effect (BE, e.g., Lorenz 1993, 2008). As shown in Fig. 2a (e.g., as discussed on page 15 of Lorenz 1993), the term ``butterfly'' was partly used due to its geometric pattern in the *Y-Z* space. A butterfly pattern with a finite size and varying curvatures within the phase space also qualitatively suggests an important feature of solution boundedness. Therefore, BE means that a tiny change in an IC can produce a very different time evolution of a solution for three variables (X, Y, Z). However, the separation (or divergence) of two orbits should be bounded by the size of a butterfly pattern.

The average separation (i.e., an average divergence) of nearby trajectories has been quantitatively measured using the Lyapunov exponent (LE, Wolf et al. 1985; Zeng et al. 1991, 1993). A positive LE suggests an exponential rate in the averaged separation of two infinitesimally nearby trajectories over an infinite period of time (e.g., Eqs. (25)-(26) of Shen 2014). Chaotic solutions within the 3DLM, as well as high-dimensional LMs, have a positive LE. Since the LE is defined as a long-term averaged separation, researchers often misinterpret the divergence of two nearby, but finitely separated, chaotic trajectories within the 3DLM as continuing over time and lasting forever. The misunderstanding also makes people believe that a blow-up solution is due to the divergent nature of chaos. In fact, in addition to a positive LE, solution boundedness is another major feature of a chaotic system. Due to solution boundedness, a trajectory should recurve within the phase space (e.g., Hilborn 2000). Therefore, time-varying (local) growth rates along a chaotic orbit are observed (e.g., Zeng et al. 1993) and may become negative, as indicated by a negative finite time LE (e.g., Fig. 7 of Nese 1989; Fig. 1 of Eckhardt and Yao 1993; p. 397 of Ding and Li 2007; Fig. 3 of Bailey 2011). In other words, the infinite-time limit in the definition of an LE does not imply a monotonically increasing separation between two nearby trajectories over a long period of time. Two initial nearby trajectories can quickly separate and reach the bound of their separation.

Coexistence of Chaos and Order

The 3DLM produces three different attractors and each attractor exclusively appears within the phase space, depending on the interval of system parameters. The 3DLM with a single-type solution suggests that either chaos or order exclusively exists. Is this realistic? Below, we present a different scenario that two different attractors may coexist and dominate system dynamics in a separate region (i.e., a different subspace) within the phase space, referred to as the first kind of attractor coexistence. A coexistence of two different solutions, appearing within the same model, and with the same parameters, but with different ICs, has been well studied using conservative Hamiltonian systems (e.g., Hilborn 2000). By comparison, earlier studies within the forced dissipative 3DLM (e.g., Yorke and Yorke 1979; p. 242 of Drazin 1992; p. 333 of Ott 2002) have also documented the coexistence of steady-state and chaotic solutions. However, such a coexistence only appears over a very small range of r, giving the length of an interval less than 0.7 (i.e., $24.06 < r < r_c = 24.74$). As a result, the characteristics of the coexistence and its potential role in revealing the nature of weather has not been well explored.

The 3DLM with the same parameters, including r = 24.4, $\sigma = 10$, and b = 8/3, but with different ICs, was used to illustrate such a coexistence in a homework problem for the course entitled Computational Ordinary Differential Equations taught by the first author at San Diego State University during Fall 2018. As simply shown in the animation, https://goo.gl/scqRBo, six different orbits can clearly be categorized into two types of solutions, chaotic or steady-state. Below, we apply the GLM in order to show that coexistence may appear within a wider interval of the heating parameter and suggest that attractor coexistence should be considered in order to refine the view of the nature of weather.

3. The Generalized Lorenz Model

Based on our recent studies (e.g., Shen, 2014-2019; Faghih-Naini and Shen 2018), we successfully developed a GLM that: (1) is derived based on partial differential equations for the Rayleigh-Benard convection[4]; (2) allows a large number of modes, say M modes, where M is an

[4] By comparison, chaotic models in Lorenz (1996/2006, 2005) were not derived from physics-based partial differential equations.

odd number greater than three; and (3) produces aggregated negative feedback[5] that is accumulated from the feedback of various smaller-scale processes, yielding a larger effective dissipation in higher dimensional LMs (Shen 2019a; Shen et al. 2019). As a result of aggregated negative feedback, a higher-dimensional LM requires a larger critical value for the Rayleigh parameter (r_c) for the onset of chaos. For example, the r_c for the 5DLM, 7DLM, and 9DLM are 42.9, 116.9, and 679.8, respectively, as compared to a r_c of 24.74 for the 3DLM (Shen 2019a). Fig. 2 displays chaotic solutions obtained from the 3D, 5D, 7D, and 9D LMs with different heating parameters. Therefore, a tiny perturbation with the same strength may play a different role within the GLM with a different value of M, showing a dependence on the dimension (or the degree of spatial complexity associated with a various number of modes) of the GLM.

Two Kinds of Attractor Coexistence

The GLM with M = 5 or M = 7 (i.e., 5DLM or 7DLM) also produces three different types of solutions, including a steady-state, chaotic, and limit cycle/torus[6]. More importantly, the GLM with M = 9 (i.e., 9DLM) displays two kinds of attractor coexistence, each with two different attractors. For the first kind of coexistence, both chaotic and steady-state solutions occur concurrently with the same model and the same parameters. The only difference is their ICs. Such a coexistence shares properties similar to that of the 3DLM but appears over a wider range of the Rayleigh parameter (e.g., $679.8 < r < 1,058$), as compared to the small interval (e.g., $24.06 < r <$

[5] Negative feedback can be found within the so-called Lorenz-Stenflo system that extends the 3DLM with one additional ODE containing one additional mode that takes rotation into consideration (e.g., Xavier and Rech 2010; Park et al. 2015, 2016).
[6] A torus is defined as a composite motion with two (or more) oscillatory frequencies whose ratio is irrational (e.g., Faghih-Naini and Shen, 2018).

24.74) for the 3DLM. In fact, since the first kind of attractor coexistence has been overlooked for decades, we became aware of such a finding by Yorke and Yorke (1979) after observing coexistence within the 9DLM and performing a literature review.

In addition to the first kind of attractor coexistence, the 9DLM is able to produce the second kind of attractor coexistence, consisting of nonlinear, periodic (i.e., limit cycle) orbits and steady-state solutions at large Rayleigh parameters (e.g., r = 1,600). The new kind of coexistence was recently documented in Shen (2019a), Shen et al. (2019), and Reyes and Shen (2019). By extending the above analysis, we now show that the 3DLM with a realistic value of $\sigma = 1$ also generates two kinds of attractor coexistence, suggesting that the features are not specific to our 9DLM. See additional information in the Supplemental Materials.

Depending on system parameters, ICs and the dimension of the model (say the value of M within the GLM), a modeling system may contain one or more attractors[7] (e.g., a point, chaotic, and/or periodic attractor) within the phase space. When both chaotic and regular attractors coexist, they occupy two different regions (or two different subspaces) within the phase space. Therefore, we observe two kinds of solution dependence on ICs, including (1) the dependence of solution types on ICs and (2) a sensitive dependence on ICs for chaotic solutions. The former suggests that an IC may lead to a chaotic or non-chaotic solution. The latter indicates that only chaotic solutions display sensitive dependence on a tiny, initial perturbation. We illustrate these below.

Two Kinds of IC Dependence and Final State Sensitivity

[7] The coexistence of chaotic and quasi-periodic orbits has been recently documented in a modified Lorenz system by Saiki et al. (2017).

Each of the three, single-type solutions exclusively appears. Among them, a steady-state or limit cycle solution has no long-term memory regarding its ICs and initial errors. Although a chaotic orbit displays the sensitivity of its time evolution to initial perturbations, its statistics (i.e., the attractor itself or the butterfly pattern within the phase space) is independent of the ICs. As long as a system's parameters are given, the long-term statistics of the single-type solution is already determined and is independent of ICs. In comparison, when two attractors coexist in two different regions within the phase space, a different IC may lead to a different type of solution with very different statistics. Thus, the impact of a tiny initial perturbation can be very different, depending on its association with a chaotic or non-chaotic orbit. A tiny initial perturbation may only have a short-term impact on the initial transient evolution of non-chaotic (e.g., the steady-state or limit cycle) solutions or lead to a very different evolution for chaotic orbits. Below, we illustrate such an impact of ICs (i.e., the location of the starting point within the phase space) on determining an orbit's subsequent evolution and final destination (i.e., a point attractor or a chaotic attractor).

Control runs apply three sets of ICs at different locations within the phase space: close to the non-trivial equilibrium point, at the origin (i.e., a saddle point), and at point (100, 100, 100, 100, 100, 100, 100, 100, 100). For parallel runs, a finite-amplitude perturbation ($\in = -0.9$) is added into the ICs. In Fig. 3, solutions of the control runs are shown in blue, while results of parallel runs are displayed in green, red, or orange. Top panels display the time evolution of Y components, while bottom panels present solutions within the *X*-*Y* space. The model with r = 680 produces the coexistence of steady-state and chaotic orbits, displaying a dependence on ICs. For the first case

(Figs. 3a and 3d) with the IC that is close to the non-trivial equilibrium point, the orbit moves toward the equilibrium point, producing steady-state solutions. Since the orbit spirals into the non-trivial equilibrium point within the *X*-*Y* space, it is also called a spiral sink solution. For the second case (Figs. 3b and 3e) where an IC is close to a saddle point at the origin but away from the non-trivial equilibrium point, solutions still approach the same non-trivial equilibrium point as a steady-state solution, while initially displaying a different time evolution as compared to the first case. On the other hand, for the third case (Figs. 3c and 3f), the model produces a chaotic solution, different from the steady-state solution. A comparison between control and parallel runs suggests that an initial perturbation only has a short-term impact on the initial transient evolution of steady-state solutions but can lead to a very different evolution for chaotic solutions[8]. As also discussed in Fig. 5 of Shen et al. (2019a), a systematic analysis of the dependence of chaotic and non-chaotic orbits on ICs was previously performed using an ensemble modeling approach with 4,096 ensemble members.

While the appearance of stable solutions may suggest better predictability, a system with coexisting solutions additionally displays final state sensitivity (e.g., Grebogi et al. 1983) when ICs start near the boundary of two different attractors (i.e., solutions). As illustrated below, such a final state sensitivity creates a different challenge for predictability.

Finite and Deterministic Predictability

[8] Such a dependence on initial conditions, close to (or away from) the non-trivial equilibrium point, can be shown by the following YouTube video for a double pendulum (between 1:00-1:20): https://www.youtube.com/watch?v=LfgA2Auyo1A. This footnote is provided only for review.

The rate of a growing initial error with time has been used to determine predictability, suggesting a finite predictability in chaotic (or unstable) systems. Such a growth rate is proportional to the divergence of two nearby trajectories measured using a Lyapunov exponent. Within the chaotic regimes of the 3DLM, as well as the GLM that contains one positive LE and solution boundedness, time-varying divergence and the convergence of nearby trajectories yields time-varying growth rates and, thus, time-varying predictability. Estimated predictability over a short period should display a dependence on various initial states[9]. By comparison, when non-chaotic (i.e., steady-state or nonlinear periodic) solutions appear as a single type of solution or coexist with another type of solution, their predictability should be deterministic (unlimited). As a result, when a system possesses the coexistence of chaotic and non-chaotic attractors, ICs determine whether finite or deterministic predictability may appear.

4. A Refined View on the Nature of Weather

Since climate and weather involve open systems, an assumption of constant parameters within numerical simulations using the 3DLM, as well as high-dimensional LMs, is not realistic and, thus, the applicability of numerical results to realistic climate or weather should be interpreted with caution. To better understand the validity of applying chaotic solutions in order to define the nature of weather, below, we provide additional comments. Within the forced dissipative 3DLM, chaotic solutions appear within a finite range of parameters (e.g., heating parameter), bounded on one side by stable, steady-state solutions and on the other side by nonlinear periodic solutions. Chaotic solutions may not be able to represent the entirety of weather. Additionally, within chaotic

[9] As a result, we agree with Prof. Arakawa that the predictability limit is not necessarily a fixed value (Lewis 2005).

solutions, a tiny perturbation can always lead to a very different time evolution. Stated alternatively, within the chaotic regime, the system does not have a mechanism for completely removing the impact of a tiny perturbation. Although these findings are interesting, it is reasonable to ask whether it is realistic to expect such an effect for any tiny perturbation (e.g., Pielke 2008).

By comparison, within the GLM with M = 9, or higher, that possesses coexisting chaotic and steady-state solutions, a tiny initial perturbation may play a very different role. A tiny perturbation may have no long-term impact when it appears to be associated with a steady-state solution, suggesting that the perturbation eventually dissipates. On the other hand, a tiny perturbation may lead to a large impact on the time evolution of the chaotic solution. As a result, the 9DLM with a dual role for a tiny initial perturbation over a wide range of the heating parameter is more realistic than the classical 3DLM. Such a comparison indicates the need to refine our view of weather by taking the dual nature associated with attractor coexistence into consideration. To this end, we suggest, contrary to the traditional view that weather is chaotic, that weather is, in fact, a superset that consists of both chaotic and non-chaotic processes, including both order and chaos.

Additional Support: Coexisting Solutions at Two Time Scales, Vacillation and Intransitivity

Coexisting solutions at two time scales, which are not the same as the coexisting attractors discussed above, have also been documented in scientific literature. Related studies additionally support the refined view on the nature of weather. For example, co-existence of fast and slow manifolds has been discussed by Lorenz (1986, 1992), Lorenz and Krishnamurthy (1987) and Curry et al. (1995). Both types of solutions in Lorenz (1986) are non-chaotic. By comparison, fast and slow "variables" that are chaotic may also coexist within coupled systems (e.g., Pena and Kalnay 2004; Mitchell and Gottwald 2012). In fact, an analysis using a singular perturbation method (Bender and Orszag 1978) indicates that the GLM also possesses the coexistence of slow and fast variables that correspond to large and very small spatial modes (e.g., Eq. (2) and Eq. (4) of Shen 2019a in a high-dimension phase space).

The (potential) occurrence of a nonlinear periodic solution (i.e., limit cycle) in the atmosphere was first illustrated by laboratory experiments using dishpans. Based on experiments by David Fultz (Fultz et al. 1959) and Raymond Hide (Hide 1953), Lorenz (1993) suggested three types of solutions, including (1) steady state solutions, (2) irregular chaotic solutions, and (3) vacillation. "Amplitude vacillation" is defined as a solution whose amplitude grows and periodically decays in a regular cycle (Lorenz 1963c; Ghil and Childress 1987; Ghil et al. 2010). Studies by Pedlosky and Smith (e.g., Pedloksy 1972; Smith 1975; Smith and Reilly 1977) found that amplitude vacillation can be viewed as a limit cycle solution.

Some people may wonder whether the appearance of LCs (i.e., nonlinear periodic solutions or vacillation) challenges the validity of the so-called error growth model (Lorenz 1969c, 1996; Nicolis 1992; Zhang et al. 2019; i.e., a logistic equation, that has been used to analyze errors in chaotic systems). Given an initial condition with a small value, the solution of the logistic equation grows at an initial larger growth rate, then a nonlinear smaller growth rate, and eventually approaches a constant defined as a saturated error. However, for periodic solutions such as vacillation (Lorenz 1969c), the error averaged over all growing and decaying components neither grows or decays. As a result, an averaged error that grows with time appears when a large number of growing errors and a small number of decaying errors are averaged. From the perspective of weather predictions, including a sufficiently large number of ensemble runs in order to obtain a forecast score that decreases monotonically with time is often required. (Note that within the logistic equation, large errors, which are larger than the value at the equilibrium point, should decay nonlinearly and then linearly.) While the error model with monotonically increasing solutions may describe the statistical behavior of the system within which the majority of small errors tends to grow, the error model cannot accurately represent the transient evolution of the specific solution consisting of decaying components or periodic solutions. In short, the decaying errors that may be associated with the steady-state solutions are not explicitly included within the error growth model.

In 1984, Lorenz proposed another idealized system of three ODEs for qualitatively depicting atmospheric circulation, known as the Lorenz (1984) model. Since detailed derivations of the Lorenz (1984) model were missing (e.g., Veen 2002a, b), it is difficult to trace the physical source of the forcing terms (parameters "F" and "G" in Eqs. (1)-(3) of Lorenz 1984) in the model. Additionally, as compared to fully dissipative systems where the time change rate of volume of the solutions is negative, the volume of the solution within the 1984 model does not necessarily shrink to zero (e.g., p. 380 of Lorenz 1990). Therefore, results obtained using the Lorenz 1984 model should be analyzed and interpreted with caution. Here, we illustrate some important features that are consistent with our findings (e.g., Shen 2019b) that support the revised view on the dual nature of weather.

Major features within the Lorenz (1984) model are summarized as follows: (1) there are three types of solutions, including steady state, periodic solutions, and chaotic solutions, that depend on the values of system parameters; (2) (some) periodic solutions can be identified as limit cycle solutions (Wang et al., 2014); multistability with coexisting limit cycle solutions gave rise a question of whether or not intransitivity may occur (i.e., whether or not any of the state solutions may last forever); (3) when a seasonally varying forcing term F with a time scale of 12 months was applied, chaotic solutions appear during winter and two different limit cycle solutions appear during active and inactive summer, respectively (e.g., Fig. 6 of Lorenz 1990); (4) a spectral analysis displays peaks at time scales of 20 days associated with solutions during the summer (e.g., Figs. 1-3 of Pielke and Zeng 1994); (5) the transition from a chaotic solution in winter to a periodic solution in summer displays a final state sensitivity in association with the coexistence of two different limit cycle solutions. Such a transition may be likely unpredictable. The final state sensitivity suggests that the system is unlikely intransitive. However, our results indicate that once summer begins and has been observed, a predictability of more than two weeks may be expected during each cycle of a periodic solution during the summer months.

The above analysis supports our revised view on the dual nature of weather and the hypothetical mechanism for the recurrence (or periodicity) of successive African Easterly Waves (AEWs), based on the GLM, in Shen (2019b). The insensitivity of limit cycles to initial conditions implies that AEW simulations could be more predictable than we assumed (i.e., a predictability of more than two weeks).

5. Concluding Remarks

The chaotic nature of weather with finite predictability has been revealed for decades using the Lorenz model (Lorenz 1963), leading to a view that weather is chaotic. By including additional small-scale processes within the generalized Lorenz model (e.g., Shen 2019a), we previously suggested the possibility of suppressing chaotic responses and, thus, incrementally improving predictability. In this study, we further discussed coexisting attractors in order to illustrate the dual nature of chaos and order in weather that leads to a different view on the intrinsic predictability of weather. As a result, we suggest that the entirety of weather is a superset that consists of both chaotic and non-chaotic processes. Specific weather systems may appear on a chaotic or non-chaotic orbit for their finite lifetime, depending on the time scales of the energy source. The refined view with a duality of chaos and order is fundamentally different from the Laplacian view of deterministic predictability and the Lorenz view of deterministic chaos. The appearance of periodic solutions (i.e., vacillation) and their transition to chaotic solutions associated with time varying parameters were indeed documented by Prof. Lorenz using different approaches (e.g., Lorenz 1969, 1984, 1990).

The refined view is not too optimistic or too pessimistic as compared to traditional views. Both potential and challenges are suggested. The refined view for a dual nature of weather permits the possibility of both finite and unlimited predictability (e.g., up to the lifetime of a dissipative system). Although chaotic solutions with BE have finite, time-varying predictability, as a result of a sensitivity to initial conditions, they do not exclusively appear but occur within a subset of the total number of solutions. By comparison, for non-chaotic processes with steady-state or nonlinear periodic solutions, their intrinsic predictability is deterministic and their practical predictability can be continuously increased by improving the accuracy of the model and the initial conditions. To this end, if we are able to identify non-chaotic solutions such as steady-state, periodic, or quasi-periodic solutions in advance, we may obtain longer predictability or better estimates on predictability. Our future work will focus on developing schemes for the detection of chaotic and non-chaotic solutions (e.g., Sprott and Xiong, 2015; Reyes and Shen 2019) in order to improve our understanding of the roles of butterfly effects in the real world and on high-resolution global models; and, thus, our understanding of the conditions under which nonlinear interactions may lead to chaotic solutions and/or non-chaotic solutions such as limit cycle solutions.

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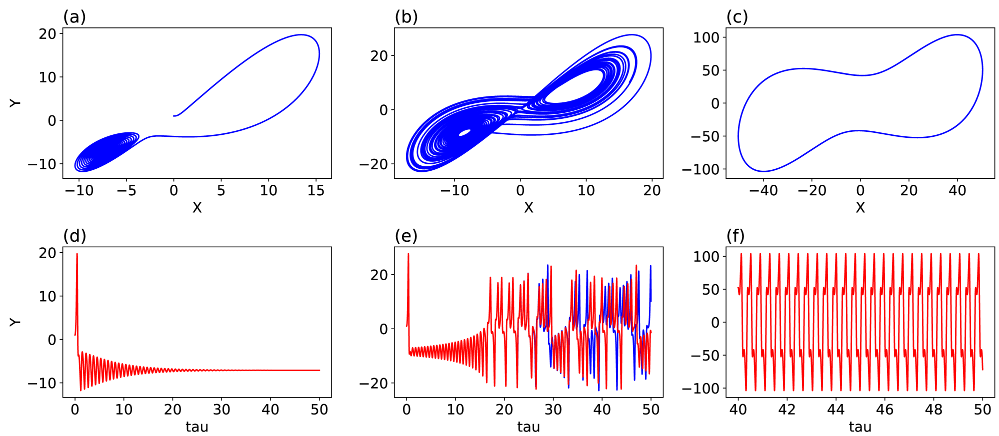

Figure 1: Three types of solutions within the 3DLM. Left, middle, and right panels display steady-state, chaotic, and limit cycle solutions at small, moderate, and large heating parameters (i.e., $r = 20, 28,$ and 350), respectively. The solutions are categorized into a point attractor, a chaotic attractor, and a periodic attractor, respectively. Top panels show orbits within the $X - Y$ space and bottom panels depict the time evolution of Y. Blue lines provide solutions from control runs. To display results from parallel runs, red lines are added in the bottom panels. Sensitive dependence on initial conditions is shown in panel (e) with two visible lines. Panels (b) and (e) are reproduced from Shen (2019b).

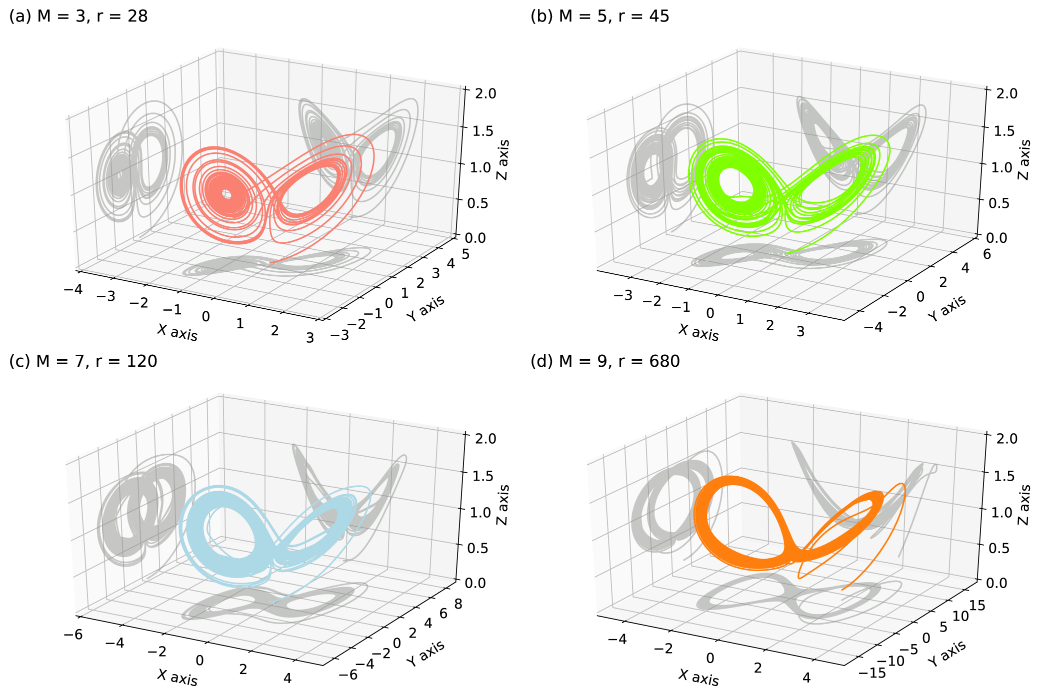

Figure 2: Chaotic solutions in the $X-Y-Z$ phase space within the 3D, 5D, 7D, and 9D Lorenz models (LMs). Panels (a)-(c) use the same initial conditions with Y = 1 and the remaining as zero, while panel (d) uses the IC with 100 for all variables. Variables (X, Y, Z) are normalized by $2\sqrt{r-1}$, $2\sqrt{r-1}$, and $(r-1)$, respectively. A larger heating parameter is required for the onset of chaos in a higher-dimensional LM. Reproduction from Shen (2016) and Shen (2019).

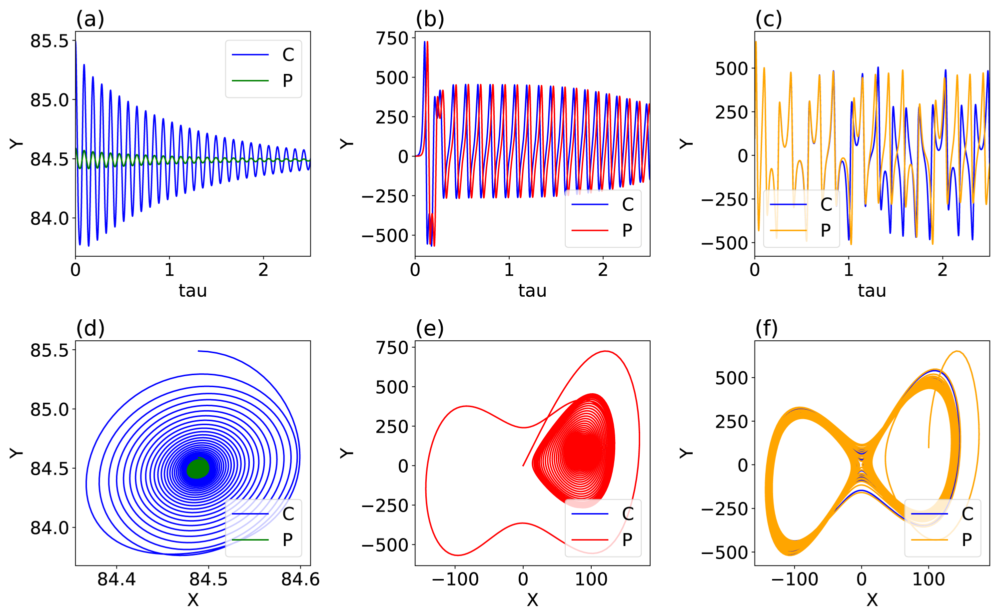

Figure 3: Solutions of the GLM with $M = 9$ and $r = 680$. Initial conditions are placed near the non-trivial critical point and the origin (i.e., trivial critical point) and at $(100, 100, 100, 100, 100, 100, 100, 100, 100)$. Top panels show the time evolution of Y for $t \in [0, 2.5]$, while bottom panels display the corresponding solutions $t \in [0, 10]$ within the $X - Y$ space. Control and parallel runs are denoted by 'C' and 'P', respectively. A finite-amplitude perturbation ($\epsilon = -0.9$) is added into the parallel runs. Panels (c) and (f) are reproduced from Shen (2019a).

Supplemental Materials: Was $\boldsymbol{\sigma = 10}$ a Magic Choice?

For the past 50 years, although various types of solutions for Lorenz (1963) have been documented, chaotic solutions have been a main focus. As discussed in the main text, since chaotic solutions appear over a finite range of parameters, their applicability in revealing the nature of weather depends on the realism of not only the models employed but also model parameter values. In his book in 1993, Lorenz humbly expressed that it may not be possible for him to discover the butterfly-pattern solution if a realistic value of $\sigma = 1$ was used, as shown below:

``

I was lucky in more ways than one. An essential constant of the model is the Prandtl number -- the ratio of the viscosity of the fluid to the thermal conductivity. Barry had chosen the value 10.0 as having the order of magnitude of the Prandtl number of water. As a meteorologist, he might well have chosen to model convection in air instead of water, in which case he would probably have used the value 1.0. With this value the solutions of the three equations would have been periodic, and I probably would never have seen any reason for extracting them from the original seven.

''

Therefore, one may wonder how fortunate Prof. Lorenz was and whether a realistic value of $\sigma = 1$ may have influenced our view on the nature of weather. We make an attempt of addressing the question by analyzing a GLM with M = 9 and examining a 3DLM with $\sigma = 1$. As discussed in Shen (2019), the GLM with M = 9 has stable, non-trivial equilibrium points for all $r > 1$ when $\sigma = 10$ and b = 8/3. To have stable, non-trivial equilibrium points for $\sigma = 1$ within the 3DLM, we chose b = 2/5. Such a choice leads to two kinds of attractor coexistence, a unique feature first identified within the 9DLM (Shen 2019). With $\sigma = 1$ in the 3DLM, the first kind of coexistence includes chaotic and steady-state solutions at a moderate heating parameter (e.g., r = 170, as shown in Fig. S1). The second kind of coexistence consists of a limit cycle and a steady-state solution at a large heating parameter (e.g., r = 250, not shown). Table S1 lists initial conditions for the results provided in Fig. S1. Thus, chaotic solutions may still appear within the 3DLM for a realistic value of $\sigma = 1$, but they coexist with steady-state solutions. The appearance of chaotic solutions depends not only on the range of the heating parameter but also on the ICs.

Both traditional and new model configurations with $(\sigma, b) = (10, 8/3)$ and $(1, 2/5)$, respectively, can produce chaotic solutions. For the traditional configuration that has been well applied in numerous studies since Lorenz (1963), all of the three equilibrium points are unstable when $r > 24.74$. The stability of three equilibrium points for $\sigma = 10$, as well as for $\sigma = 1$, is illustrated in Fig. S2. The non-existence of stable equilibrium points within the chaotic regime makes it easier to obtain chaotic solutions. However, no tiny, initial perturbation can completely lose its impact within the chaotic regime. We may interpret this as a finding that a tiny, initial perturbation cannot completely dissipate (before leading to a large impact). By comparison, for the new configuration, while the origin is still a saddle point, the two, non-trivial equilibrium points are stable (Fig. S2b). The existence of stable equilibrium points enables the coexistence of chaotic and steady-state solutions, the latter of which has no long-term memory regarding a tiny, initial perturbation.

As a result of coexistence for $\sigma = 1$ within the 3DLM, a proper choice of initial conditions is required in order to simulate a chaotic solution. Without knowing this, Prof. Lorenz thought that it may be impossible to obtain a “strange'' solution if $\sigma = 1$ was first used in the Saltzman (1962) model, giving no motivation for him to work on the 3DLM. In other words, the value of $\sigma = 10$ used in the original study (e.g., Saltzman, 1962) was indeed a “fortunate” choice so that an unexpected irregularly oscillatory solution could be revealed, inspiring Prof. Lorenz to develop the 3DLM to discover the interesting chaotic features. However, on the other hand, we now understand that such a configuration can only depict a partial picture for the nature of weather. Based on our results and analysis, a realistic system should include physical processes for (some of) the tiny disturbances in order to completely dissipate. Since it produces the coexistence of chaotic and steady-state solutions and since the steady-state solution has no long-term memory of tiny perturbations, the 3DLM with the new configuration of $\sigma = 1$ satisfies the objective. Such a system, which is similar to the 9DLM that produces two kinds of coexisting attractors, provides a more realistic view on the true nature of weather than the original 3DLM with a typical configuration. The above analysis supports our refined view that weather is a superset that consists of chaotic (with BE) and non-chaotic (without BE) processes.

Table S1: Initial conditions (ICs) for revealing the coexistence of two attractors for $\sigma = 1$, $b = 0.4$, and $r = 170$ within the 3DLM. $X_c = Y_c = \sqrt{b(r-1)}$ and $Z_c = (r-1)$. The six rows provide the ICs for Fig. S1.

X	Y	Z
X_c	$Y_c + 1$	Z_c
-X_c	$-Y_c + 1$	Z_c
0	1	0
-76.72346293	37.62433028	-146.96230812
-27.75526885	167.67883615	3.66782724
136.44623635	99.45689394	-19.76741851

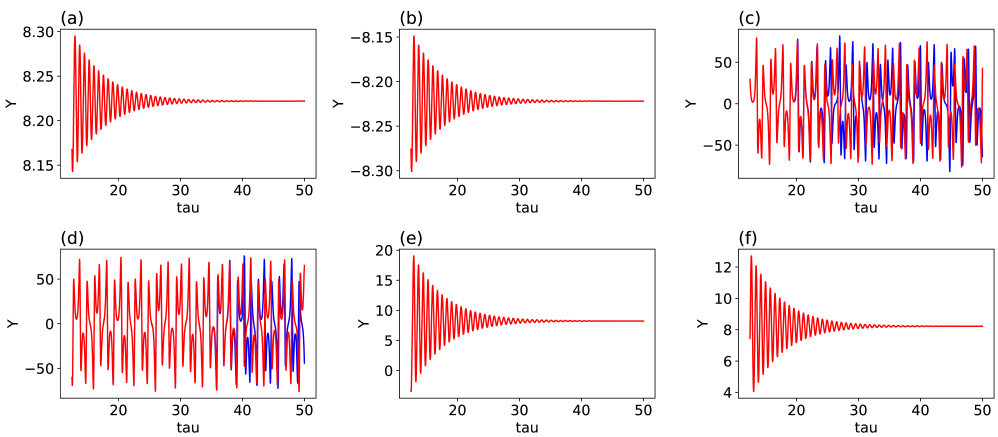

Figure S1: A co-existence of chaotic (c, d) and non-chaotic (a, b, e, f) solutions using the same parameters for $\sigma = 1$, $b = 0.4$, and $r = 170$ within the 3DLM. Blue and red lines display solutions from the control and parallel runs, respectively. Initial conditions for the results in six panels are listed in Table S1.

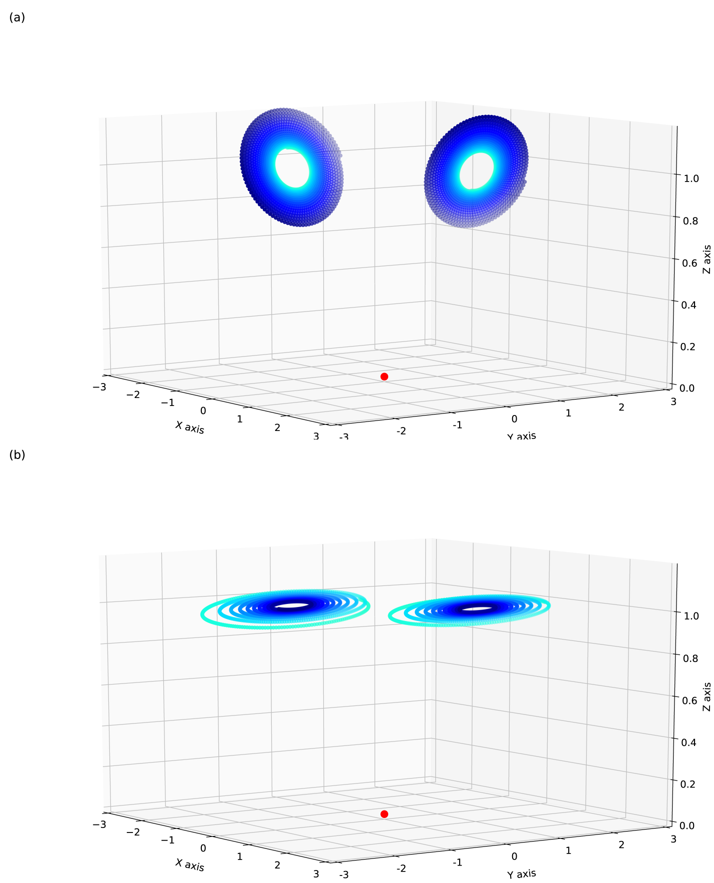

Figure S2: Local behavior near the two non-trivial critical points for the 3DLM with $\sigma = 10$ (a) and $\sigma = 1$ (b). Lighter blue dots indicate the locations of orbits at earlier times. A red dot indicates the origin, which is a saddle point. Orbits in panel (a) spiral away from the non-trivial critical points while orbits in panel (b) spiral toward the non-trivial critical points.