Diagnosing eddy diffusivities via one-shot optimization

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Introduction

- To test parameterizations, one often takes a forward modeling approach, comparing the statistics generated from low-resolution simulation (with parameterization scheme) with those from a high-resolution simulation.
- Here, we diagnose eddy diffusivities via an optimization procedure.
- Match a measure based on the divergent eddy flux, which is defined via the definition of a force function Ψ_e (Maddison, Marshall & Shipton, in review with *Ocean Model.*), where

$$\overline{\boldsymbol{u}'q'} = -\nabla\Psi_e + \boldsymbol{e}_z \times \nabla\Phi_e + \boldsymbol{H}_e.$$

- Decomposition is optimal in that $-\nabla\Psi_e$ has minimal L^2 norm, and as a result Ψ_e itself is smooth, which aids inversion.

Simulation details

- Three layer QG equations, closed basin, asymmetric wind forcing in the upper layer, friction in the bottom layer, and partial-slip boundary conditions on lateral boundaries.
- Please see Maddison, Marshall & Shipton (in review with *Ocean Model.*) and Marshall, Maddison & Berloff (2012, *J. Phys. Oceanogr.*) for precise details.

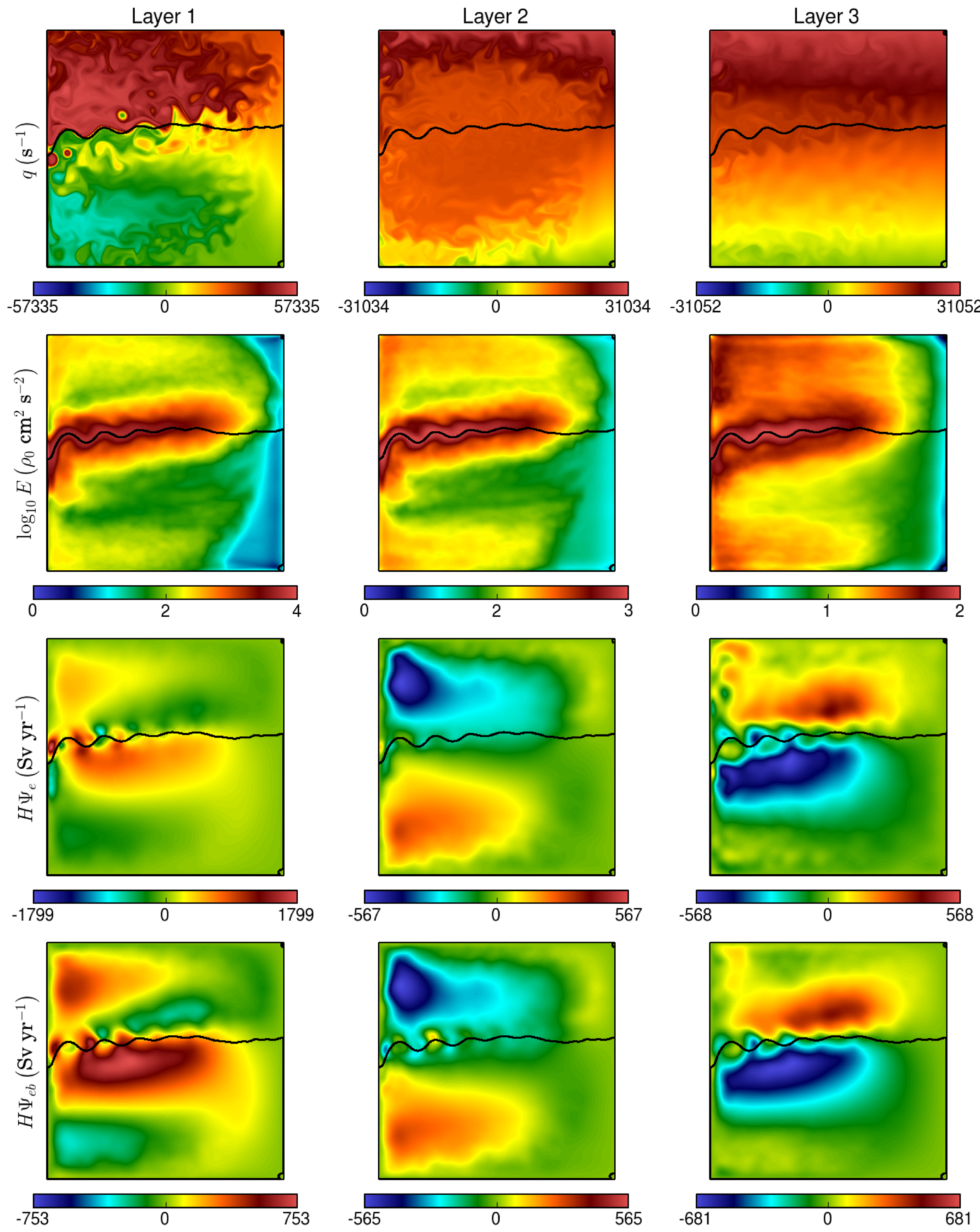

Figure 1: Data and force functions from simulation data (black contour is the location of the mean jet). From top to bottom: potential vorticity (PV) q; total eddy energy E; eddy force function Ψ_e; eddy force function Ψ_{eb} due to buoyancy flux.

Implementation and PV mixing parameterization

- Consider PV mixing $\overline{\boldsymbol{u}'q'} = -\kappa\nabla\overline{q}$. We seek a stationary point of

$$J = \|\Psi_{e,i} - \Psi_{p,i}\|^2_{L^2} + \langle\nabla\lambda_i, \nabla\Psi_{p,i} - \kappa\nabla\overline{q}_i\rangle_{L^2} + \epsilon\|\nabla\kappa_i\|^2_{L^2},$$

where $\nabla^2\Psi_e = -\nabla\cdot\overline{\boldsymbol{u}'q'}$.

- The terms in J are the mismatch, (weak form of) constraint, and a regularization respectively.

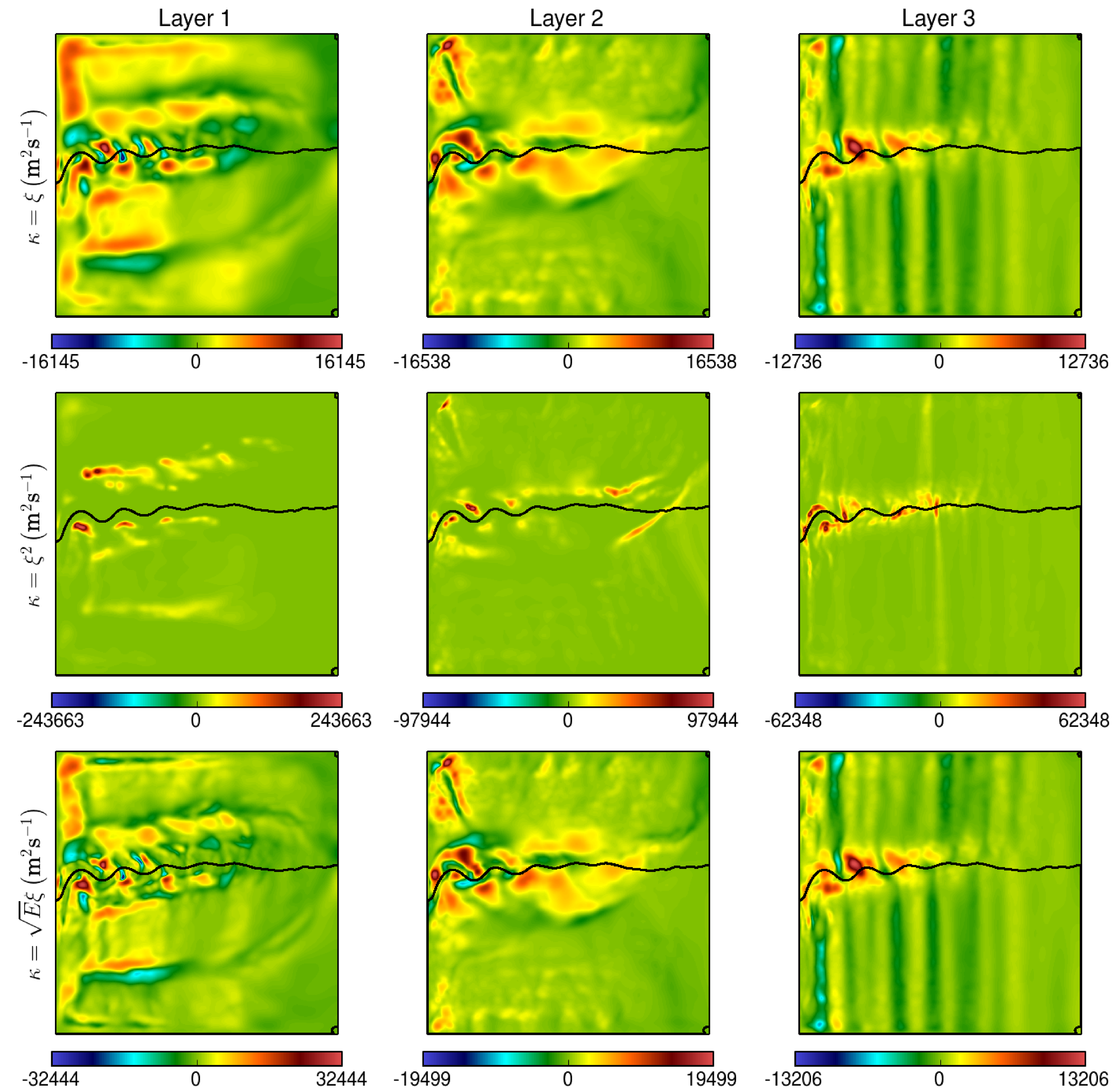

Figure 2: κ field of the three parameterization variants tested: unsigned $\kappa = \xi(\boldsymbol{x})$; $\kappa = \xi(\boldsymbol{x})^2$ with κ constrained to be positive-definite; $\kappa = \sqrt{E}\xi(\boldsymbol{x})$, where E is the eddy energy field output from simulation and is a mixing-length type variant.

	relative error			mean (m^2 s^{-1})		
	Layer 1	Layer 2	Layer 3	Layer 1	Layer 2	Layer 3
$\kappa(\boldsymbol{x})$	14.08%	3.01%	1.47%	608	759	91
$\kappa(\boldsymbol{x}) = \xi^2 \geq 0$	32.59%	3.78%	2.48%	2665	1370	1154
$\kappa(\boldsymbol{x}) = \xi\sqrt{E}$	8.35%	2.45%	1.36%	415	746	76

Table 1: Table of relative error and mean κ value associated with the parameterization variants.

- Implemented using FEniCS (Logg *et al.*, 2012, *Springer*) and solved using one shot (e.g., Ta'asan, 1991, *ICASE Report*).

```
kappa = xi # specify form of kappa
J_1 = ((psi - fns["ffd_empb_%i" % (l + 1)]) ** 2) * dx
J_2_res = grad(psi) - kappa * grad(fns["q_%i_n_mean" % (l + 1)])
J_2 = inner(grad(lam), J_2_res) * dx
J_3 = eps * (grad(xi) ** 2) * dx
J = J_1 + J_2 + J_3
dJ = derivative(J, X, du = tests) # compute directional derivative
solve(dJ == 0, X, boundary_conditions, solver_options) # solve for the system
```

Figure 3: Sample of the Python code implementing the calculation (e.g., Logg *et al.*, 2012, *Springer*; Alnæs *et al.*, 2014, *ACM TOMS*).

Gent–McWilliams parameterization

- Similar calculations for the Gent–McWilliams parameterization (Gent & McWilliams, 1990, *J. Phys. Oceanogr.*), where $\overline{\boldsymbol{u}'b'} = -\kappa_{\mathrm{GM}}\nabla\overline{b}$. J is now

$$J = \sum_{i=1}^{3}\|\Psi_{eb,i} - \Psi_{p,i}\|^2 + \sum_{i=1}^{2}\langle\nabla\lambda_i, \nabla\Psi_{p,i} - H_i s_i^+(\kappa_{\mathrm{GM}})_{i+1/2}\nabla(\overline{\psi}_i - \overline{\psi}_{i+1})\rangle_{L^2}$$
$$+\,\epsilon\sum_{i=1}^{2}\left(\frac{H_i + H_{i+1}}{2}\right)\|\nabla(\kappa_{\mathrm{GM}})_{i+1/2}\|^2_{L^2}, \qquad s_i^+ = \frac{f_0^2}{g'_{i\pm1/2}H_i},$$

where $\nabla^2\Psi_{eb} = -\nabla\cdot(\partial/\partial z)(f_0/N_0^2)\overline{\boldsymbol{u}'b'}$.

- Results below. No $\kappa \geq 0$ variant here as trivial solution is returned in lower interface despite multiple initializations.

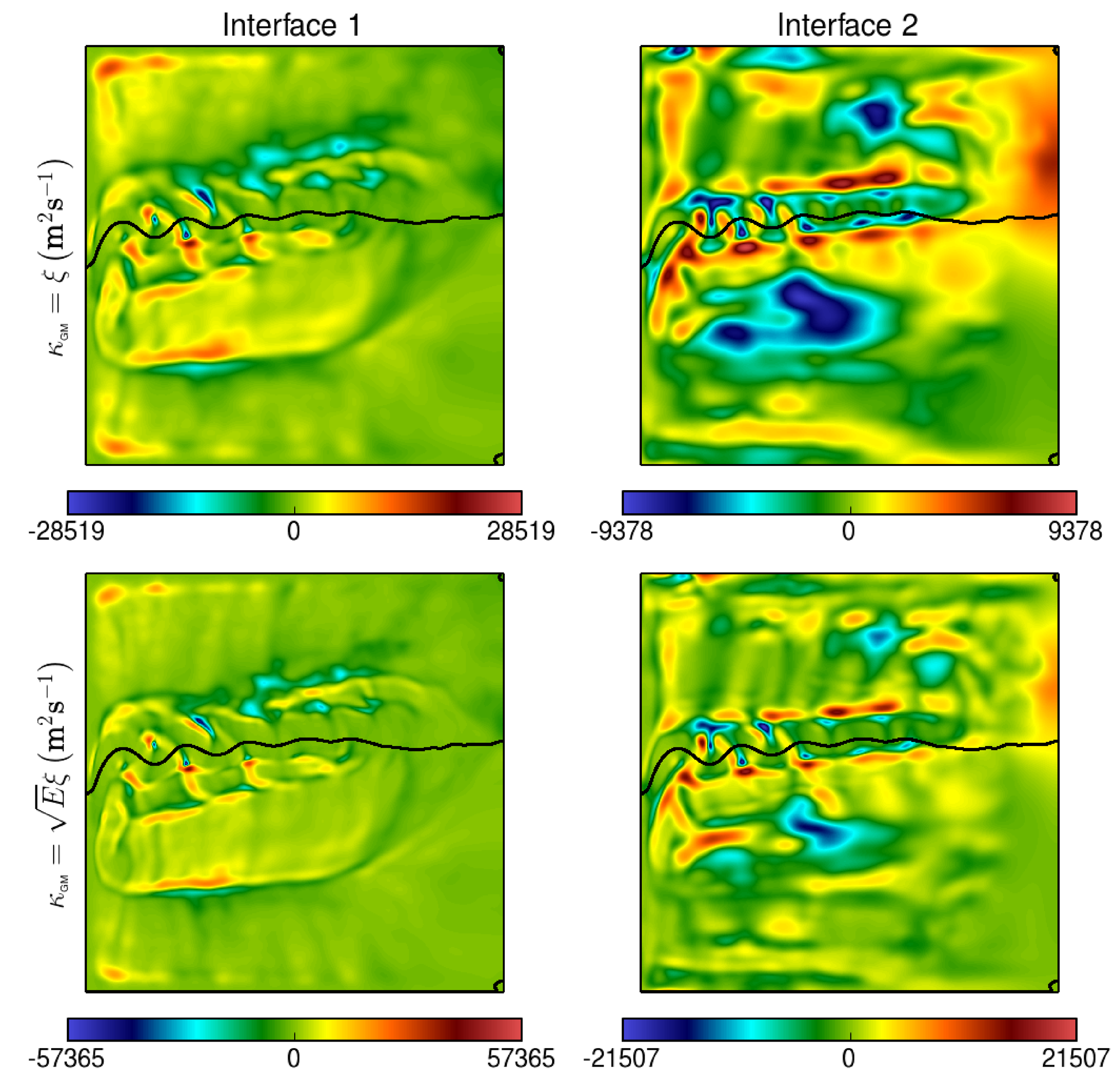

Figure 4: $(\kappa_{\mathrm{GM}})_{i+1/2}$ field for the same three parameterization variants.

	relative error			mean (m^2 s^{-1})	
	Layer 1	Layer 2	Layer 3	Interface 1	Interface 2
$\kappa_{\mathrm{GM}}(\boldsymbol{x})$	6.07%	9.21%	10.25%	45	−282
$\kappa_{\mathrm{GM}}(\boldsymbol{x}) = \xi\sqrt{E}$	3.99%	5.96%	6.66%	−97	−290

Table 2: Table of relative error and mean κ_{GM} value associated with the parameterization variants.

Conclusions

- κ varies in space, is generally locally negative, with strength correlated with eddy energy.
- Mean of κ may be negative in the GM case.
- $\kappa = E\xi$ variant is seen to be the most successful variant (not shown here).
- Calculations based on data from simulations with different configuration (from ARCHER) show qualitatively similar results.
- Multiple parameter optimizations potentially possible (e.g., eddy suppressed diffusivity; Ferrari & Nikurashin, 2010, *J. Phys. Oceanogr.*).

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