Why the bolus velocity deserved to survive and how we use it?

A 3D EP theory for all waves at all latitudes as given by (what we call) the impulse-bolus (IB) pseudomomentum

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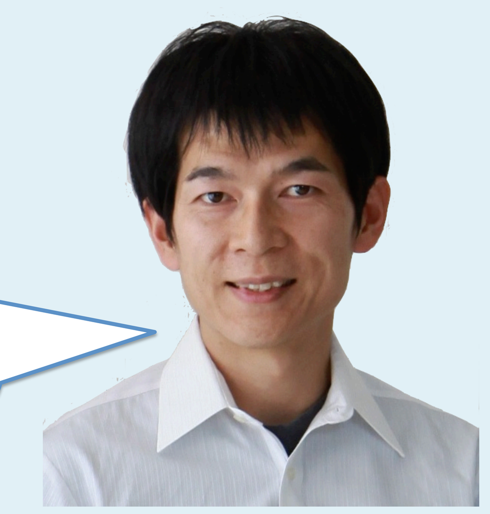

- The IB pseudomoemtnum equation has previously been derived using a shallow-water model
- **Pressure flux does not point in the direction of the group velocity of Rossby waves**
- **The IB flux points in the direction of the group velocity of all waves at all latitudes** which is attributed to a divergence-form wave-induced pressure associated with the virial threorem

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Aiki, Takaya, and Greatbatch (Journal of the Atmospheric Sciences, 2015, in press) doi:10.1175/JAS-D-14-0172.1

$$Q^\dagger = \psi_{xx} + \psi_{yy} + (\psi_z f_0^2/N^2)_z$$
$$(\partial_t + \overline{u}\partial_x)[\psi_{xx} + \psi_{yy} + (\psi_z f_0^2/N^2)_z] + \psi_x(\beta - \overline{u}_{yy}) = 0$$

deviation from Eulerian zonal mean

$$A^\dagger = A - \overline{A}$$
$$\nabla_{yz} = \langle\langle \partial_y, \partial_z \rangle\rangle$$

QG Taylor–Bretherton identity (meridional flux of QGPV)

$$\overline{Q^\dagger \psi_x} = -\nabla_{yz} \cdot \langle\langle -\overline{\psi_x \psi_y}, \overline{-\psi_x \psi_z f_0^2/N^2} \rangle\rangle$$

QG Eliassen–Palm relation (prognostic equation for the QG pseudomomentum)

$$\partial_t[(1/2)\overline{Q^{\dagger 2}}/(\overline{u}_{yy} - \beta)] + \nabla_{yz} \cdot \langle\langle -\overline{\psi_x \psi_y}, -\overline{\psi_x \psi_z f_0^2/N^2} \rangle\rangle = 0$$

Allows for the group velocity vector to be diagnosed from model output without performing a Fourier analysis

Governing equation for neutral waves at all latitudes:

MIGWs (mid-latitude inertia-gravity waves)
MRWs (mid-latitude Rossby waves)
EQWs (all types of equatorial waves)

$$u'_t - (f_0 + \beta y)v' = -p'_x,$$
$$v'_t + (f_0 + \beta y)u' = -p'_y,$$
$$\rho'_t + w'\overline{\rho}_z = 0,$$
$$u'_x + v'_y + w'_z = 0,$$
$$\langle\langle u', v', w' \rangle\rangle = \langle\langle \xi'_t, \eta'_t, \zeta'_t \rangle\rangle,$$

$$\zeta' \equiv -\rho'/\overline{\rho}_z = -p'_z/N^2,$$
$$K = (u'^2 + v'^2)/2, \quad G = (N^2/2)\zeta'^2,$$
$$q' \equiv v'_x - u'_y - (f_0 + \beta y)\zeta'_z,$$
$$q'_t + \beta v' = 0,$$
$$\eta' = -q'/\beta,$$

Ertel's potential vorticity

$$A' = A - \overline{A}$$
$$\nabla = \langle\langle \partial_x, \partial_y, \partial_z \rangle\rangle$$

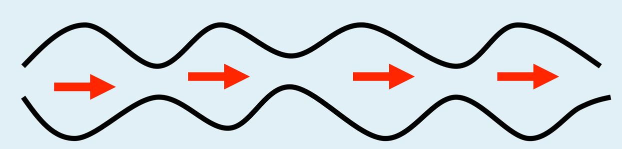
in height coordinates and with overbar and single prime Indicating Eulerian time mean and deviation from it

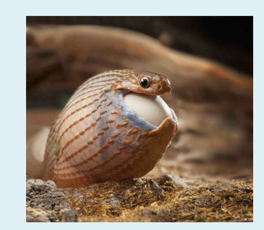
Approximated expressions using a Taylor expansion in height-coordinates

(overbar and single prime indicating an Eulerian time mean and deviation from it)

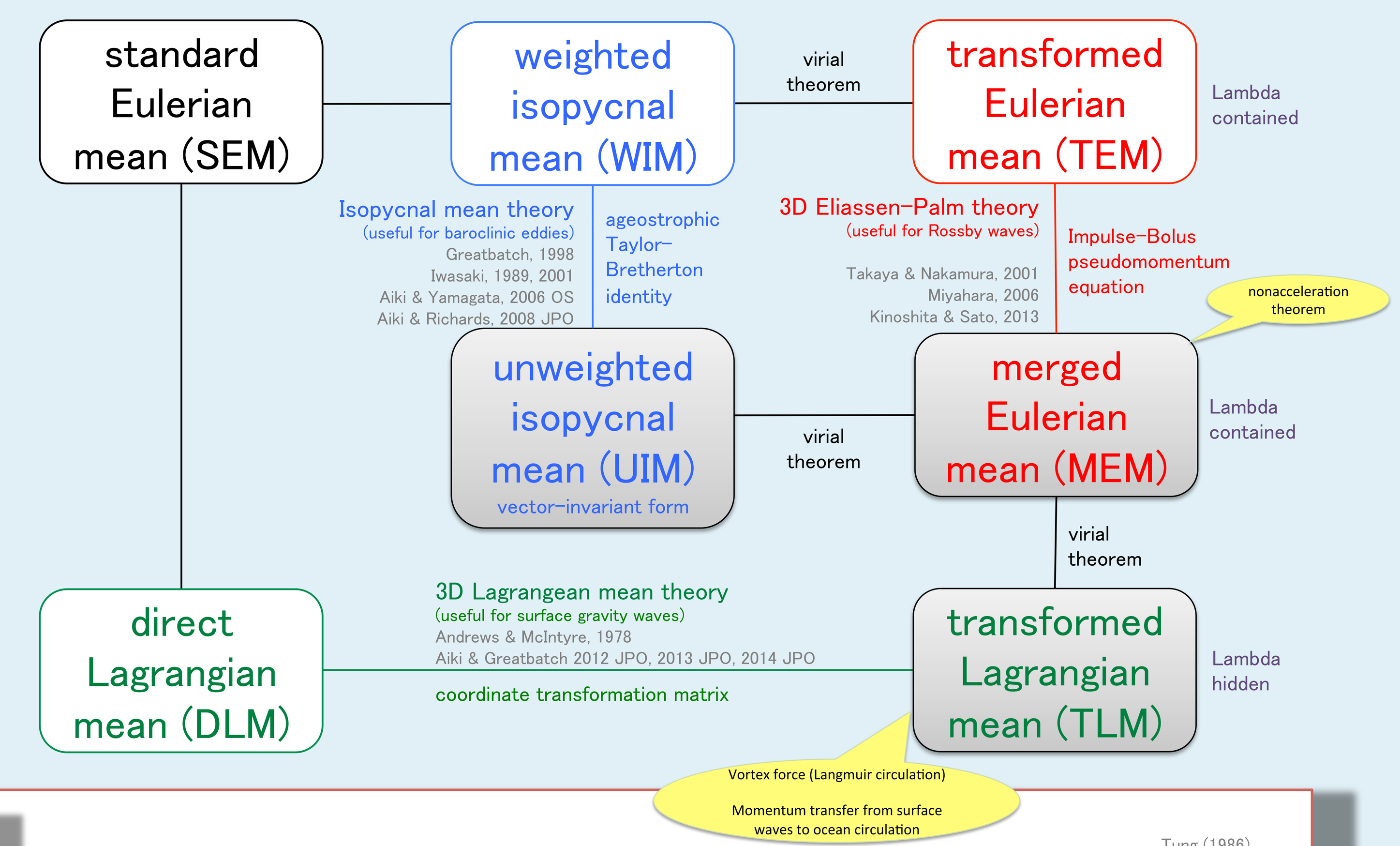
$$\overline{A}^L = \overline{A} + \overline{\xi' A'_x} + \overline{\eta' A'_y} + \overline{\zeta' A'_z}$$ 3D Lagrangian mean
$$\widetilde{A} = \overline{A} + \overline{\zeta' A'_z}$$ Unweighted isopycnal mean
$$\widehat{A} = \widetilde{A} + \overline{\zeta'_z A'} = \overline{A} + (\overline{\zeta' A'})_z$$ Thickness-weighted isopycnal mean

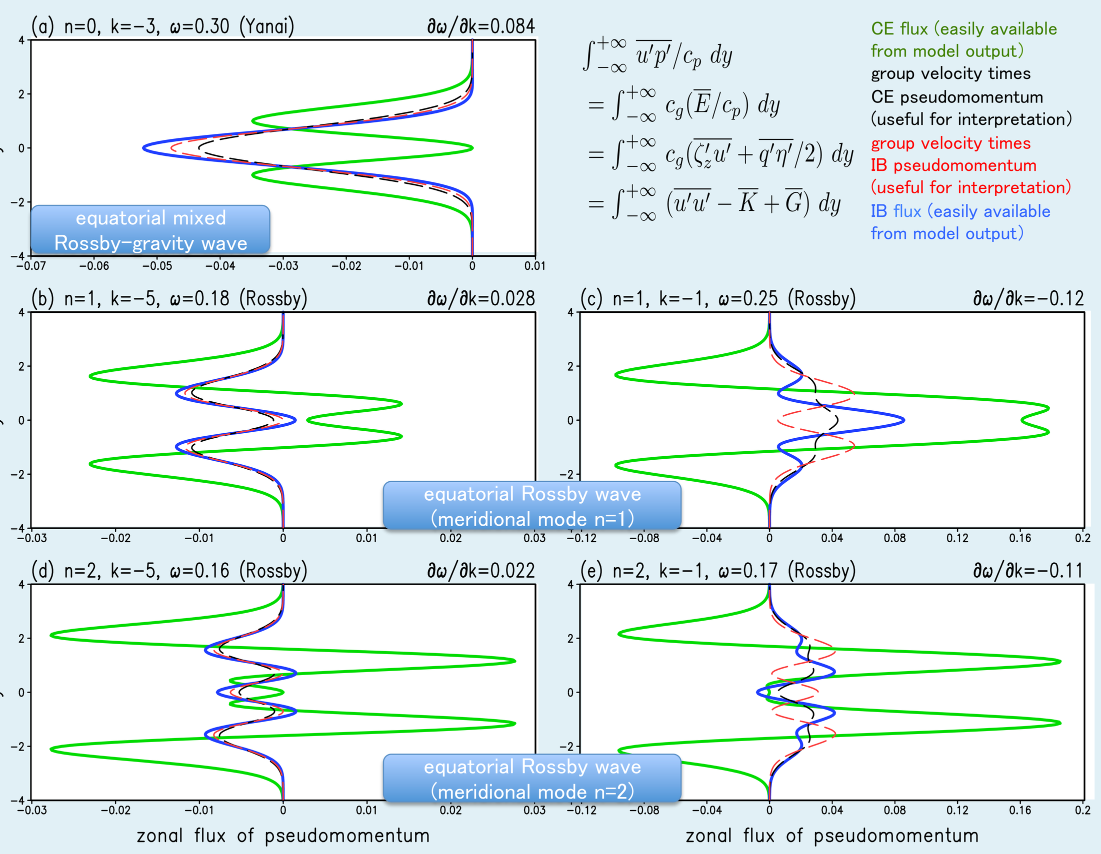
$$u^{Stokes} \equiv \overline{u}^L - \overline{u} = \overline{\xi' u'_x} + \overline{\eta' u'_y} + \overline{\zeta' u'_z}$$ Stokes-drift velocity
$$u^{qs} \equiv \widehat{u} - \overline{u} = (\overline{\zeta' u'})_z$$ quasi-Stokes velocity
$$u^{bolus} \equiv \widehat{u} - \widetilde{u} = \overline{\zeta'_z u'}$$ bolus velocity (Rhines, 1982)





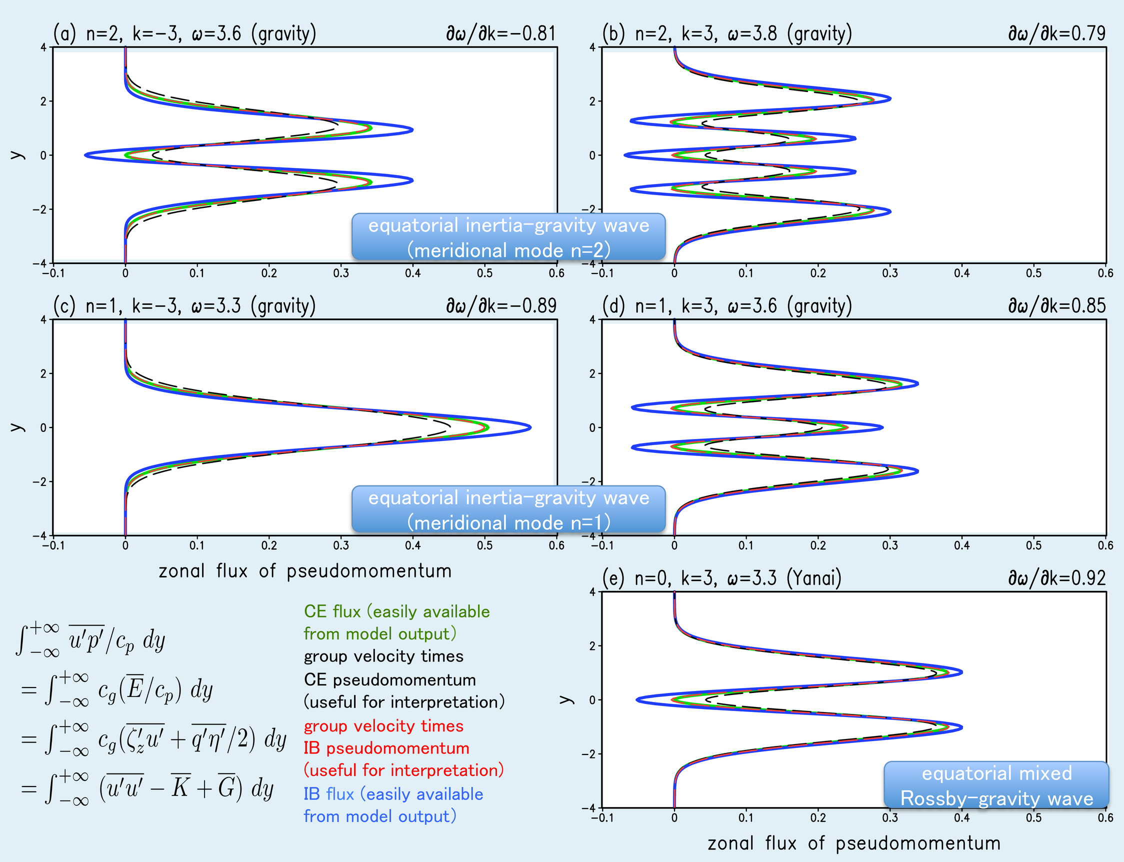
Relationships between the low-pass filtered momentum equations in different theories for representing the effect of waves on the mean flow





$$\int_{-\infty}^{+\infty} \overline{u'p'}/c_p \, dy$$ CE flux (easily available from model output)
$$= \int_{-\infty}^{+\infty} c_g(\overline{E}/c_p) \, dy$$ group velocity times CE pseudomomentum (useful for interpretation)
$$= \int_{-\infty}^{+\infty} c_g(\overline{\zeta'_z u'} + \overline{q'\eta'}/2) \, dy$$ group velocity times IB pseudomomentum (useful for interpretation)
$$= \int_{-\infty}^{+\infty} (\overline{u'u'} - \overline{K} + \overline{G}) \, dy$$ IB flux (easily available from model output)

What it the origin of the difference of the CE and IB fluxes



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$$= \int_{-\infty}^{+\infty} (\overline{u'u'} - \overline{K} + \overline{G}) \, dy$$ IB flux (easily available from model output)

Prognostic equation for the classical Energy–based (CE) pseudomomentum

$$(\underbrace{K + G}_{E})_t + \nabla \cdot \langle\langle \overline{u'p'}, \overline{v'p'}, \overline{w'p'} \rangle\rangle = 0, \qquad c_p \equiv \omega/k, \qquad c_g \equiv \partial\omega/\partial k$$
$$(\overline{E}/c_p)_t + \nabla \cdot \langle\langle \overline{u'p'}/c_p, \overline{v'p'}/c_p, \overline{w'p'}/c_p \rangle\rangle = 0,$$
$$\int_{-\infty}^{+\infty} \overline{u'p'}/c_p \, dy = \int_{-\infty}^{+\infty} c_g(\overline{E}/c_p) \, dy, \qquad \overline{u'p'}/c_p \neq c_g(\overline{E}/c_p)$$

Prescription for the phase velocity in the denominator to be rewritten

See green and black lines In the left figure

$$\pi' \equiv \int^t p' dt,$$
$$u' - f\eta' = -\pi'_x,$$
$$v' + f\xi' = -\pi'_y,$$

$$E \equiv K + G$$
$$= (u'^2 + v'^2 + N^2\zeta'^2)/2$$
$$= (u'\xi'_t + v'\eta'_t - \zeta\pi'_{zt})/2,$$

$$[\underbrace{(-\overline{u'\xi'_x} - \overline{v'\eta'_x} + \overline{\zeta'\pi'_{zx}})/2}_{\text{CE pseudomomentum}}]_t$$
$$= -\nabla \cdot \langle\langle -(\overline{\xi'_x p'} + \overline{u'\pi'_x})/2, -(\overline{\eta'_x p'} + \overline{v'\pi'_x})/2, -(\overline{\zeta'_x p'} + \overline{w'\pi'_x})/2 \rangle\rangle,$$

Redirecting the CE flux by singling out $\Lambda \equiv [(\overline{\xi' p'})_x + (\overline{\eta' p'})_y + (\overline{\zeta' p'})_z]/2$

$$[\underbrace{(-\overline{u'\xi'_x} - \overline{v'\eta'_x} + \overline{\zeta'\pi'_{zx}})/2}_{\text{CE pseudomomentum}}]_t$$
$$= -\nabla \cdot \langle\langle (\overline{\xi' p'_x} - \overline{u'\pi'_x})/2 - \Lambda, (\overline{\eta' p'_x} - \overline{v'\pi'_x})/2, (\overline{\zeta' p'_x} - \overline{w'\pi'_x})/2 \rangle\rangle,$$

see virial theorem

$$[\underbrace{(-\overline{u'\xi'_x} - \overline{v'\eta'_x} + \overline{\zeta'\pi'_{zx}})/2}_{\text{CE pseudomomentum}}]_t$$
$$= -\nabla \cdot \langle\langle \overline{E} - \overline{v'v'} + (\overline{v'\eta'})_t/2, \overline{v'u'} - (\overline{u'\eta'})_t/2, \overline{\zeta' p'_x} - (\overline{\zeta'\pi'_x})_t/2 \rangle\rangle,$$

see gauge transformation

Impulse–Bolus (IB) pseudomomentum equation

$$\partial_t(\overline{\zeta'_z u'} + \overline{q'\eta'}/2) + \nabla \cdot \langle\langle \overline{u'u'} - \overline{K} + \overline{G}, \overline{v'u'}, \overline{\zeta' p'_x} \rangle\rangle = 0,$$

Two separate approaches for the ageostrophic Eliassen-Palm relation

prognostic equation for Classical Energy–based (CE) Pseudomomentum

pro: directly related to wave energetics
con: the pressure–based flux does not point in the direction of the group velocity of Rossby waves

Prognostic equation for Impulse–bolus (IB) Pseudomomentum

pro: IB flux is parallel to the group velocity of all MIGWs, MRWs, EQWs
con: thought to be available only in a shallow water model or isentropic coordinates

main outcome of this study

Gauge Transformation between CE and IB pseudomomenta is found

Helpful for understanding what is the origin of the difference between the CE and IB fluxes

CE pseudomomentum — IB pseudomoemtum — 3D-divergence

$$(-\overline{u'\xi'_x} - \overline{v'\eta'_x} + \overline{\zeta'\pi'_{zx}})/2 = \overline{\zeta'_z u'} + \overline{q'\eta'}/2 + \nabla \cdot \langle\langle -\overline{v'\eta'}, \overline{u'\eta'}, \overline{\zeta'\pi'_x} \rangle\rangle/2$$

Ageostrophic Taylor–Bretherton identity

$$\partial_t(\overline{\zeta'_z u'}) + \overline{q'v'} = -\nabla \cdot \langle\langle \overline{u'u'} - \overline{K} + \overline{G}, \overline{v'u'}, \overline{\zeta' p'_x} \rangle\rangle,$$
$$\partial_t(\overline{\zeta'_z v'}) - \overline{q'u'} = -\nabla \cdot \langle\langle \overline{u'v'}, \overline{v'v'} - \overline{K} + \overline{G}, \overline{\zeta' p'_y} \rangle\rangle,$$

Tung (1986)
Hayashi and Young (1987)
McPhaden and Ripa (1990)
Takehiro and Hayashi (1992)

Prognostic equation for (what we call) the impulse–bolus (IB) pseudomoemntum

$$\partial_t(\overline{\zeta'_z u'} + \overline{q'\eta'}/2) + \nabla \cdot \langle\langle \overline{u'u'} - \overline{K} + \overline{G}, \overline{v'u'}, \overline{\zeta' p'_x} \rangle\rangle = 0,$$
$$\partial_t(\overline{\zeta'_z v'} - \overline{q'\xi'}/2) + \nabla \cdot \langle\langle \overline{u'v'}, \overline{v'v'} - \overline{K} + \overline{G}, \overline{\zeta' p'_y} \rangle\rangle = \beta(\overline{v'\xi'} - \overline{u'\eta'})/2,$$

Ripa (1982, JPO)
Andrews (1983, JAS)
Haynes (1988)
Brunet and Haynes (1996)

bolus velocity — Impulse of planetary waves

$$\overline{\zeta'_z u'} + \overline{q'\eta'}/2$$

$$\overline{\zeta'(u'_z - w'_x)}$$ Impulse of gravity waves (nonhydrostatic)

$$\int_{-\infty}^{+\infty} \overline{u'p'}/c_p \, dy$$
$$= \int_{-\infty}^{+\infty} c_g(\overline{E}/c_p) \, dy$$
$$= \int_{-\infty}^{+\infty} c_g(\overline{\zeta'_z u'} + \overline{q'\eta'}/2) \, dy$$
$$= \int_{-\infty}^{+\infty} (\overline{u'u'} - \overline{K} + \overline{G}) \, dy$$

Virial Theorem for fluid waves

$$\Lambda \equiv [(\overline{\xi' p'})_x + (\overline{\eta' p'})_y + (\overline{\zeta' p'})_z]/2$$
$$= \overline{K} - \overline{G} + f(\overline{\xi' v'} - \overline{\eta' u'})/2 - [(\overline{\xi' u'} + \overline{\eta' v'})/2]_t$$
$$= -\overline{E} + (\overline{u'u'} - \overline{\xi' u'_t} + \overline{v'v'} - \overline{\eta' v'_t})/2 + f(\overline{\xi' v'} - \overline{\eta' u'})/2,$$

See second line (left) is the virial theorem represents a well-known equipartition statement between KE and PE associated with linear waves in a nonrotating frame

Lamda has been ignored in almost all previous studies by assuming three-dimensionally homogeneous waves

Previous knowledge for Lambda

Lambda: a previously-uninvestigated key quantity

$$\Lambda = \underbrace{(-\overline{\pi'_y p'_x} + \overline{\pi'_x p'_y})/(2f)}_{\simeq 0 \text{ for plane waves}} + \underbrace{\overset{\simeq \overline{E} \text{ for MIGWs}}{(\overline{u'_t v'} - \overline{u' v'_t})/(2f)}}_{\simeq 0 \text{ for MRWs}} - \overline{E}.$$

Lambda = 0 for MIGWs: the CE and IB fluxes point in the same direction
Lambda = -E for MRWs: the CE and IB fluxes point in different directions

$$\Lambda \simeq \overline{K} - \overline{G} + (f/\beta)\overline{q'u'}$$ an expression to be used for model diagnosis in a future study

New knowledge for Lambda

Virial Theorem provides a link between the pressure gradient terms in MEM and TLM

$$\overline{p} + \overline{\Lambda} = \overline{p} + \overline{\xi' p'_x} + \overline{\eta' p'_y} + \overline{\zeta' p'_z} - \overline{\Lambda}$$
$$= \overline{p} + \overline{\xi' p'_x} + \overline{\eta' p'_y} - \overline{\zeta'^2} N^2 + \overline{G} - \overline{K} - f(\overline{\xi' v'} - \overline{\eta' u'})/2 + [(\overline{\xi' u'} + \overline{\eta' v'})/2]_t$$
$$= \underbrace{(\overline{p} + \overline{\xi' p'_x} + \overline{\eta' p'_y})}_{\text{LM pressure}} \underbrace{- \overline{K} - f(\overline{\xi' v'} - \overline{\eta' u'})/2}_{\text{Bernoulli head}} + [(\overline{\xi' u'} + \overline{\eta' v'})/2]_t,$$

Bernouli head in the Lagrangian mean framewrok has been thought to be formidable

We suggest to understand it using Lambda in the virial therem

Merged Eulerian mean (MEM) momentum equations

this study (Aiki et al., 2015)

$$[\overline{u} + u^{qs} - \underbrace{(\overline{\zeta'_z u'} + \overline{q'\eta'}/2)}_{\text{IB pseudomomentum}} + \underbrace{(\overline{\xi' u'} + \overline{\eta' v'})_x/2}_{\simeq 0}]_t + \nabla \cdot (\overline{\mathbf{U}}\overline{u}) - f[\overline{v} + v^{qs} + \underbrace{(\overline{\xi' v'} - \overline{\eta' u'})_x/2}_{\text{DI wave activity}}] = -(\overline{p} + \overline{\Lambda})_x,$$

$$[\overline{v} + v^{qs} - \underbrace{(\overline{\zeta'_z v'} - \overline{q'\xi'}/2)}_{\text{IB pseudomomentum}} + \underbrace{(\overline{\xi' u'} + \overline{\eta' v'})_y/2}_{\simeq 0}]_t + \nabla \cdot (\overline{\mathbf{U}}\overline{v}) + f[\overline{u} + u^{qs} - \underbrace{(\overline{\xi' v'} - \overline{\eta' u'})_y/2}_{\text{DI wave activity}}] = -(\overline{p} + \overline{\Lambda})_y,$$

Transformed Lagrangian mean (TLM) momentum equations

Andrews & McIntyre, 1978

$$[\overline{u} + u^{Stokes} - \underbrace{[-\overline{u'\xi'_x} - \overline{v'\eta'_x} + f(\overline{\xi'\eta'_x} - \overline{\xi'_x\eta'})/2]}_{\text{GL pseudomomentum}}]_t$$
$$+ \nabla \cdot (\overline{\mathbf{U}}\overline{u}) - f(\overline{v} + v^{Stokes}) - \beta(\overline{\eta'^2 v'})$$
$$= -[\underbrace{(\overline{p} + \overline{\xi' p'_x} + \overline{\eta' p'_y})}_{\text{LM pressure}} \underbrace{- \overline{K} - f(\overline{\xi' v'} - \overline{\eta' u'})/2}_{\text{Bernoulli head}}]_x,$$

$$[\overline{v} + v^{Stokes} - \underbrace{[-\overline{u'\xi'_y} - \overline{v'\eta'_y} + f(\overline{\xi'\eta'_y} - \overline{\xi'_y\eta'})/2]}_{\text{GL pseudomomentum}}]_t$$
$$+ \nabla \cdot (\overline{\mathbf{U}}\overline{v}) + f(\overline{u} + u^{Stokes}) + \beta(\overline{\eta' u'}) + \beta(\overline{\xi' v'} - \overline{\eta' u'})/2$$
$$= -[\underbrace{(\overline{p} + \overline{\xi' p'_x} + \overline{\eta' p'_y})}_{\text{LM pressure}} \underbrace{- \overline{K} - f(\overline{\xi' v'} - \overline{\eta' u'})/2}_{\text{Bernoulli head}}]_y,$$