Challenges Associated with Adapting the Governing Flow Equations for Coordinate Systems Aligned with Steep Slopes

Holly J. Oldroyd, Eric R. Pardyjak, Hendrik Huwald,
and Marc B. Parlange

ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE
THE UNIVERSITY OF UTAH
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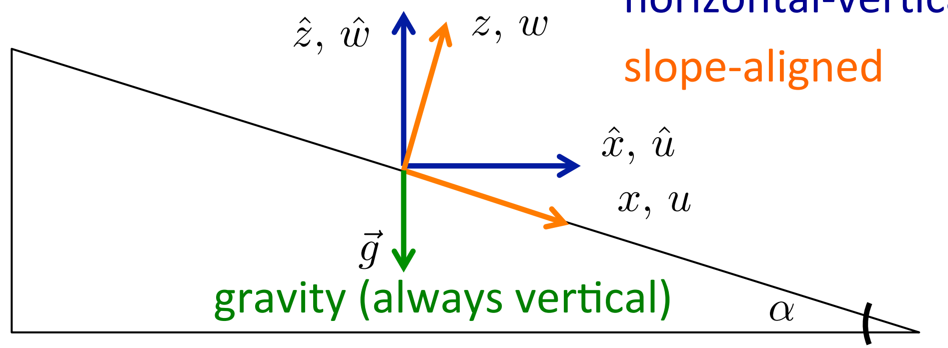
Slope-Aligned Coordinate System



- Convenient for momentum budget equations:
- Note: shear and buoyancy mechanisms are <u>not orthogonal</u>

Stream-wise momentum budget

$$\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y} + W\frac{\partial U}{\partial z} = -\frac{1}{\rho_o}\frac{\partial P}{\partial x} + \boxed{g\frac{\Delta\theta}{\theta_o}\sin(\alpha)} - \boxed{\frac{\partial \overline{u'w'}}{\partial z}}$$

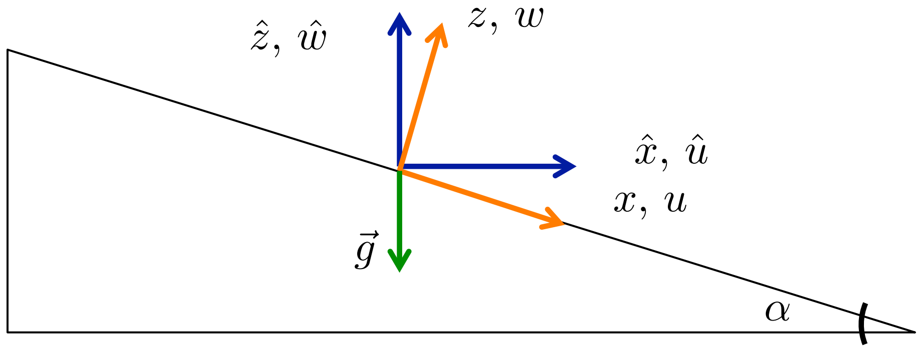
Slope-normal momentum budget

$$\frac{\partial W}{\partial t} + U\frac{\partial W}{\partial x} + V\frac{\partial W}{\partial y} + W\frac{\partial W}{\partial z} = -\frac{1}{\rho_o}\frac{\partial P}{\partial z} - \boxed{g\frac{\Delta\theta}{\theta_o}\boxed{\cos(\alpha)}} - \boxed{\frac{\partial \overline{w'w'}}{\partial z}}$$

--Following Manins and Sawford (1979)

Slope-Aligned Coordinate System

horizontal-vertical



Turbulence Kinetic Energy (TKE):

$$\overline{e} = 0.5\overline{u_i'^2}$$

TKE budget equation:

$$\frac{\partial \overline{u_i'^2}}{\partial t} + \overline{U_j}\frac{\partial \overline{u_i'^2}}{\partial x_j} = +2\delta_{i3}\frac{g\overline{(u_i'\theta_v')}}{\overline{\theta_v}} - 2\overline{u_i'u_j'}\frac{\partial \overline{U_i}}{\partial x_j} - \frac{\partial \overline{(u_j'u_i'^2)}}{\partial x_j} - \frac{2}{\overline{\rho}}\frac{\partial \overline{(u_i'P')}}{\partial x_i} - 2\varepsilon$$

--for flat terrain (Stull; 1988)

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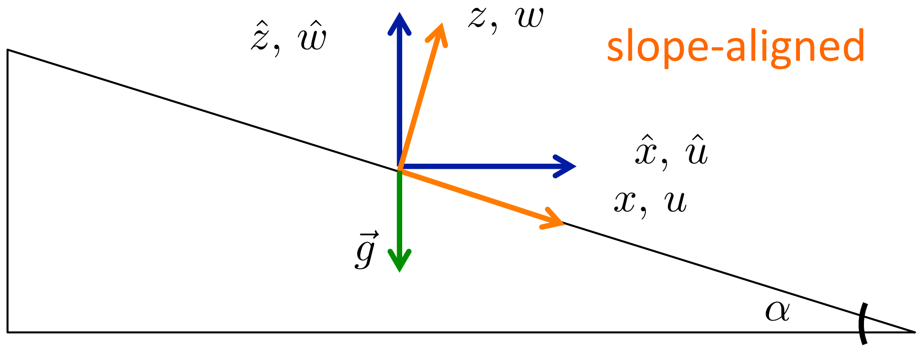

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Variety of forms:
--Manins and Sawford (1979)
--Nadeau et al. (2013)

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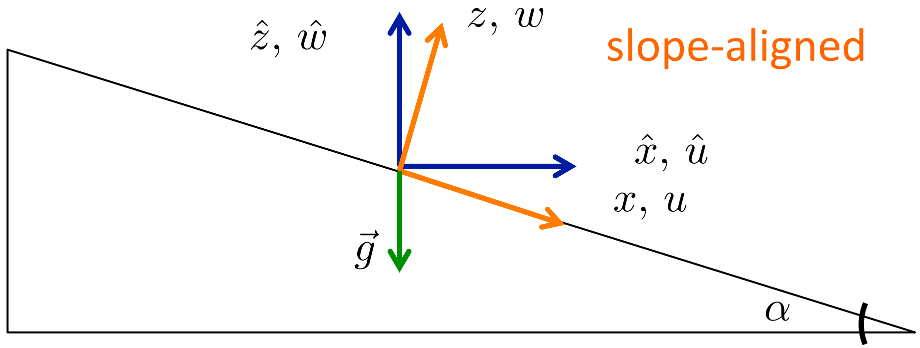

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General, orthogonal coordinate system: $g(\delta_{i1} sin\alpha_1 + \delta_{i2} sin\alpha_2 + \delta_{i3} cos\alpha_3)$

$$\frac{\partial \overline{e}}{\partial t} + \overline{U_j}\frac{\partial \overline{e}}{\partial x_j} = +sin\alpha_1\frac{g\overline{(u_1'\theta_v')}}{\overline{\theta_v}} + sin\alpha_2\frac{g\overline{(u_2'\theta_v')}}{\overline{\theta_v}} + cos\alpha_3\frac{g\overline{(u_3'\theta_v')}}{\overline{\theta_v}}$$

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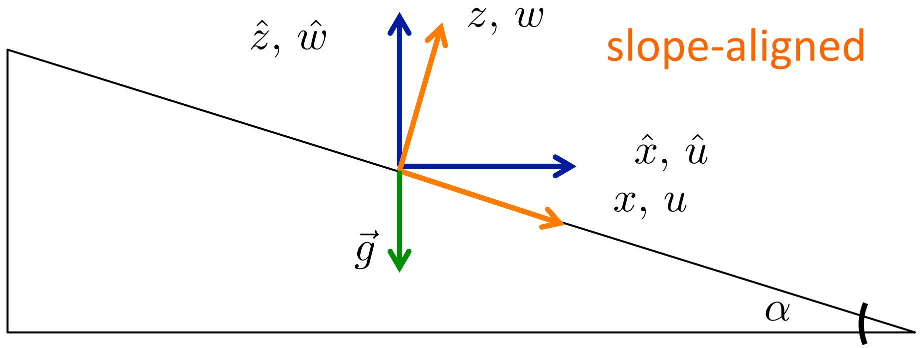

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For flat terrain:

$\alpha_1 = \alpha_2 = 0^\circ$ (from horizontal)

$\alpha_3 = 0^\circ$ (from vertical)

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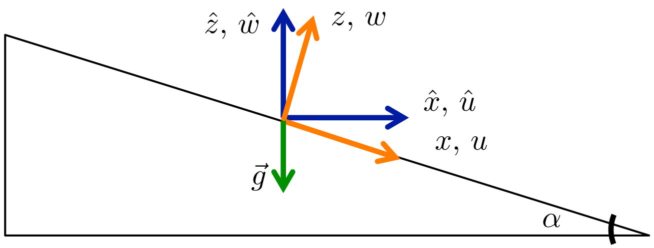

For steep, complex, variable topography:

Part I: challenges for momentum fluxes
- Tilt correction methodology

Part II: challenges for buoyancy fluxes
- Slope alignment

Stream-wise momentum budget

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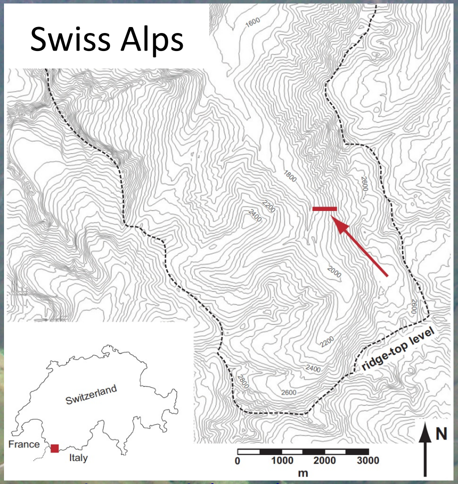
Field Experiment Setup

- Sonic anemometers slope-parallel
- 20 Hz sampling
- Slope angle = 35.5^{o}



--Nadeau et al. (2011)

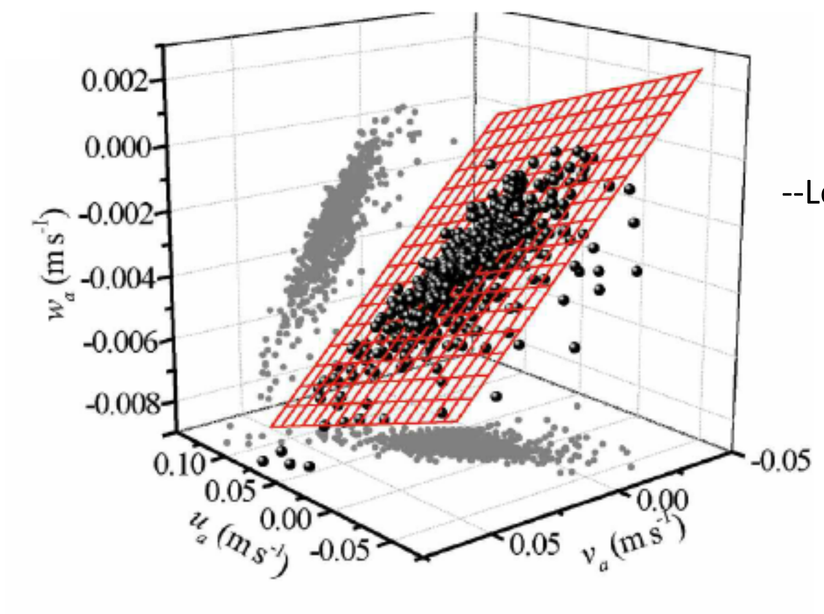
September
Z = 1.27m

10 m Flux Tower

Part I: momentum fluxes
Revisiting tilt corrections for complex topography

1. Purposes (i.e. a sonic anemometer):
   a) set a coordinate system for analysis
      - typically assumed, aligned with terrain
      - 'terrain-following' near the surface (--Sun,; 2007)
      - w-component (z-direction) is surface normal

   b) reduce cross-contamination between components for fluxes
2. 'Planar fit' tilt correction (PF)
   - Fits ensemble of streamlines, $\langle w \rangle = 0$ (but $\bar{w}_i = 0$ or $\bar{w}_i \neq 0$)
   - 'best fit' minimizes *S*



--Lorke et al., (2013)

---Wilczak et al., (2001)

Part I: momentum fluxes

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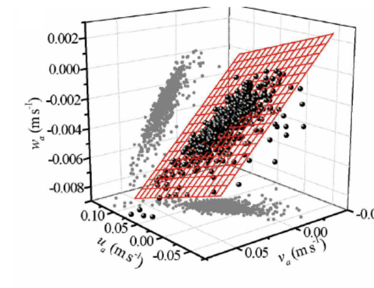

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$$S = \sum_n (\overline{w}_i - b_0 - b_1 \overline{u}_i - b_2 \overline{v}_i)^2$$

Shift (m/s)

Set the transformation matrix such that:

$$\sin \alpha = -b_1 / \sqrt{1 + b_1^2}$$

pitch angle

$$\sin \beta = b_2 / \sqrt{1 + b_2^2}$$

Roll angle

---Wilczak et al., (2001)

Revisiting tilt corrections for complex topography

- For complex terrain: Sector-wise planar fit (SPF)
  - Account for local terrain-induced perturbations

    ---i.e.; Yuan (2011)

  - How to choose the averaging time for the PF, τ_{PF} ?
  - How to choose the sector size, Λ ?
  - Do these choices matter? (--Vickers and Mahrt (2006))

Hypotheses:

Average time: τ_{PF}

- Not necessarily same as for fluxes, τ_f
- Long enough for *mean streamlines* to converge
- Shorter times
  - increase the maximum number of PF segments, N
  - increase statistical significance of the PF

Sector size: Λ

- Small enough to reduce the terrain-induced perturbations
- Large enough to encompass majority of fluctuations in wind direction, σ_{WD}

Main hypothesis:

- We can derive *'cost functions'*
  - create objective methods for decision-making

Cost functions for SPF

Goal: Choose sector sizes, Λ and PF average times, τ_{PF}

- in an objective way
- know the implications/tradeoffs of these choices

$$S_{rms} = \frac{\sqrt{S}}{\sqrt{N}} = \frac{\sqrt{\sum_n (\overline{w}_i - b_0 - b_1\overline{u}_i - b_2\overline{v}_i)^2}}{\sqrt{N}}$$

Evaluates how well the streamlines define the fitting plane

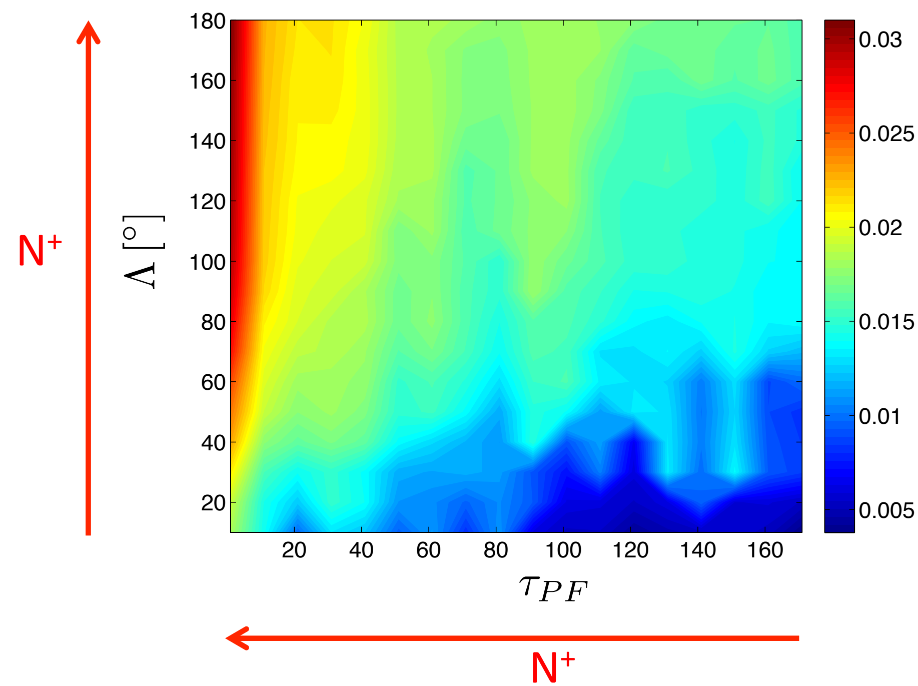
minimize →

$$S_{rms} = f(\tau_{PF}, \Lambda)$$

Example:

Wind sector centered at 85° from North (nighttime; downslope)

A helpful, but not sufficient Criteria!



Cost functions for SPF

Goal: Choose sector sizes and PF average times

- in an objective way
- know the implications/tradeoffs of these choices

Evaluate the sensitivity of the b-coefficients:

$$\frac{\partial(b_{0,1,2})}{\partial\Lambda} = f(\tau_{PF}, \Lambda) \qquad \frac{\partial(b_{0,1,2})}{\partial\tau_{PF}} = f(\tau_{PF}, \Lambda)$$

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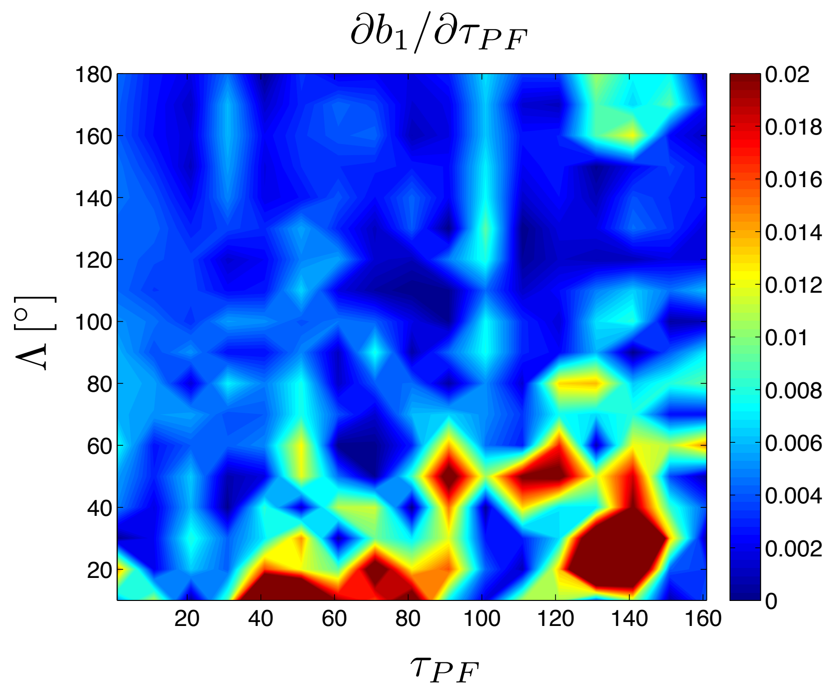


Methodology:

- Evaluate all cost functions
- Look for global minima

Caution for main wind directions:

- Insensitive to increasing Λ
- Adding tails of the distribution
- Use smallest reasonable Λ

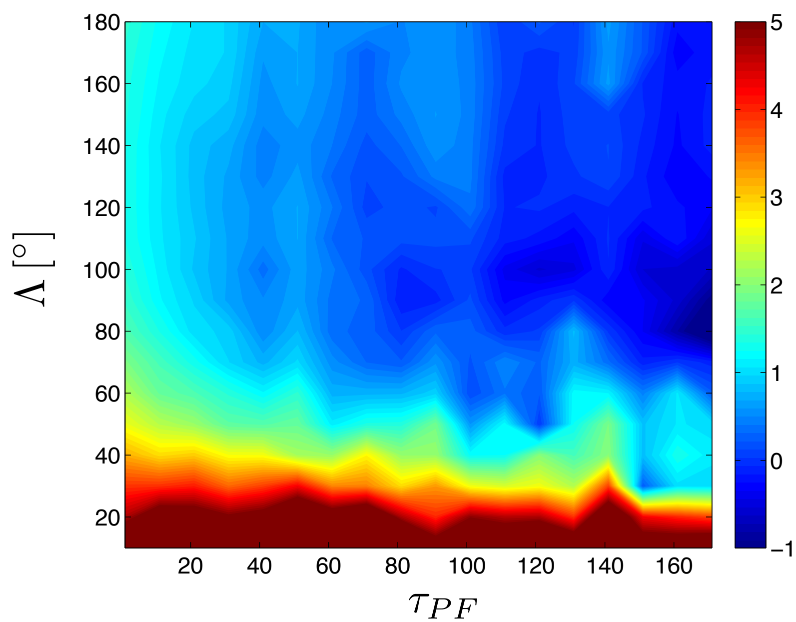
Variability for tilt angles

Example: Wind sector centered at 85° from North (nighttime; downslope)

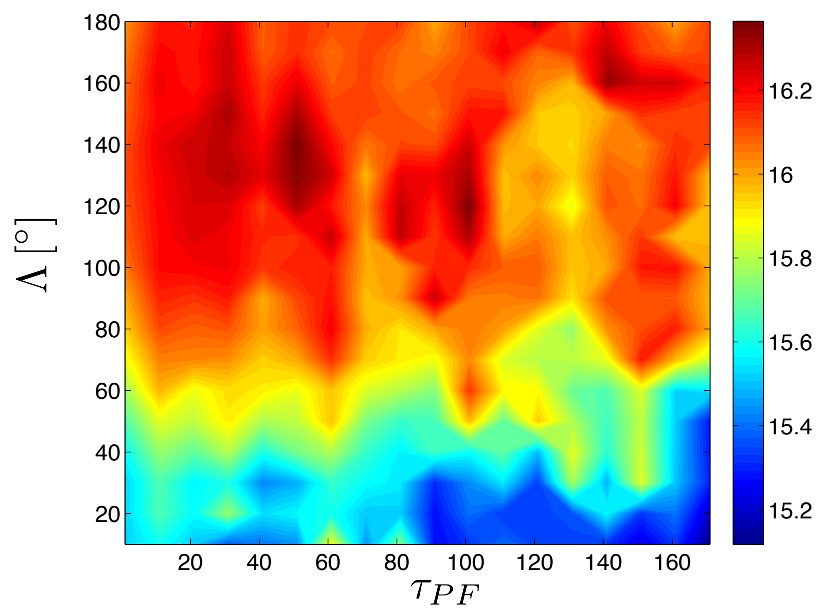
$$\sin\alpha = -b_1/\sqrt{1+b_1^2}$$

$$\sin\beta = b_2/\sqrt{1+b_2^2}$$

pitch angle



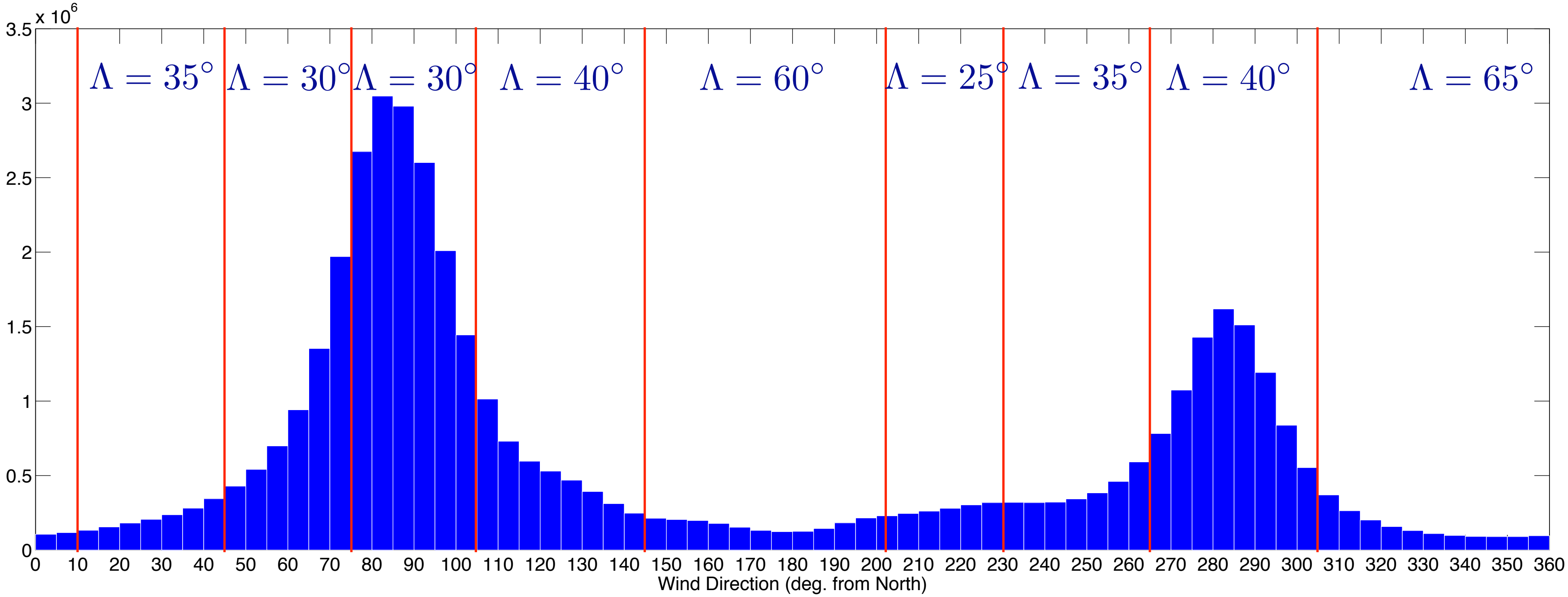
Roll angle



'Select' SPF
(from the optimization methodology)

Site Specific!

Planar-fit averaging time: $\tau_{PF} = 45\ (min)$

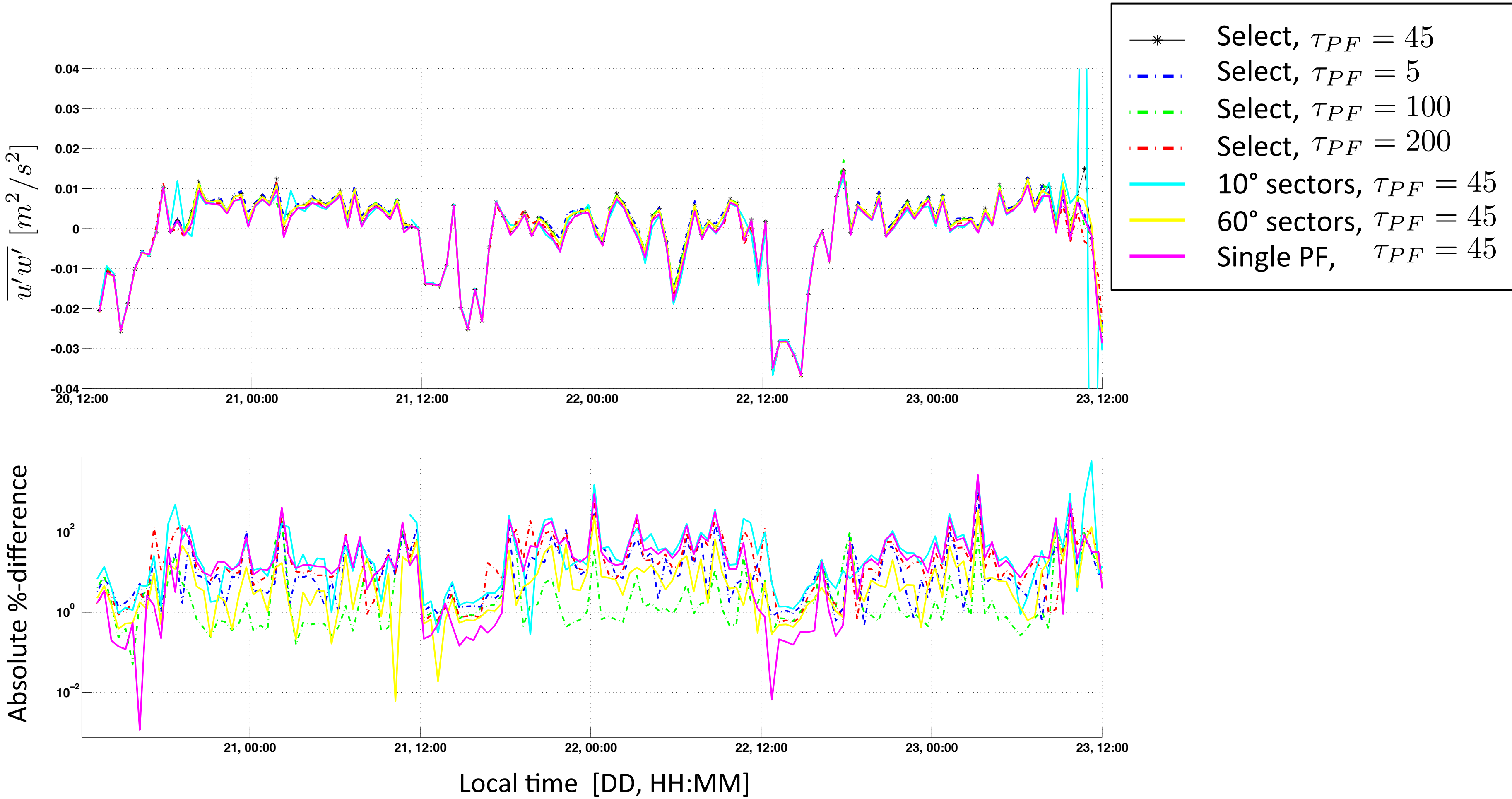


Nighttime winds
Downslope flow

daytime winds
Up slope/valley flow

Implications of SPF decisions

Momentum Fluxes

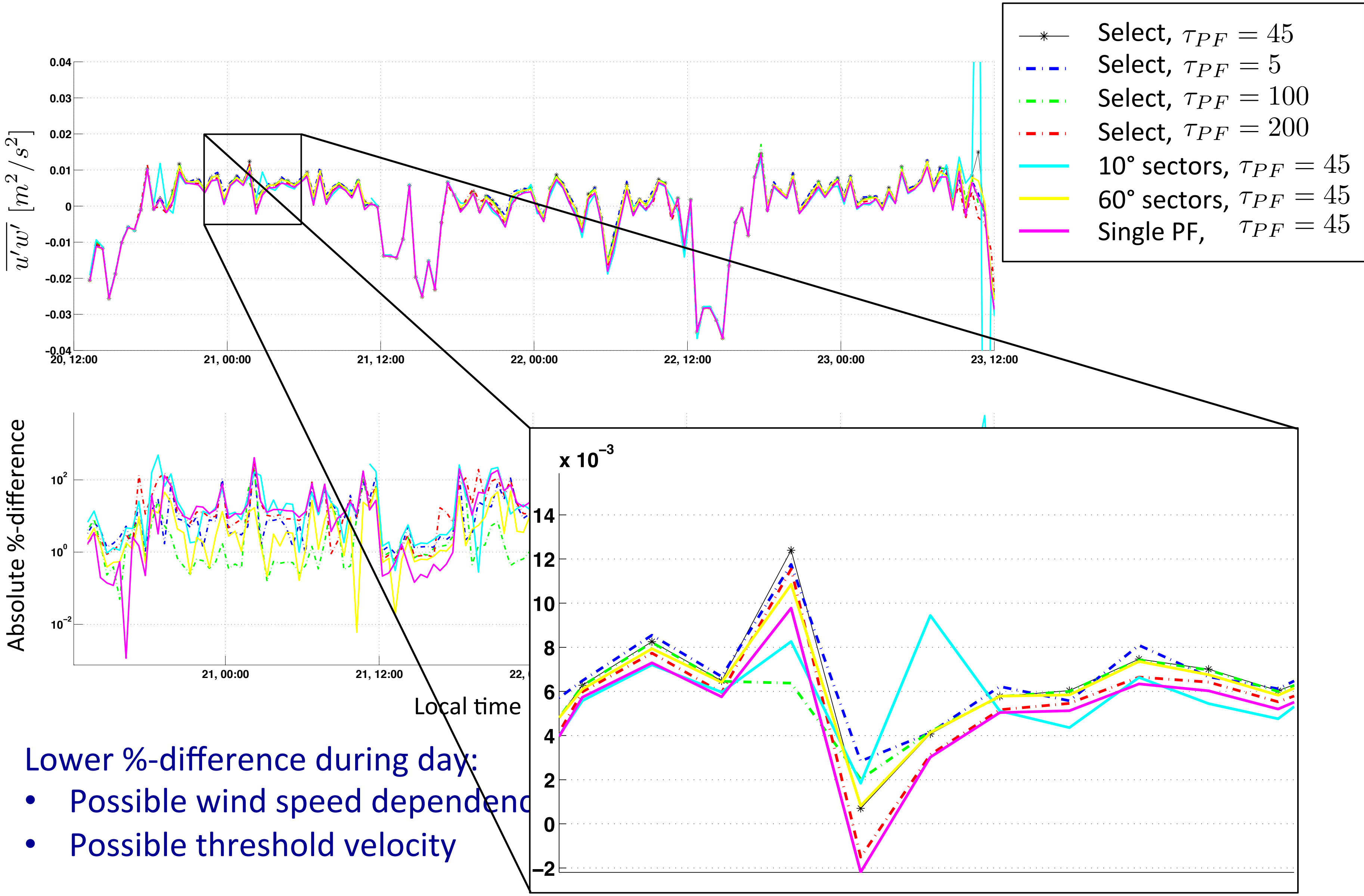


Lower %-difference during day:

- Possible wind speed dependence
- Possible threshold velocity

Implications of SPF decisions

Momentum Fluxes



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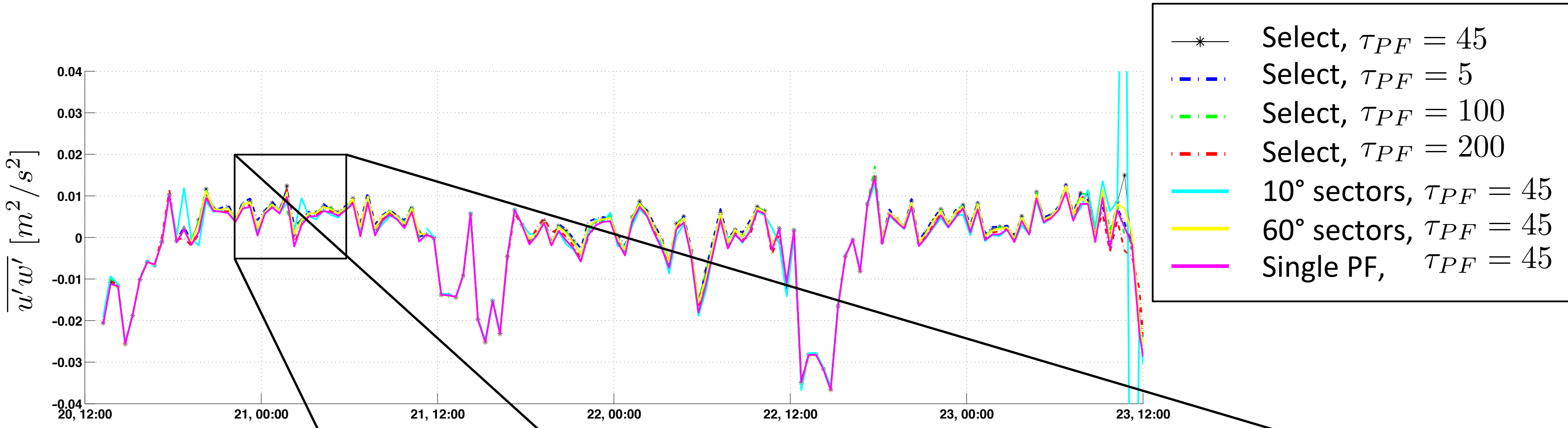

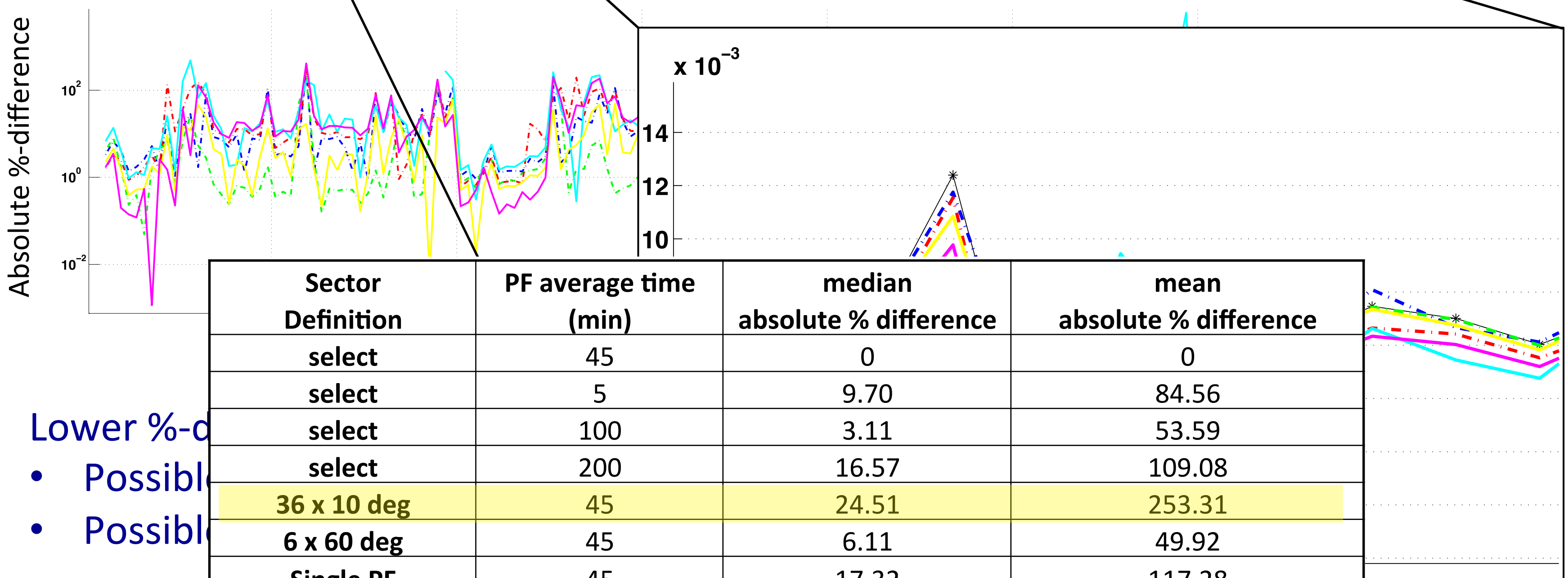

Sector Definition	PF average time (min)	median absolute % difference	mean absolute % difference
select	45	0	0
select	5	9.70	84.56
select	100	3.11	53.59
select	200	16.57	109.08
36 x 10 deg	45	24.51	253.31
6 x 60 deg	45	6.11	49.92
Single PF	45	17.32	117.28

Lower %-d

- Possibl
- Possibl

Part II: Buoyancy fluxes

TKE budget equation:

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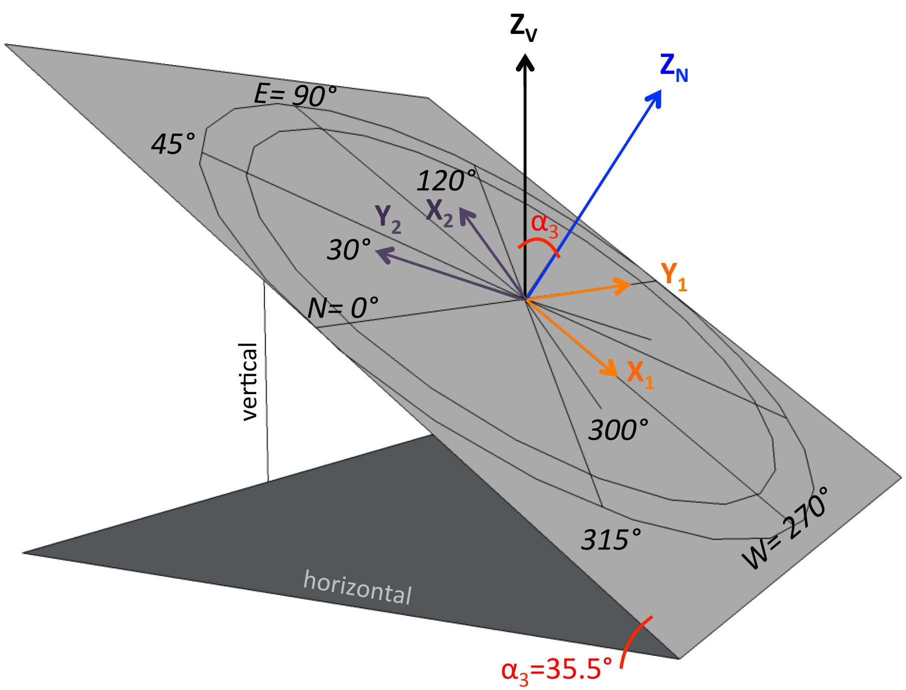

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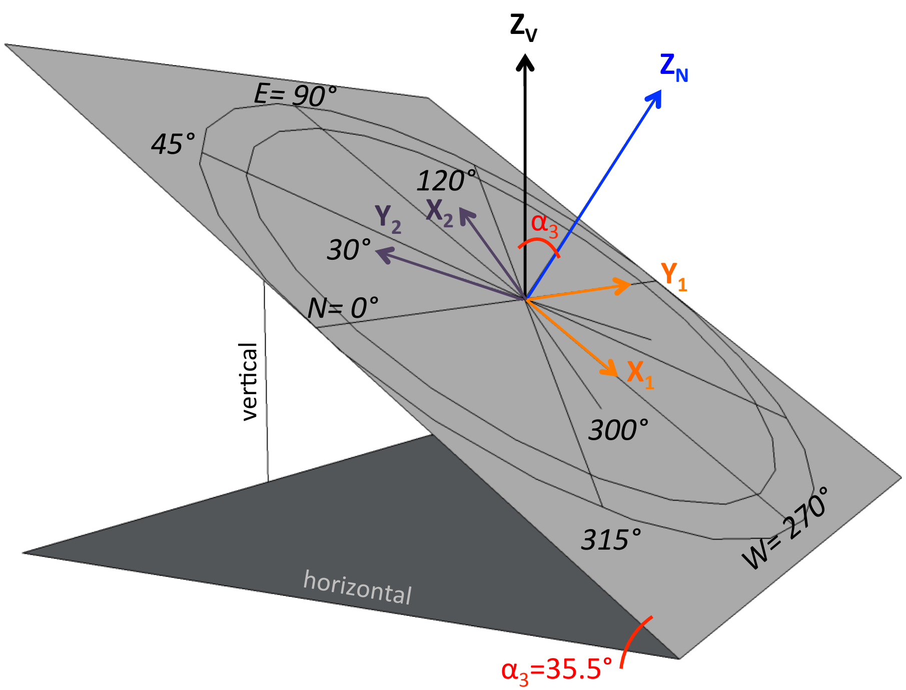
Night: Down slope flow

$\alpha_2 = 0°;\ \alpha_1 = \alpha_3 = 35.5°$

--Manins and Sawford (1979)



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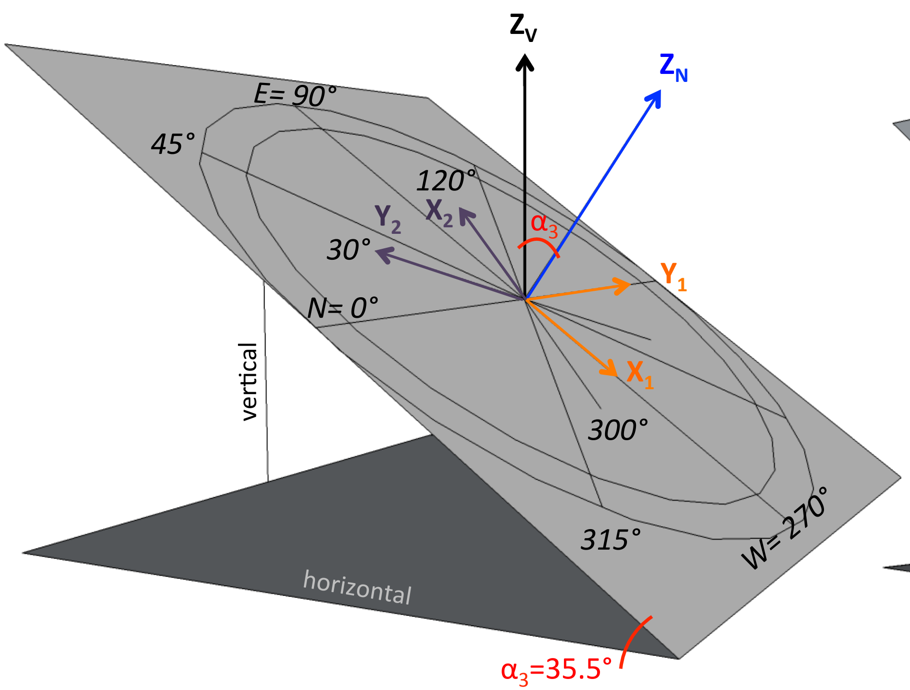
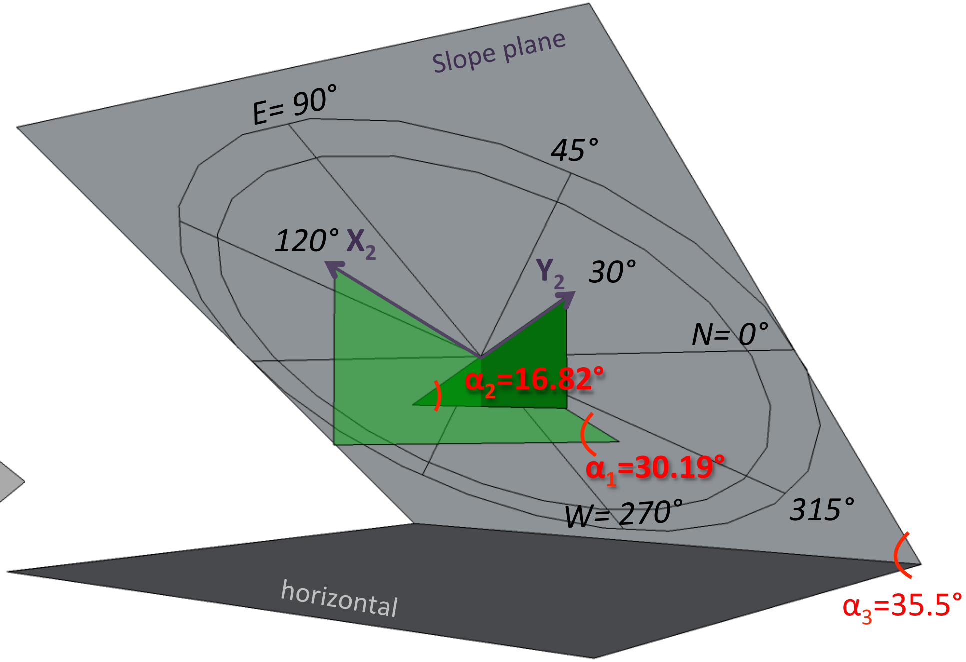
$$-\overline{u_i'u_j'}\frac{\partial \overline{U_i}}{\partial x_j} - \frac{\partial \overline{(u_j'e)}}{\partial x_j} - \frac{1}{\overline{\rho}}\frac{\partial \overline{(u_i'P')}}{\partial x_i} - \varepsilon$$

Day: up slope/up valley flow

$\alpha_1 = 30.2°;\ \alpha_2 = 16.8°;\ \alpha_3 = 35.5°$





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Part II: Buoyancy fluxes

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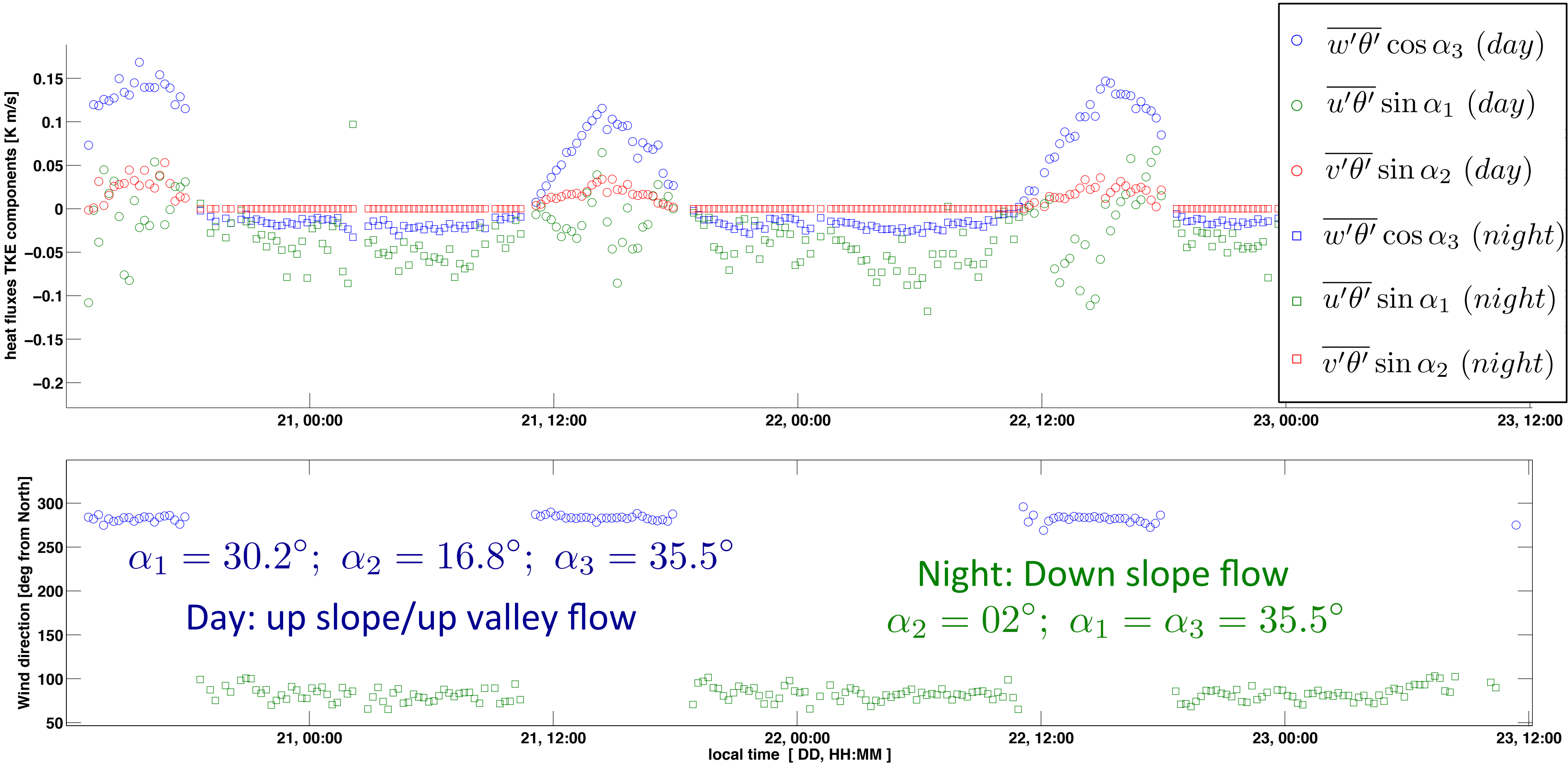

Part II: Buoyancy fluxes

Implications for:

$$Ri_f = \frac{\frac{g}{\theta_v}\left(sin\alpha_1\,\overline{u_1'\theta_v'} + sin\alpha_2\,\overline{u_2'\theta_v'} + cos\alpha_3\,\overline{u_3'\theta_v'}\right)}{\overline{u_i^{'}u_j^{'}}\,\frac{\partial \overline{U_i}}{\partial x_j}}$$

$$L = -\frac{u_*^3}{\kappa}\,\frac{\overline{\theta_v}}{g(\sin\alpha_1\,\overline{u_1'\theta_v'} + \sin\alpha_2\,\overline{u_2'\theta_v'} + \cos\alpha_3\,\overline{u_3'\theta_v'})}$$

Conclusions

- Clearly for SPF, the choice of sector sizes and average time:
  - can greatly change tilt correction angles
  - can effect momentum flux estimates

- Methodology to objectively evaluate the degrees of freedom for SPF

- Revisited governing flow equations for steep slopes:
  - keep all the buoyancy components for TKE in theory
  - AND in and practice
  - implications for *Ri* and *L*

$$\frac{g}{\theta_v}(\overline{u'\theta_v'}\sin\alpha_1 + \overline{v'\theta_v'}\sin\alpha_2 + \overline{w'\theta_v'}\cos\alpha_3)$$

- Results will be site-dependent

Thanks

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