Challenges Associated with Adapting the Governing Flow Equations for Coordinate Systems Aligned with Steep Slopes

Holly J. Oldroyd, Eric R. Pardyjak, Hendrik Huwald, and Marc B. Parlange
Slope-Aligned Coordinate System

- Convenient for momentum budget equations:
- Note: shear and buoyancy mechanisms are not orthogonal

Stream-wise momentum budget

\[
\frac{\partial U}{\partial t} + U \frac{\partial U}{\partial x} + V \frac{\partial U}{\partial y} + W \frac{\partial U}{\partial z} = - \frac{1}{\rho_o} \frac{\partial P}{\partial x} + g \frac{\Delta \theta}{\theta_o} \sin(\alpha) - \frac{\partial \overline{u'w'}}{\partial z}
\]

Slope-normal momentum budget

\[
\frac{\partial W}{\partial t} + U \frac{\partial W}{\partial x} + V \frac{\partial W}{\partial y} + W \frac{\partial W}{\partial z} = - \frac{1}{\rho_o} \frac{\partial P}{\partial z} - g \frac{\Delta \theta}{\theta_o} \cos(\alpha) - \frac{\partial \overline{w'w'}}{\partial z}
\]

--Following Manins and Sawford (1979)
TKE (Turbulence Kinetic Energy): 

\[ \bar{e} = 0.5\bar{u}'^2 \]

TKE budget equation:

\begin{align*}
\frac{\partial \bar{u}'^2}{\partial t} + U_j \frac{\partial \bar{u}'^2}{\partial x_j} &= + 2\delta_{ij} \frac{g(u'_i \theta'_v)}{\theta_v} - 2u'_i u'_j \frac{\partial U_i}{\partial x_j} - \frac{\partial (u'_j u'_i)^2}{\partial x_j} - \frac{2}{\bar{\rho}} \frac{\partial (u'_i P')}{\partial x_i} - 2\varepsilon \\
\text{--for flat terrain (Stull; 1988)}
\end{align*}
Slope-Aligned Coordinate System

TKE budget equation:

\[
\frac{\partial u'_i^2}{\partial t} + U_j \frac{\partial u'_i^2}{\partial x_j} = 2\delta_{ij} \frac{g(u'_i \theta'_v)}{\theta_v} - 2u'_iu'_j \frac{\partial U_i}{\partial x_j} - \frac{\partial (u'_j u'_i^2)}{\partial x_j} - \frac{2}{\rho} \frac{\partial (u'_i P')}{\partial x_i} - 2\bar{\varepsilon}
\]

Variety of forms:

-- Manins and Sawford (1979)
-- Nadeau et al. (2013)
Slope-Aligned Coordinate System

TKE budget equation:
\[
\frac{\partial \overline{u_i'^2}}{\partial t} + \overline{U_j \frac{\partial u_i'^2}{\partial x_j}} = +2\delta_{i3} \frac{g(u_i'\theta'_v)}{\theta_v} - 2u_i'u_j \frac{\partial \overline{U_i}}{\partial x_j} - \frac{\partial (u_j'u_i'^2)}{\partial x_j} - \frac{2}{\rho} \frac{\partial (u_i'P')}{\partial x_i} - 2\varepsilon
\]

General, orthogonal coordinate system:
\[
\frac{\partial \overline{e}}{\partial t} + \overline{U_j \frac{\partial \overline{e}}{\partial x_j}} = +sin\alpha_1 \frac{g(u_1'^\theta'_v)}{\theta_v} + sin\alpha_2 \frac{g(u_2'^\theta'_v)}{\theta_v} + cos\alpha_3 \frac{g(u_3'^\theta'_v)}{\theta_v}
\]
\[
-u_i'u_j \frac{\partial \overline{U_i}}{\partial x_j} - \frac{\partial (u_j'e)}{\partial x_j} - \frac{1}{\rho} \frac{\partial (u_i'P')}{\partial x_i} - \varepsilon
\]

Turbulence Kinetic Energy (TKE):
\[
\overline{e} = 0.5u_i'^2
\]
Slope-Aligned Coordinate System

TKE budget equation:

\[ \frac{\partial \overline{u_i'^2}}{\partial t} + \overline{U}_j \frac{\partial \overline{u_i'^2}}{\partial x_j} = + 2\delta_{i3} \frac{g(\overline{u_i' \theta'_v})}{\partial \overline{\theta_v}} - 2\overline{u_i' u_j'} \frac{\partial \overline{U_i}}{\partial x_j} - \frac{\partial (\overline{u_j' u_i'^2})}{\partial x_j} - \frac{2}{\overline{\rho}} \frac{\partial \overline{(u_i' P')}}{\partial x_i} - 2\varepsilon \]

General, orthogonal coordinate system:

\[ \frac{\partial \overline{e}}{\partial t} + \overline{U}_j \frac{\partial \overline{e}}{\partial x_j} = + \sin \alpha_1 \frac{g(\overline{u_1' \theta'_v})}{\partial \overline{\theta_v}} + \sin \alpha_2 \frac{g(\overline{u_2' \theta'_v})}{\partial \overline{\theta_v}} + \cos \alpha_3 \frac{g(\overline{u_3' \theta'_v})}{\partial \overline{\theta_v}} \]

For flat terrain:

\[ \alpha_1 = \alpha_2 = 0^\circ \quad \text{(from horizontal)} \]
\[ \alpha_3 = 0^\circ \quad \text{(from vertical)} \]
Slope-Aligned Coordinate System

For steep, complex, variable topography:

Part I: challenges for momentum fluxes
- Tilt correction methodology

Part II: challenges for buoyancy fluxes
- Slope alignment

Stream-wise momentum budget

\[ \frac{\partial U}{\partial t} + U \frac{\partial U}{\partial x} + V \frac{\partial U}{\partial y} + W \frac{\partial U}{\partial z} = - \frac{1}{\rho_o} \frac{\partial P}{\partial x} + g \frac{\Delta \theta}{\theta_o} \sin(\alpha) - \frac{\partial u'w'}{\partial z} \]

TKE budget equation:

\[ \frac{\partial \bar{e}}{\partial t} + \bar{U}_j \frac{\partial \bar{e}}{\partial x_j} = + \sin \alpha_1 \frac{g(u'_1 \theta'_v)}{\theta_v} + \sin \alpha_2 \frac{g(u'_2 \theta'_v)}{\theta_v} + \cos \alpha_3 \frac{g(u'_3 \theta'_v)}{\theta_v} \]

\[ -u'_i u'_j \frac{\partial \bar{U}_i}{\partial x_j} - \frac{\partial (u'_j \bar{e})}{\partial x_j} - \frac{1}{\bar{\rho}} \frac{\partial (u'_i P')}{\partial x_i} - \varepsilon \]
Field Experiment Setup

- Sonic anemometers slope-parallel
- 20 Hz sampling
- Slope angle = 35.5°

Swiss Alps

September

Z = 1.27m

10 m Flux Tower

---Nadeau et al. (2011)
Part I: momentum fluxes
Revisiting tilt corrections for complex topography

1. Purposes (i.e. a sonic anemometer):
   a) set a coordinate system for analysis
      • typically assumed, aligned with terrain
      • ‘terrain-following’ near the surface (--Sun,; 2007)
      • w-component (z-direction) is surface normal
   b) reduce cross-contamination between components for fluxes

2. ‘Planar fit’ tilt correction (PF)
   • Fits ensemble of streamlines, $\langle w \rangle = 0$ (but $\bar{w}_i = 0$ or $\bar{w}_i \neq 0$)
   • ‘best fit’ minimizes $S$

---Wilczak et al., (2001)
Part I: momentum fluxes
Revisiting tilt corrections for complex topography

1. Purposes (i.e. a sonic anemometer):
   a) set a coordinate system for analysis
      • typically assumed, aligned with terrain
      • ‘terrain-following’ near the surface (Sun, 2007)
      • w-component (z-direction) is surface normal
   b) reduce cross-contamination between components for fluxes

2. ‘Planar fit’ tilt correction (PF)
   • Fits ensemble of streamlines, $\langle w \rangle = 0$ (but $\bar{w}_i = 0$ or $\bar{w}_i \neq 0$)
   • ‘best fit’ minimizes $S$

$$S = \sum_n (\bar{w}_i - b_0 - b_1 \bar{u}_i - b_2 \bar{v}_i)^2$$

Set the transformation matrix such that:

$$\sin \alpha = -\frac{b_1}{\sqrt{1 + b_1^2}}$$  \hspace{1cm} \sin \beta = \frac{b_2}{\sqrt{1 + b_2^2}}$$

pitch angle  \hspace{1cm} Roll angle

---Wilczak et al., (2001)
For complex terrain: **Sector-wise planar fit (SPF)**

- Account for local terrain-induced perturbations

--- i.e.; Yuan (2011)

- How to choose the **averaging time** for the PF, $\tau_{PF}$?
- How to choose the **sector size**, $\Lambda$?
- Do these choices matter? (--Vickers and Mahrt (2006))

**Hypotheses:**

**Average time:** $\tau_{PF}$
- Not necessarily same as for fluxes, $\tau_f$
- Long enough for *mean streamlines* to converge
- Shorter times
  - increase the maximum number of PF segments, $N$
  - increase statistical significance of the PF

**Sector size:** $\Lambda$
- Small enough to reduce the terrain-induced perturbations
- Large enough to encompass majority of fluctuations in wind direction, $\sigma_{WD}$

**Main hypothesis:**
- We can derive ‘*cost functions*’
  - create **objective** methods for decision-making
Cost functions for SPF

**Goal:** Choose sector sizes, $\Lambda$ and PF average times, $\tau_{PF}$
- in an objective way
- know the implications/tradeoffs of these choices

$$S_{rms} = \frac{\sqrt{S}}{\sqrt{N}} = \sqrt{\frac{\sum_{n}(\bar{w}_i - b_0 - b_1 \bar{u}_i - b_2 \bar{v}_i)^2}{\sqrt{N}}}$$

Evaluates how well the streamlines define the fitting plane

minimize $S_{rms} = f(\tau_{PF}, \Lambda)$

Example:
Wind sector centered at 85° from North (nighttime; downslope)

A helpful, but not sufficient Criteria!
Goal: Choose sector sizes and PF average times
• in an objective way
• know the implications/tradeoffs of these choices

Evaluate the sensitivity of the b-coefficients:

\[ \frac{\partial (b_{0,1,2})}{\partial \Lambda} = f(\tau_{PF}, \Lambda) \]
\[ \frac{\partial (b_{0,1,2})}{\partial \tau_{PF}} = f(\tau_{PF}, \Lambda) \]

Example:
Wind sector centered at 85° from North
(nighttime; downslope)

Methodology:
• Evaluate all cost functions
• Look for global minima

Caution for main wind directions:
• Insensitive to increasing \( \Lambda \)
• Adding tails of the distribution
• Use smallest reasonable \( \Lambda \)
Variability for tilt angles

**Example:** Wind sector centered at 85° from North (nighttime; downslope)

\[
\sin \alpha = -\frac{b_1}{\sqrt{1 + b_1^2}} \quad \sin \beta = \frac{b_2}{\sqrt{1 + b_2^2}}
\]

- **pitch angle**
- **Roll angle**
'Select' SPF  
(from the optimization methodology)  

Site Specific!  

Planar-fit averaging time: $\tau_{PF} = 45 \text{ (min)}$ 

Nighttime winds  
Downslope flow  

daytime winds  
Up slope/valley flow
Implications of SPF decisions
Momentum Fluxes

Lower %-difference during day:
• Possible wind speed dependence
• Possible threshold velocity
Implications of SPF decisions
Momentum Fluxes

Select, $\tau_{PF} = 45$
Select, $\tau_{PF} = 5$
Select, $\tau_{PF} = 100$
Select, $\tau_{PF} = 200$
10° sectors, $\tau_{PF} = 45$
60° sectors, $\tau_{PF} = 45$
Single PF,

Lower %-difference during day:
• Possible wind speed dependence
• Possible threshold velocity

$x \cdot 10^{-3}$

Local time

Absolute % difference

$u'w'$ [$m^2/s^2$]
Implications of SPF decisions
Momentum Fluxes

Select, $\tau_{PF} = 45$
Select, $\tau_{PF} = 5$
Select, $\tau_{PF} = 100$
Select, $\tau_{PF} = 200$

10° sectors, $\tau_{PF} = 45$
60° sectors, $\tau_{PF} = 45$
Single PF, $\tau_{PF} = 45$

Lower %-difference during day:
• Possible wind speed dependence
• Possible threshold velocity

<table>
<thead>
<tr>
<th>Sector Definition</th>
<th>PF average time (min)</th>
<th>median absolute % difference</th>
<th>mean absolute % difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>select</td>
<td>45</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>select</td>
<td>5</td>
<td>9.70</td>
<td>84.56</td>
</tr>
<tr>
<td>select</td>
<td>100</td>
<td>3.11</td>
<td>53.59</td>
</tr>
<tr>
<td>select</td>
<td>200</td>
<td>16.57</td>
<td>109.08</td>
</tr>
<tr>
<td>36 x 10 deg</td>
<td>45</td>
<td>24.51</td>
<td>253.31</td>
</tr>
<tr>
<td>6 x 60 deg</td>
<td>45</td>
<td>6.11</td>
<td>49.92</td>
</tr>
<tr>
<td>Single PF</td>
<td>45</td>
<td>17.32</td>
<td>117.28</td>
</tr>
</tbody>
</table>
Part II: Buoyancy fluxes

TKE budget equation:

$$\frac{\partial e}{\partial t} + U_j \frac{\partial e}{\partial x_j} = \sin \alpha_1 \frac{g(u_1' \theta_v')}{\theta_v} + \sin \alpha_2 \frac{g(u_2' \theta_v')}{\theta_v} + \cos \alpha_3 \frac{g(u_3' \theta_v')}{\theta_v}$$

\[ -u'_i u'_j \frac{\partial U_i}{\partial x_j} - \frac{\partial (u'_j e)}{\partial x_j} - \frac{1}{\rho} \frac{\partial (u'_i P')}{\partial x_i} - \varepsilon \]
Part II: Buoyancy fluxes

TKE budget equation:

\[
\frac{\partial \bar{e}}{\partial t} + U_j \frac{\partial \bar{e}}{\partial x_j} = \sin \alpha_1 \frac{g(u_1' \theta'_v)}{\theta_v} + \sin \alpha_2 \frac{g(u_2' \theta'_v)}{\theta_v} + \cos \alpha_3 \frac{g(u_3' \theta'_v)}{\theta_v}
\]

Night: Down slope flow

\(\alpha_2 = 0^\circ; \ \alpha_1 = \alpha_3 = 35.5^\circ\)

\[
- u'_i u'_j \frac{\partial \bar{U}_i}{\partial x_j} - \frac{\partial (u'_j e)}{\partial x_j} - \frac{1}{\rho} \frac{\partial (\bar{u}_i P')}{\partial x_i} - \varepsilon
\]

--Manins and Sawford (1979)
Part II: Buoyancy fluxes

TKE budget equation:

$$\frac{\partial e}{\partial t} + U_j \frac{\partial e}{\partial x_j} = +\sin \alpha_1 \frac{g(u_1' \theta'_v)}{\theta_v} + \sin \alpha_2 \frac{g(u_2' \theta'_v)}{\theta_v} + \cos \alpha_3 \frac{g(u_3' \theta'_v)}{\theta_v}$$

Day: up slope/up valley flow

$$\alpha_1 = 30.2^\circ; \quad \alpha_2 = 16.8^\circ; \quad \alpha_3 = 35.5^\circ$$
Part II: Buoyancy fluxes

Wind direction [deg from North]

Day: up slope/up valley flow

Night: Down slope flow

\[ \alpha_1 = 30.2^\circ; \alpha_2 = 16.8^\circ; \alpha_3 = 35.5^\circ \]

\[ \alpha_2 = 02^\circ; \alpha_1 = \alpha_3 = 35.5^\circ \]
Part II: Buoyancy fluxes

Implications for:

\[
Ri_f = \frac{g}{\theta_v} \left( \sin \alpha_1 \bar{u}_1' \theta_v' + \sin \alpha_2 \bar{u}_2' \theta_v' + \cos \alpha_3 \bar{u}_3' \theta_v' \right)
\]

\[
L = -\frac{u^3}{\kappa} \frac{\theta_v}{g(\sin \alpha_1 \bar{u}_1' \theta_v' + \sin \alpha_2 \bar{u}_2' \theta_v' + \cos \alpha_3 \bar{u}_3' \theta_v')}
\]
Conclusions

- Clearly for SPF, the choice of sector sizes and average time:
  - can greatly change tilt correction angles
  - can effect momentum flux estimates

- Methodology to **objectively evaluate** the degrees of freedom for SPF

- Revisited governing flow equations for steep slopes:
  - keep all the buoyancy components for TKE in theory
  - AND in and practice
  - implications for $Ri$ and $L$
    \[
    \frac{g}{\theta_v} (u'\theta'_v \sin \alpha_1 + v'\theta'_v \sin \alpha_2 + w'\theta'_v \cos \alpha_3)
    \]

- Results will be site-dependent
References


