Investigation of Vortex Profiles via 2-D Vortex Mergers

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Introduction

In atmospheric research, explicit assumptions are often made about the shapes of velocity or vorticity profiles of axisymmetric tornado-like flows. Examples of such "vortices" are, for example, the potential vortex, the Rankine-combined vortex, the Burgers–Rott vortex, the Sullivan vortex, the Kuo vortex, the Serrin vortex, the Lamb–Oseen vortex etc. In most of such models, the velocity decay away from the vortex axis behaves like r^{-1} where r is the distance from the vortex axis.

We undertake a two-dimensional numerical study in which several initial configurations of point vortices (approximating Rankine-combined vortex blobs) are developed using the Hamiltonian equations until a nearly steady state is achieved. The resulting point-vortex distributions and velocity fields are then analyzed under the assumption of axisymmetry.

Recent radar studies [1, 5, 6] (and some older vortex-chamber analyses [4]) indicate that tornado-related vorticity and velocity fields exhibit a power-law dependence on the distance r from the vortex axis; in particular, $v \propto r^{-b}$ for the azimuthal velocity and $\zeta \propto r^{-(1+b)}$ for the vorticity with $0 < b \leq 1$. One such study is illustrated in the figure below, taken from [5]:

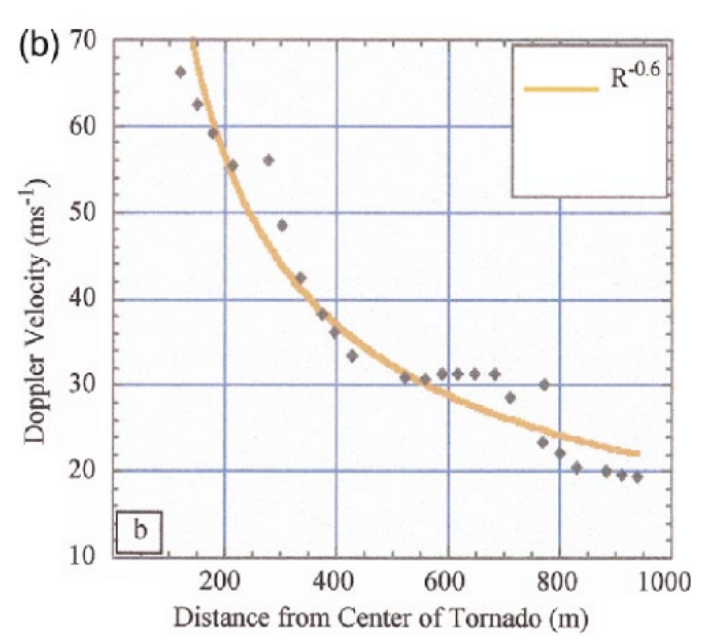

Figure 1: Doppler velocity decay for the Dimmit, TX, 1995, tornado [5].

The Main Question

What is the resulting "natural" rate of decay of velocity away from the center of the vortex? What is the resulting velocity profile?

The Short Answer

We consistently discover velocity decay of the form r^{-b} with $0 < b < 1$ between a solid-core inner flow and a irrotational outer flow.

Methodology

- Specify various initial vorticity distributions
- Develop the vortices using Hamiltonian equations
- Post-process raw data assuming axisymmetry
- Analyze velocity decay rates
- Study implications for vortex models

Governing Equations

Consider N point vortices with strengths $\Gamma_i > 0$ and positions $\mathbf{x}_i(t) = \{x_i(t), y_i(t)\}$ in $\mathbb{R}^2$ that depend on time, t.

The Hamiltonian equations

This system is governed by the equations for $i = 1, \dots, N$

$$\dot{x}_i = \frac{1}{2\pi}\sum_{j\neq i}\frac{\Gamma_j(y_j - y_i)}{r_{ij}^2}, \quad \dot{y}_i = -\frac{1}{2\pi}\sum_{j\neq i}\frac{\Gamma_j(x_j - x_i)}{r_{ij}^2}$$

where $r_{ij} = \|\mathbf{x}_i - \mathbf{x}_j\|$ and $\mathbf{x}_i(0)$ are given.

These equations form a Hamiltonian system that has rigorous connections with the Euler equation [2, 3], and the corresponding Hamiltonian is $H = -\frac{1}{4\pi}\sum_{\substack{1\leq i,j\leq N \\ i\neq j}} \Gamma_i\Gamma_j \log\|\mathbf{x}_i - \mathbf{x}_j\|$. This Hamiltonian is independent of time, i.e., $\frac{dH}{dt} = 0$.

Sample Initial & Final Configurations

We consider various initial circular arrangements of uniformly, randomly distributed point vortices with equal strengths, modeling two, three, or four 2D Rankine-combined vortices. These are then developed using the Hamiltonian equations and the resulting configurations are analyzed under the assumption of axisymmetry as described below. Some examples of initial and final configurations are given in the figure below.

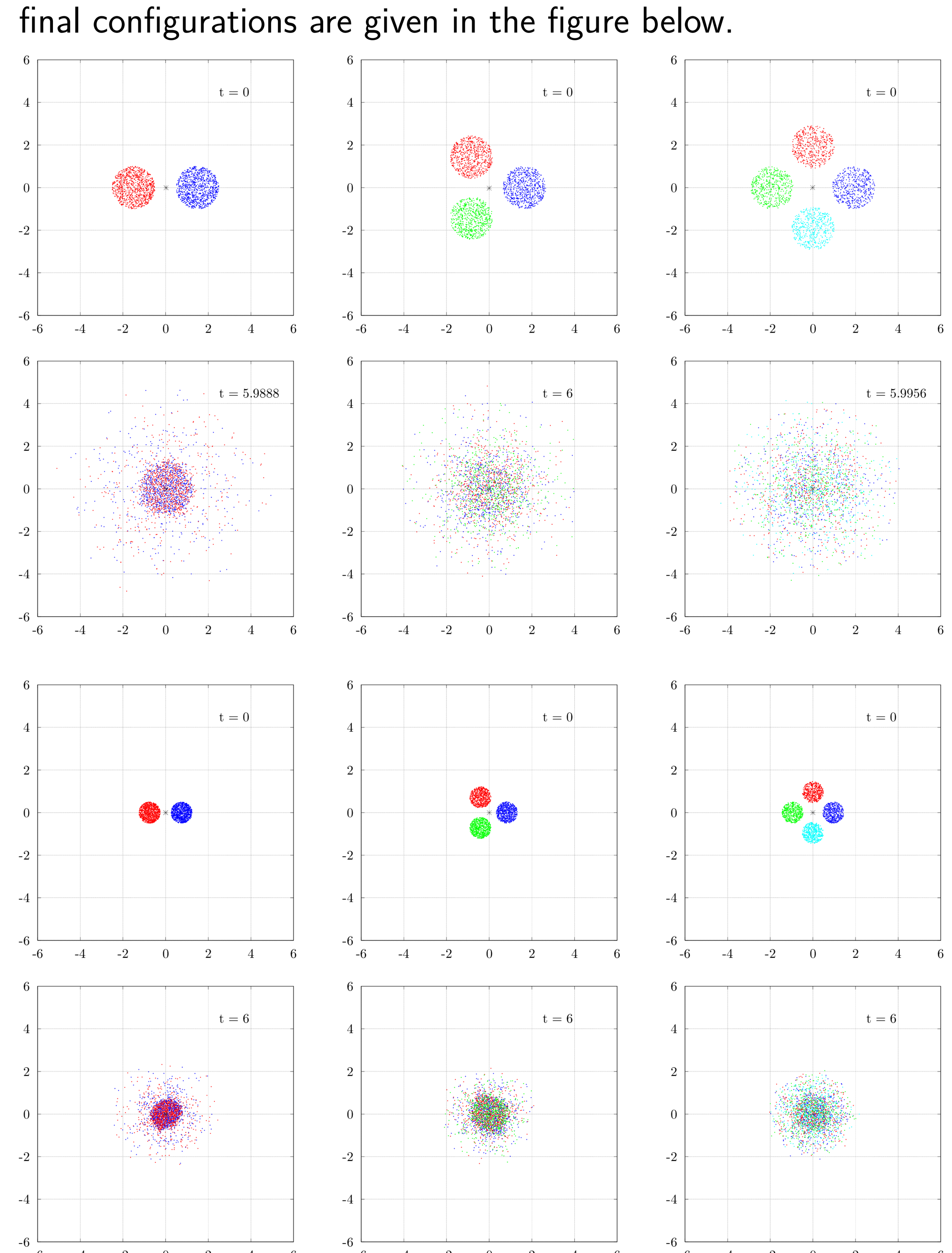

Figure 2: Initial and resulting configurations modeling the merger of 2, 3, and 4 Rankine-combined vortex blobs of equal strength.

Axisymmetric Analysis

In most of our simulations, the point vortices align into a nearly axisymmetric shape as seen in the previous figure, so we analyze the results assuming axisymmetry. The azimuthal velocity, v, and the vertical component of vorticity, ξ, are then functions of r only, and they satisfy $v' + \frac{v}{r} = \xi$, $v(0) = 0$. By integrating, we can obtain

Velocity & Vorticity Computation

$$v(r) = \frac{1}{2\pi r}\iint_{B_r} \xi\, dA, \quad \xi(r) = \frac{1}{2\pi r}\frac{d}{dr}\iint_{B_r} \xi\, dA,$$

where $B_r \subset \mathbb{R}^2$ is the disk of radius r centered at the center of symmetry (center of vorticity), and

$$\iint_{B_r} \xi\, dA = \int_{\partial B_r} \mathbf{u}\cdot d\mathbf{s}$$

is the circulation at radius r. This circulation is easy to compute for point vortex distributions as a finite sum.

Resulting Velocity Profiles: Set 1

Most results with closely packed initial arrangements of vortices suggest a merger and convergence to a steady-state axisymmetric solution with the azimuthal velocity approximated by

$$v(r) = \begin{cases} c_1 r & \text{for } 0 < r < r_0, \\ c_2 r^{-b} & \text{for } r_0 < r < r_1 \text{ and } 0 < b < 1, \\ c_3 r^{-1} & \text{for } r_1 < r. \end{cases}$$

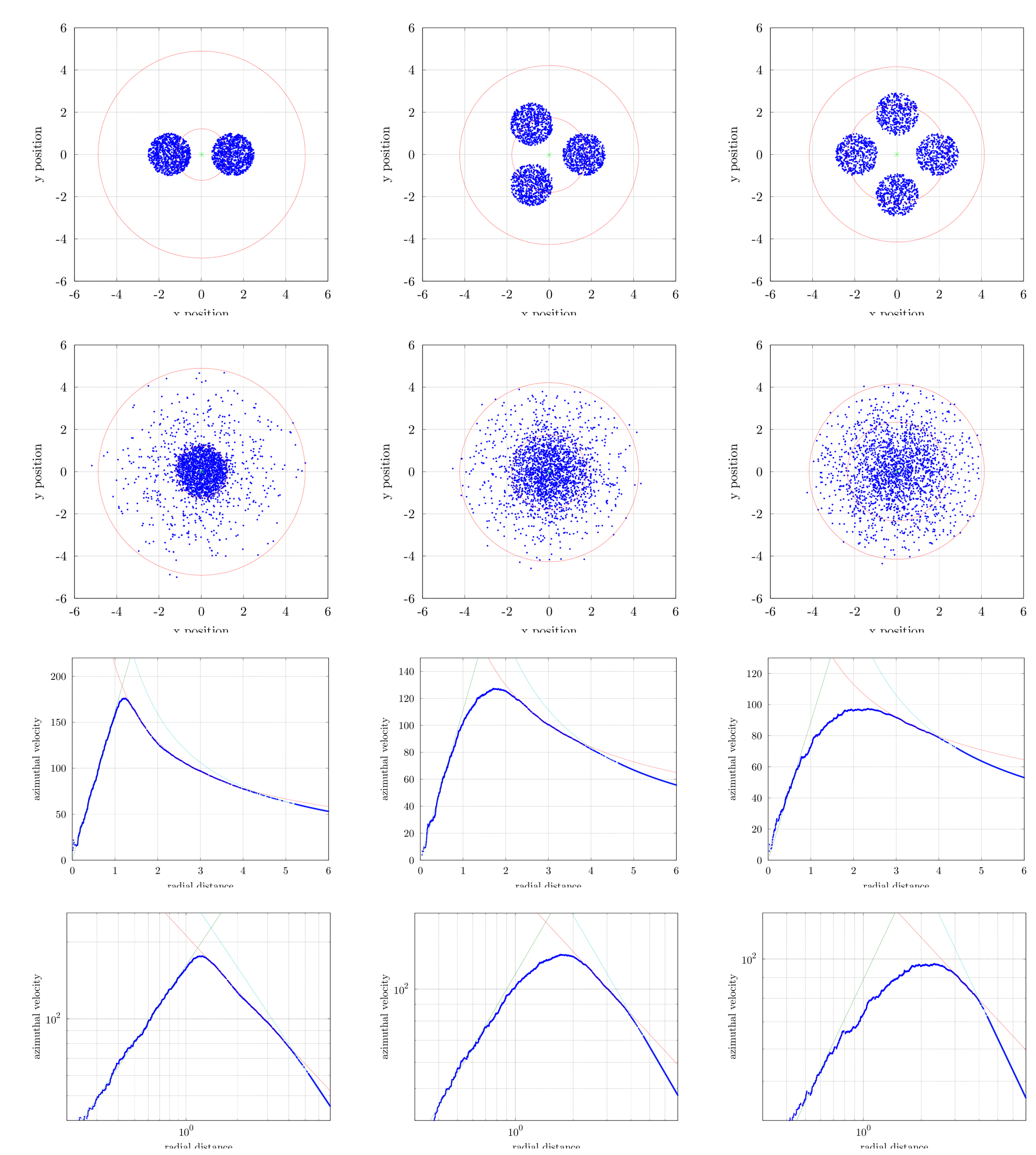

Figure 3: Initial and resulting configurations modeling the merger of 2, 3, and 4 Rankine-combined vortex blobs of equal strength; azimuthal velocity plotted on a normal and a log-log scale.

Resulting Velocity Profiles: Set 2

Results with more intense vortices (same total vorticity as in the previous set, but more concentrated near the center of vorticity):

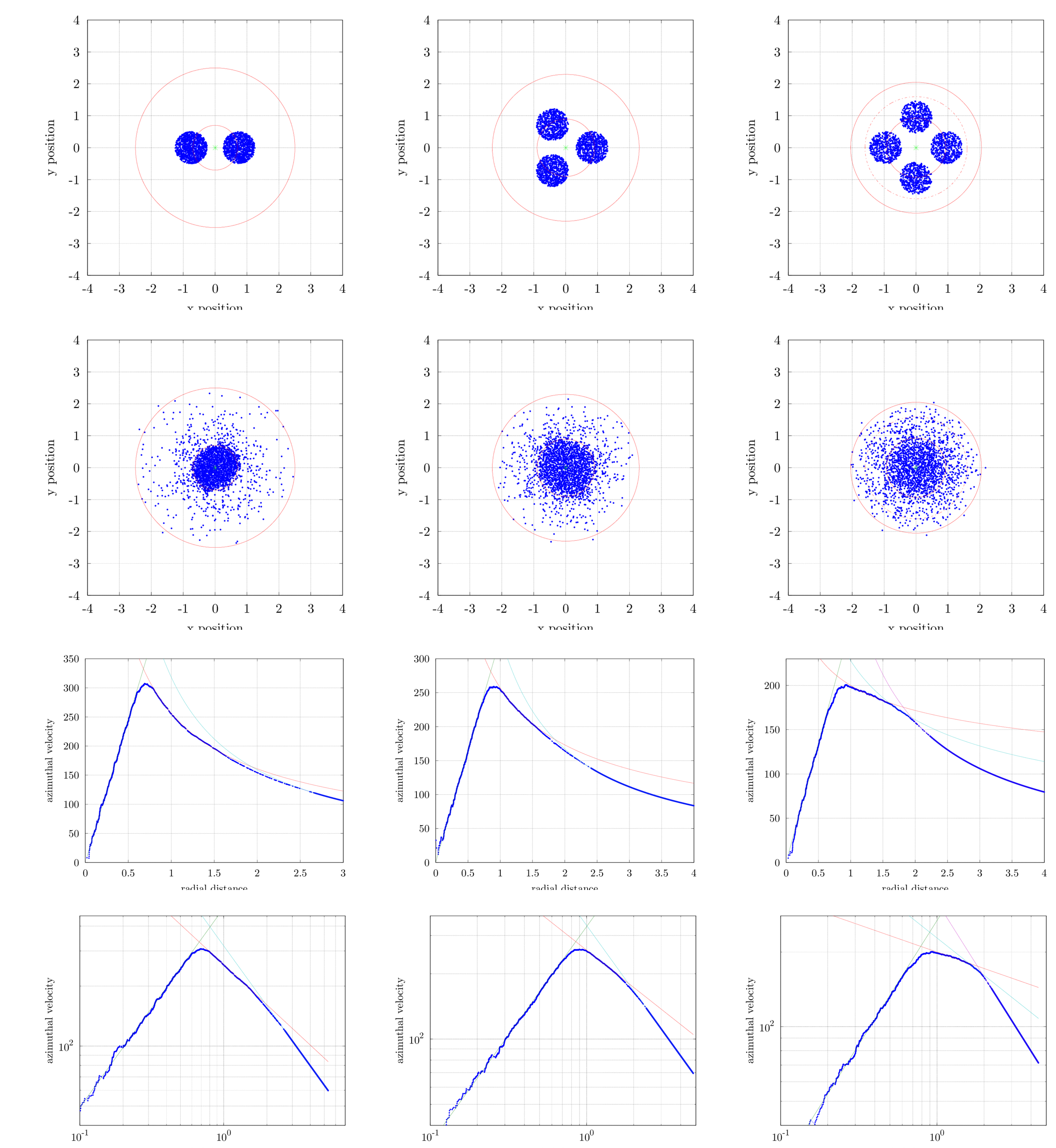

Figure 4: Initial and resulting configurations modeling the merger of 2, 3, and 4 Rankine-combined vortex blobs of equal strength; azimuthal velocity plotted on a normal and a log-log scale.

Summary

Our results indicate that a "natural" azimuthal velocity profile to model a vortex blob is one in which an inner-most solid-core region (where the velocity is proportional to r) is surrounded by an annulus in which velocity decays like r^{-b} with $0 < b < 1$, outside of which the flow is irrotational (and velocity decays like r^{-1}). These three regions are connected with "transition regions" so that the overall velocity (and vorticity) profiles are smooth.

A similar study with continuous distributions of vorticity within the merging blobs should be performed to support the conclusions of our point-vortex study.

References

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