EQUILIBRIUM TROPICAL CYCLONE SIZE IN AN IDEALIZED STATE OF AXISYMMETRIC RADIATIVE-CONVECTIVE EQUILIBRIUM

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1. Introduction

The past three decades have seen tremendous advances in our theoretical understanding of the thermodynamic structure of a mature tropical cyclone. Most recently, Emanuel and Rotunno (2011) derived a full analytical solution for the radial structure of the axisymmetric balanced tropical cyclone wind field at the top of the boundary layer. However, this solution remains defined relative to a single free parameter: the outer radius r_0. Indeed, real storms in nature are observed to span a wide range of sizes (Chavas and Emanuel 2010), yet in practice size remains largely unpredictable and relatively little research has been performed to elucidate the factors underlying its variability.

A dynamical systems approach may provide a path forward. Tang and Emanuel (2010) demonstrated analytically that tropical cyclone intensity may be viewed as a non-linear dynamical system that evolves towards a stable equilibrium whose value depends on the local environmental and initial conditions. This behavior has been verified in a modeling context on both short (e.g. Rotunno and Emanuel (1987)) and long time-scales (Hakim 2010). However, the tropical cyclone at statistical structural equilibrium remains unexplored.

Thus, in this work we systematically explore the sensitivity of the structure of an axisymmetric tropical cyclone at statistical equilibrium to the set of relevant model, initial, and environmental dimensional variables. Expanding on the work of Hakim (2010), we perform our analysis in the simplest possible model and physical environment: a highly-idealized state of radiative-convective equilibrium (RCE). The results of the sensitivity analysis are then synthesized via dimensional analysis to quantify how, at equilibrium, each structural variable of interest scales with the set of relevant input parameters.

2. Methodology

a. Idealized model/environmental RCE set-up

We construct a highly-idealized model and environmental configuration to explore equilibrium tropical cyclone structure in RCE using version 15 of the Bryan Cloud Model (CM1; Bryan and Fritsch (2002)), which incorporates state of the art numerics and physics, in particular for moist processes, while satisfying near-exact conservation of both mass and energy in a reversible saturated environment. The radial and vertical grid spacings are set to $dr = 4\ km$ and $dz = .625\ km$, respectively. Radiation is represented simply by imposing a constant cooling rate, Q_{cool}, to the potential temperature everywhere in the domain where the absolute temperature exceeds a threshold value, T_{tpp}; below this value, newtonian relaxation back to T_{tpp} is applied with a relaxation timescale of 40 days. Thus, all water-radiation feedbacks are neglected. The sea surface temperature, T_{sst}, is set constant. Surface fluxes of enthalpy and momentum are calculated using standard bulk aerodynamic formulae where the exchange coefficients of momentum, C_d, and enthalpy, C_k, are set constant. Finally, galilean invariance dictates that the effect of a uniform background wind speed, u_s, is manifest solely through the surface enthalpy fluxes, and thus we add u_s to the near-surface wind velocity, **u**, in their calculation. This set-up is conceptually similar to that of Hakim (2010) with the exception of the non-interactive radiative scheme and the inclusion of background surface fluxes.

Each axisymmetric simulation is initialized with the 70-100 day time- and horizontal-mean vertical profiles of temperature and water vapor from the corresponding 3D simulation on a 196x196x40 km domain with doubly-periodic horizontal boundary conditions and identical T_{sst}, T_{tpp}, Q_{cool}, and u_s. This ensures that each axisymmetric simulation begins very close to its "natural" model-equilibrated background state, which consists of a troposphere capped by a nearly isothermal stratosphere. The tropopause, and therefore the approximate convective outflow temperature, is set by T_{tpp}. Furthermore, RCE provides the additional constraint that surface enthalpy fluxes are exactly balanced by column-integrated radiative cooling. Combining this energetic constraint with the equation for the generalized potential intensity (Emanuel 1995), V_p, and the simple formulation for radiative cooling gives

$$V_p^2 = \frac{T_{sst} - T_{tpp}}{T_{tpp}} \frac{C_p Q_{cool} \overline{\Delta p}}{g \rho C_d |\mathbf{u}|} \tag{1}$$

where C_p is the specific heat of air, g is the acceleration due to gravity, ρ is the near-surface air density, and $\overline{\Delta p}$ is a measure of the pressure thickness of the troposphere. Thus, potential intensity in RCE with constant tropospheric cooling is a simple function of four externally-

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defined parameters: T_{sst}, T_{tpp}, u_s, and Q_{cool}, with the tropospheric thickness $\overline{\Delta p}$ primarily a function of T_{tpp}.

For an initial disturbance, the air above the boundary layer is saturated at constant virtual temperature in a region bounded by $z = [1, 9.375]\ km$ and $r = (0, r_{0_q})$ within a quiescent environment. We also test an initial mid-level vortex, characterized by a radius of vanishing wind r_{0_u} and a peak wind of $V_{m_0} = 12.5\ ms^{-1}$ at $r_{m_0} = r_{0_u}/5$, that leads to similar results, and thus we elect to use the moisture perturbation in subsequent experiments for dimensional simplicity. All simulations are run for 100 days.

b. Control run parameters

The control run parameters are: [Model] $C_d = C_k = .0015$, radial mixing length $l_h = 1.5\ km$, vertical mixing length $l_h = .1\ km$; [Environmental] $T_{sst} = 300\ K$, $T_{tpp} = 200\ K$, $f = 5 * 10^{-5}\ s^{-1}$, $Q_{cool} = 1\ \frac{K}{day}$, $u_s = 3\ ms^{-1}$; [Initial perturbation] $r_{0_q} = 200\ km$, $r_{0_u} = 400\ km$. The potential intensity of the initial RCE sounding with zero boundary layer wind speed reduction and including dissipative heating is $V_p = 93\ ms^{-1}$, which is nearly identical to the prediction of (1). Storm size is found to be highly sensitive to the location of the outer domain wall below some limiting value, and so this outer wall is placed conservatively at $12288\ km$.

c. Characterizing equilibrium storm structure

Following Emanuel and Rotunno (2011), we characterize the complete structure of the tropical cyclone wind field at the top of the boundary layer ($z \approx 1\ km$) with three variables: the maximum gradient wind speed, V_m, the radius of maximum gradient wind, r_m, and the outer radius, r_0. The equilibrium value of each variable is defined as the 70-100 day mean calculated from the 2-day running mean radial gradient wind profiles, allowing one to verify that each variable has independently reached statistical equilibrium. To reduce model noise, r_0 is calculated from r_{12} by numerically solving the outer wind structure model of Emanuel (2004). This model assumes radiative-subsidence balance within this outer region where, at equilibrium, the subsidence rate, w_{cool}, must match the rate of Ekman suction-induced entrainment of free tropospheric air into the boundary layer. The radial profile of azimuthal velocity is therefore determined as that which provides the required Ekman suction, and is given by

$$\frac{\partial (rV)}{\partial r} = \frac{2r^2 C_d V^2}{w_{cool}(r_0^2 - r^2)} - fr \tag{2}$$

where r is the radius and V is the azimuthal wind speed. The value of w_{cool} is calculated from the assumed balance between subsidence and radiative cooling

$$w_{cool}\frac{\partial \theta}{\partial z} = Q_{cool} \tag{3}$$

where $\frac{\partial \theta}{\partial z}$ is set to its mean value in the layer $z = 1.5 - 5\ km$ (i.e. directly above the boundary layer) in the RCE initial sounding. For the control run, this gives $w_{cool} = .27\ cms^{-1}$, which agrees well with the value of $.23$ obtained by calculating the mean (negative) vertical velocity in the region $r = [400, 800]\ km$ and $z = [1.5, 5]\ km$ from the equilibrium state of the control simulation. Finally, we solve for r_0 in (2) using a shooting method.

d. Experimental approach

We perform experiments in which we independently and systematically vary all dimensional parameters deemed relevant to the dynamics of the system: l_h, f, r_{0_q}, r_{0_u}, T_{sst}, T_{tpp}, Q_{cool}, and u_s; the latter four are ultimately subsumed within V_p as discussed below. For each of l_h, f, r_{0_q}, and r_{0_u}, we run six simulations relative to the control case: three with the parameter successively halved and three successively doubled from the control value. For V_p, we perform a suite of simulations varying its four input external parameters (Eq. (1)) that spans a reasonable range of values of V_p (see below). The final scaling results indicate to which dimensional variables the equilibrium storm structure is systematically sensitive. Dimensional analysis is then applied to quantify the scaling relationship between each structural variable of interest and all relevant dimensional variables simultaneously.

3. Results

a. Control run

The equilibrium storm structure of the control simulation is characterized by $V_m^* = 73\ ms^{-1}, r_m^* = 53\ km, r_0^* = 1150\ km$. Importantly, the time-scale to statistical equilibrium, defined as the starting time of the first 30-day interval, iterating backwards from day 70, whose mean value is within 10% of the equilibrium value, is significantly longer for size variables ($\sim 60\ days$) than for intensity ($\sim 30\ days$), suggesting that storms modeled to achieve quasi-equilibrium in intensity (typically 10-20 days), as is commonly done in the literature, may not have achieved structural equilibrium or else may do so artificially due to the domain-limitation imposed by the model's outer wall.

b. Sensitivity to potential intensity

Based on Eq. (1), we hypothesize that the primary role of the dimensional parameters T_{sst}, T_{tpp}, Q_{cool}, and u_s is to modulate the potential intensity. To test this hypothesis, we explore the sensitivity of storm structure to the potential intensity by independently modulating each parameter over the following ranges: $T_{sst} = 295, 297.5, 300, 302.5, 305\ K$; $T_{tpp} = 250, 225, 200, 175, 150\ K$; $u_s = 5, 4, 3, 2, 1\ ms^{-1}$; $Q_{cool} = .25, .5, 1, 2, 4\ Kday^{-1}$. Indeed, we find not only that the scaling for V_m is linear with V_p as expected, but that the scalings for r_m and r_0 also approximately collapse to unique scalings with V_p, particularly for r_0, as shown in Figure 1. The scaling for r_0 diverges for low values of Q_{cool} due to the direct dependence of the radiative subsidence rate on Q_{cool}. Thus, we hereafter use V_p in lieu of the four parameters on which it depends in subsequent sensitivity experiments.

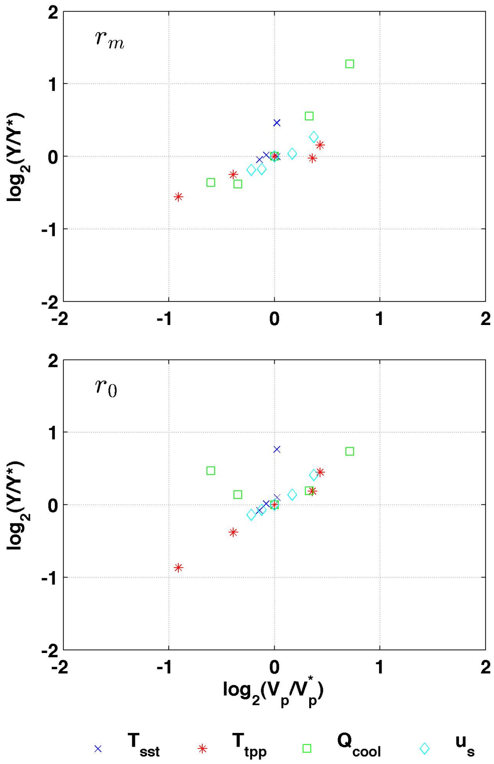

FIG. 1. Scaling of the equilibrium values of r_m (top) and r_0 (bottom) with the potential intensity, each normalized by their respective control values. A 1-unit increase (decrease) represents doubling (having). Colored shape denotes the parameter modified from control.

c. Structural scalings

All three structural variables are found to exhibit systematic sensitivities of differing magnitudes to precisely three parameters (not shown): the potential intensity, V_p, the Coriolis parameter, f, and the turbulent radial mixing length, l_h. Meanwhile, the equilibrium structure is insensitive to the initial disturbance structure, regardless of type, as well as to the vertical mixing length over the range of values tested here, though for sufficiently large (and likely unphysical) values on the order of the depth of the troposphere, storm structure does indeed become sensitive to this parameter (not shown) as strong vertical mixing across sloped angular momentum contours within the eyewall has a strong impact on the structure of a mature storm.

We quantitatively synthesize these sensitivities via dimensional analysis. The Buckingham-Pi theorem states that the number of independent non-dimensional parameters in a dimensional system is equal to the difference between the number of independent dimensional parameters and the number of fundamental measures. Moreover, any output dimensional quantity, suitably non-dimensionalized, can be expressed as a function of the set of non-dimensional parameters whose functional form can only be determined by experimentation.

For our system, this gives only one independent non-dimensional parameter, C, such that

$$Y = f(C), \quad C = \frac{V_p}{fl_h} \tag{4}$$

where Y is the structural variable non-dimensionalized as follows: V_m by V_p; r_m and r_0 by $\frac{V_p}{f}$. The scalings between each equilibrium non-dimensional variable and C for a large set of experiments varying two or more of V_p, f, or l_h are displayed in Figure 2.

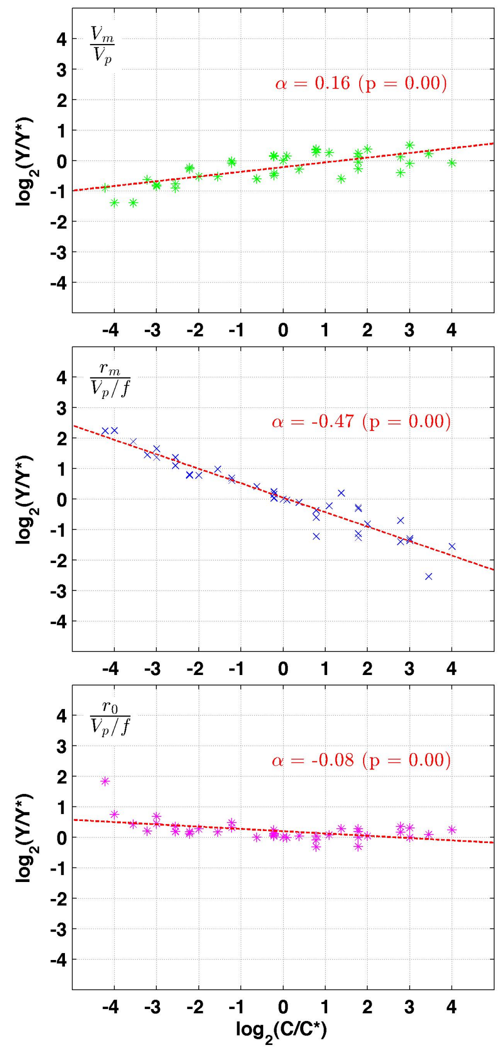

FIG. 2. Scaling of the equilibrium values of the non-dimensionalized structural variables V_m (top), r_m (middle), and r_0 (bottom) with the non-dimensional number $C = \frac{V_p}{fl_h}$. All quantities are normalized by their respective control values. Linearly-regressed slopes, corresponding to the estimated scaling exponent in (5), and associated p-values shown in red.

A linear relation in log-log space corresponds to a power-law scaling whose exponent equals the linear slope, i.e.

$$Y = C^{\alpha} \tag{5}$$

The estimated exponents are $\alpha_{V_m} = .16$, $\alpha_{r_m} = -.47$, and $\alpha_{r_0} = -.08$ and are all statistically-significantly different from zero.

Finally, we solve Eq. (5) for the approximate dimensional scalings for each structural variable:

$$\begin{aligned} V_m &\sim V_p^{1.15} (f l_h)^{-.15} \\ r_m &\sim \left(\frac{V_p}{f}\right)^{\frac{1}{2}} (l_h)^{\frac{1}{2}} \\ r_0 &\sim \left(\frac{V_p}{f}\right)^{.9} (l_h)^{.1} \end{aligned} \tag{6}$$

Thus, equilibrium storm intensity is found to scale super-linearly with the potential intensity and, more weakly, inversely with both the background rotation rate and the radial turbulent mixing length. The equilibrium radius of maximum gradient wind is found to elegantly scale as the geometric mean of the ratio of the potential intensity to the Coriolis parameter and the radial turbulent mixing length. Finally, the equilibrium outer radius is found to follow a simple quasi-linear scaling with the ratio of the potential intensity to the Coriolis parameter, with a slight expansion with increasing radial turbulent mixing length.

4. Discussion and Conclusions

A few key conclusions can be drawn from these results. First and foremost, the scaling for r_0 appears to confirm (with slight modification by radial turbulence) the predicted scaling of $\frac{V_p}{f}$ for the upper bound on tropical cyclone size given in Emanuel (1995) that is a consequence of the energetic contribution of outflow work in the Carnot framework.

Second, these results corroborate prior work demonstrating the importance of radial turbulence in determining storm structure (Bryan and Rotunno 2009). In particular, the scaling for r_m reflects the critical role of radial turbulence in counteracting eyewall frontogenesis by the secondary circulation that, in the inviscid limit, would lead to frontal collapse (Emanuel 1997). However, there is currently no accepted theory for the proper value of the radial mixing length in axisymmetric geometry. In principle l_h represents the length scale of the largest eddy since none are resolved. Notably, application of the simple ansatz that the largest eddy scales with the radius of maximum wind reduces the scalings to $V_m \sim V_p$ and $r_m \sim r_0 \sim \frac{V_p}{f}$ as would be expected from dimensional considerations alone. This structural uncertainty, coupled with the vagaries of modeling storm size, render real-world prediction of r_m and V_m very difficult in an axisymmetric framework.

Third, the combination of the long time-scales required to reach structural equilibrium and the high sensitivity to the location of the outer wall suggests that prior work modeling tropical cyclones out to statistical steady state in intensity are likely not at statistical steady state in structure, or else have equilibrated artificially.

Finally, these results indicate that real storms on Earth rarely, if ever, reach structural equilibrium; indeed, the large range in the observed size distribution likely cannot be explained by equilibrium dynamics. Analysis of the transient phase is currently a work in progress.

REFERENCES

Bryan, G. H. and J. M. Fritsch, 2002: A benchmark simulation for moist nonhydrostatic numerical models. *Monthly Weather Review*, **130 (12)**, 2917–2928, doi:10.1175/1520-0493(2002)130⟨2917:ABSFMN⟩2.0.CO;2.

Bryan, G. H. and R. Rotunno, 2009: The maximum intensity of tropical cyclones in axisymmetric numerical model simulations. *Monthly Weather Review*, **137 (6)**, 1770–1789, doi:10.1175/2008MWR2709.1.

Chavas, D. R. and K. A. Emanuel, 2010: A quikscat climatology of tropical cyclone size. *Geophysical Research Letters*, **37 (18)**, 10–13.

Emanuel, K. and R. Rotunno, 2011: Self-stratification of tropical cyclone outflow. part I: Implications for storm structure. *Journal of the Atmospheric Sciences*, **68 (10)**, 2236–2249, doi:10.1175/JAS-D-10-05024.1.

Emanuel, K. A., 1995: Sensitivity of tropical cyclones to surface exchange coefficients and a revised steady-state model incorporating eye dynamics. *Journal of the Atmospheric Sciences*, **52 (22)**, 3969–3976.

Emanuel, K. A., 1997: Some aspects of hurricane inner-core dynamics and energetics. *Journal of the Atmospheric Sciences*, **54 (8)**, 1014–1026.

Emanuel, K. A., 2004: Tropical cyclone energetics and structure. *Atmospheric Turbulence and Mesoscale Meteorology*, Cambridge University Press.

Hakim, G. J., 2010: The mean state of axisymmetric hurricanes in statistical equilibrium. *Journal of the Atmospheric Sciences*, **68 (6)**, 101227102212 037.

Rotunno, R. and K. A. Emanuel, 1987: An air-sea interaction theory for tropical cyclones. part ii: Evolutionary study using a nonhydrostatic axisymmetric numerical model. *Journal of the Atmospheric Sciences*, **44 (3)**, 542–561.

Tang, B. and K. Emanuel, 2010: Midlevel ventilation's constraint on tropical cyclone intensity. *Journal of the Atmospheric Sciences*, **67 (6)**, 1817–1830, doi:10.1175/2010JAS3318.1.