Nonlinear traveling wave solutions for the MJO skeleton model

Shengqian Chen[†] and Samuel N. Stechmann[‡]

[†] sqchen@math.wisc.edu
Department of Mathematics
University of Wisconsin at Madison, USA.

[‡] stechmann@wisc.edu
Department of Mathematics
Department of Atmospheric and Oceanic Sciences
University of Wisconsin at Madison, USA.

Introduction

Exact nonlinear traveling wave solutions are presented for the skeleton model of the Madden-Julian Oscillation (MJO), capturing its fundamental features on intraseasonal/planetary scales. Energetics, allowed traveling wave speeds and nonlinear features are discussed, such as nonlinear dispersion relation, and a short enhanced region with a long suppressed region for strongly active MJO event.

MJO skeleton model

$$u_t - yv - \theta_x = 0$$
$$yu - \theta_y = 0$$
$$\theta_t - u_x - v_y = \bar{H}a - F$$
$$q_t + \tilde{Q}(u_x + v_y) = -\bar{H}a + F$$
$$a_t = \boxed{\Gamma q a} \quad \text{nonlinear interaction}$$

Variables:

- u/v – zonal/meridional velocity;
- θ – potential temperature anomaly;
- q – lower troposphere moisture anomaly;
- a – convective activity envelope.

After meridional truncation:

$$K_t + K_x = -\frac{1}{2}(\bar{H}A - F)$$
$$R_t - \frac{1}{3}R_x = -\frac{1}{3}(\bar{H}A - F)$$
$$Q_t + \tilde{Q}K_x - \frac{\tilde{Q}}{3}R_x = (\frac{\tilde{Q}}{6} - 1)(\bar{H}A - F)$$
$$A_t = \Gamma Q A$$

Energetics

The system has energy conservation,
despite of the present of F
– radiative/cooling source terms:

$$\partial_t\Big[\ \tfrac{1}{2}u^2 \qquad \text{dry kinetic energy}$$
$$+\ \tfrac{1}{2}\theta^2 \qquad \text{dry potential energy}$$
$$+\ \frac{\tilde{Q}}{2(1-\tilde{Q})}\left(\theta + \frac{q}{\tilde{Q}}\right)^2 \qquad \text{moisture potential energy}$$
$$+\ \frac{\bar{H}}{\Gamma\tilde{Q}}a - \frac{F}{\Gamma\tilde{Q}}\log a\ \Big] \qquad \text{convective energy}$$
$$-\ \partial_x(u\theta) - \partial_y(v\theta) = 0$$

Nonlinear traveling wave solutions

Traveling wave ansatz, $A = A(x - st)$, $Q = Q(x - st)$, etc:

$$Q' = \frac{f(s)}{6s}(\bar{H}A - F)$$
$$A' = -\frac{\Gamma}{s}QA$$

- Allowed traveling wave speed criterion:

$$f(s) > 0.$$

- Four groups of allowed traveling wave speed:

  $s < -1/3$: dry Rossby;

  $s_- < s < 0$: moist Rossby;

  $0 < s < s_+$: MJO;

  $s > 1$: dry Kelvin.

- Exact traveling wave solution:

$$x - st = \pm\frac{\sqrt{3}s}{\sqrt{\Gamma f(s)}}\int_A^{A_{\max}} g(a)\mathrm{d}a,$$
$$g(a) = a^{-1}\left[\bar{H}(A_{\max} - a) - F(\log A_{\max} - \log a)\right]^{-1/2}$$

Nonlinear dispersion relation

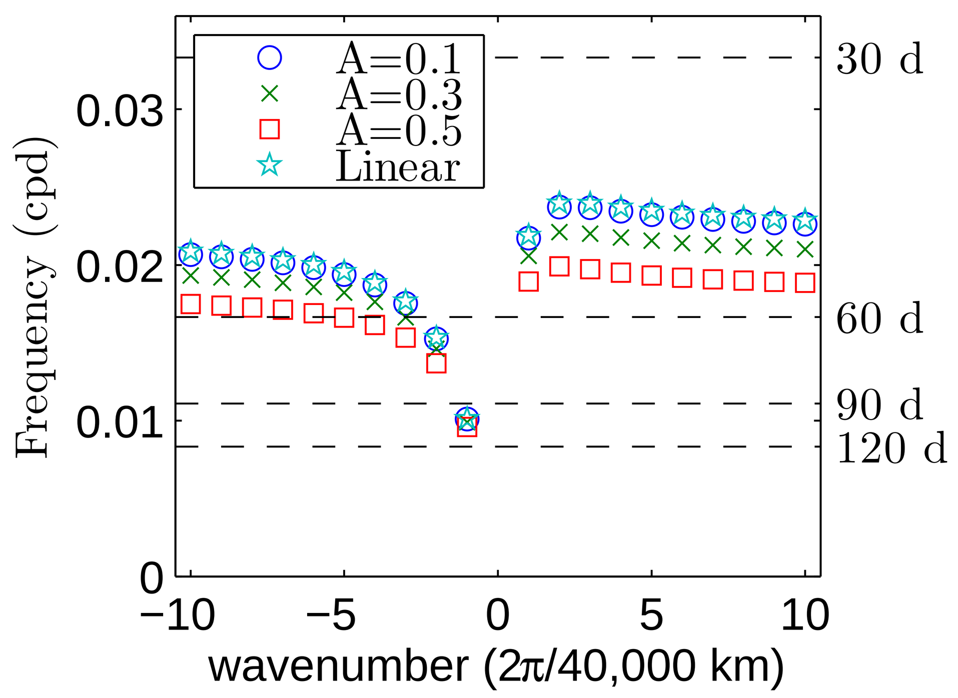

FIGURE 1: Oscillation frequency $\omega(k)$ for linear solutions and nonlinear solutions with different amplitude $\mathcal{A}$. Circle: $\mathcal{A} = 0.1$; cross: $\mathcal{A} = 0.3$; square: $\mathcal{A} = 0.5$; star: linear waves.

Wavenumber= 2

- Nonlinear phase speed = 4.61 m/s for amplitude $\mathcal{A} = 0.5$
- Linear phase speed = 5.55 m/s.

Physical quantities from the exact solution

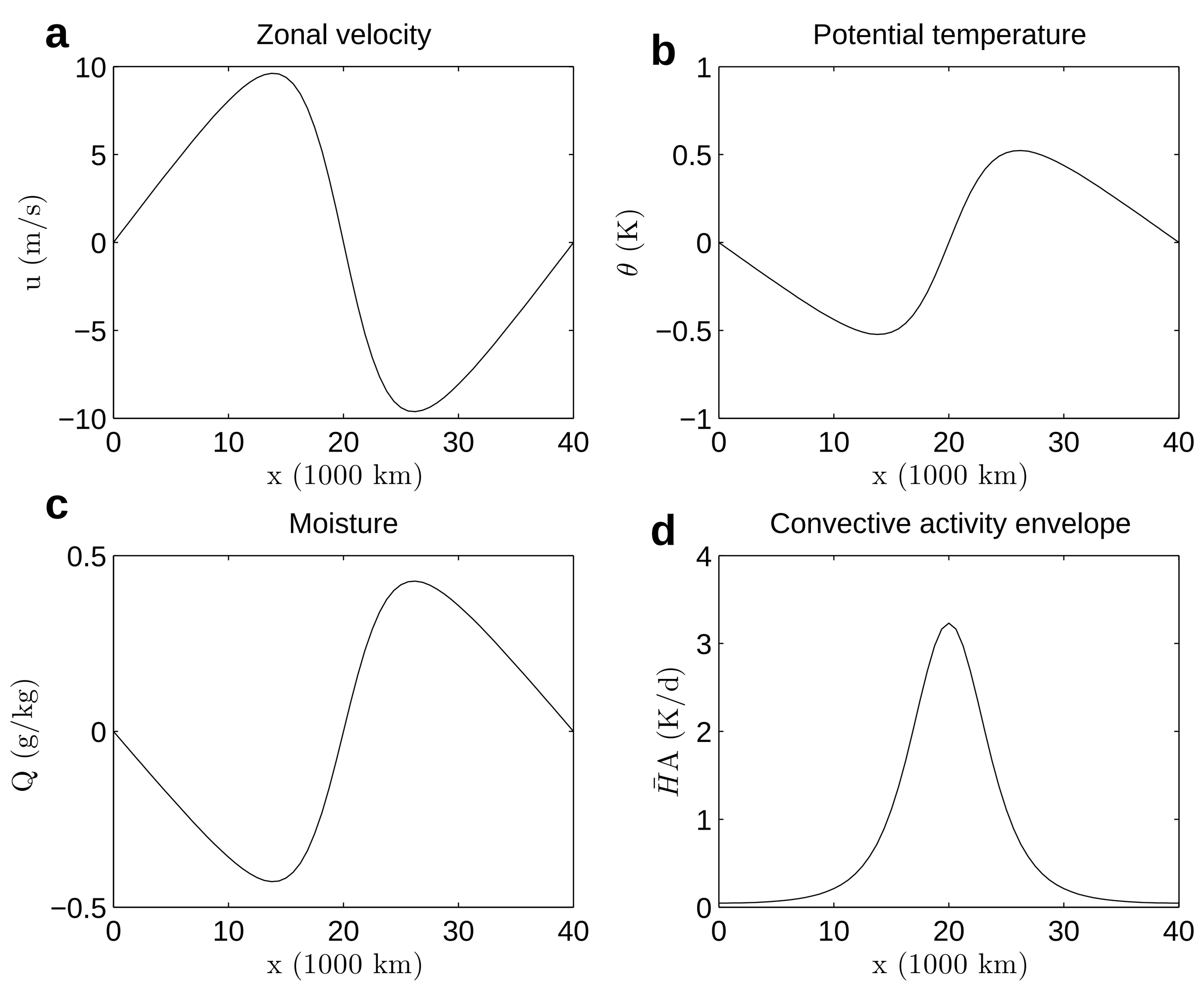

FIGURE 2: Variables at equator (y=0).

Nonlinearity and the shape of A

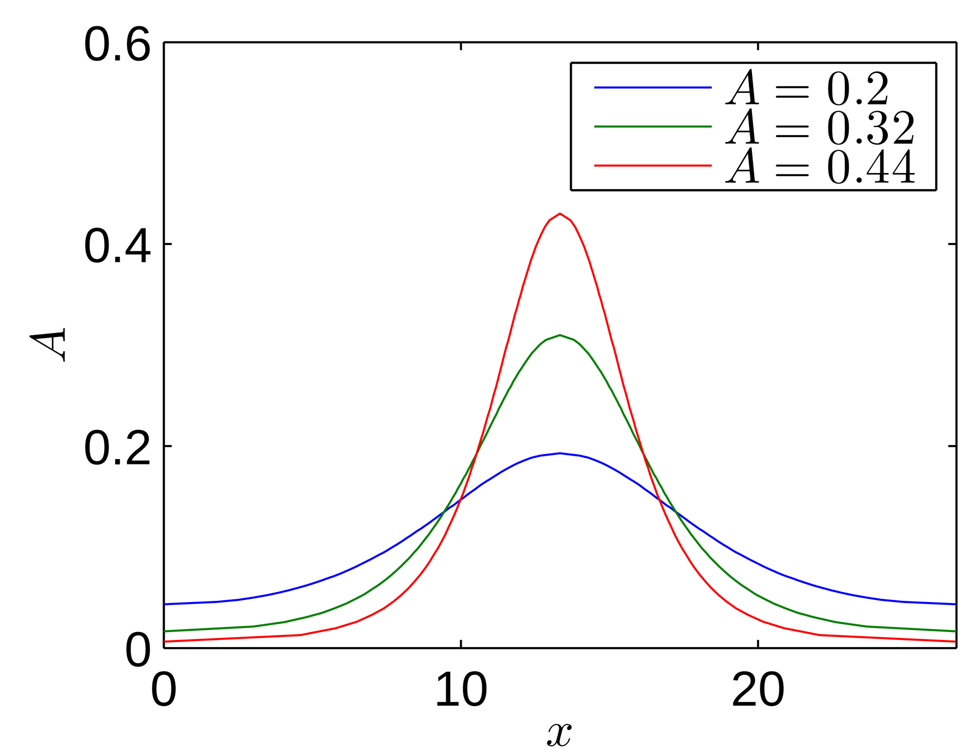

FIGURE 3: Plots of A with different amplitude for MJO mode. Equatorial wave number $k = 1$. Blue: $\mathcal{A} = 0.2$; green: $\mathcal{A} = 0.32$; red: $\mathcal{A} = 0.44$.

The shape of A transforms from a sinusoidal wave to a pulse, as the amplitude increases.

Meridional-zonal structure

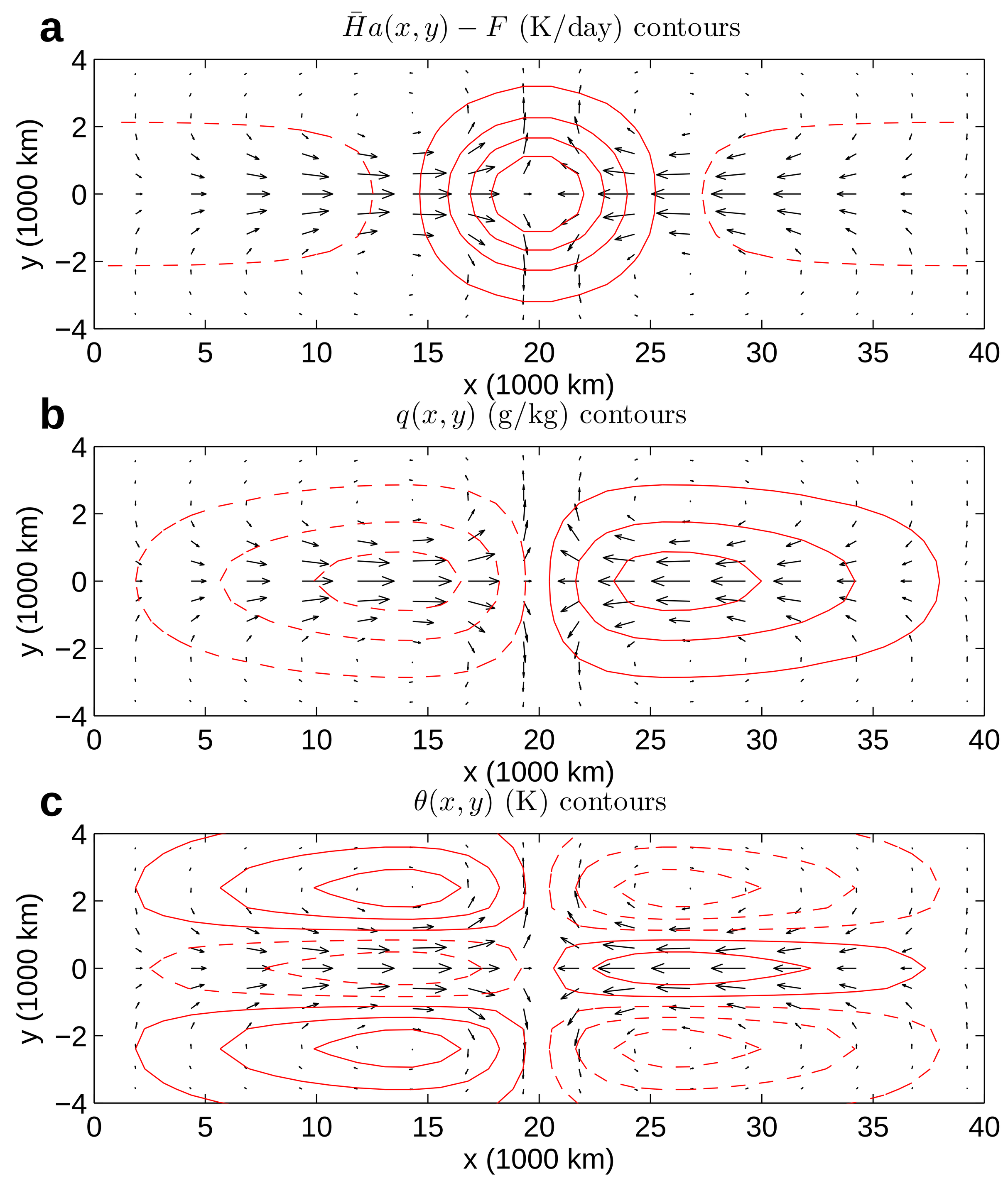

FIGURE 4: (a): zonal-meridional structure. Low-level zonal and meridional velocity are shown with contours of the amplitude of the convective activity envelope. (b): same as (a), except contours of lower tropospheric moisture, $q(x, y)$. (c): same as (a), except contours of potential temperature anomaly, $\theta(x, y)$. All positive (negative) contours are shown by solid (dashed) lines. For convective heating, moisture, and potential temperature, the contour intervals are 0.55 K/day, 0.15 g/kg, and 0.24 K, respectively. Maximum zonal and meridional velocities are 9.76 and 0.86 m/s, respectively.

Summary and conclusions

- Exact traveling wave solutions for the MJO skeleton model is presented in analytical form.
- Dispersion relation of nonlinear waves show a slower frequency and phase speed than linear waves.
- Nonlinear solutions suggest a shorter enhanced region and a longer suppressed region for strong convective activities.

References

Majda, A. J. and Stechmann, S. N. The skeleton of tropical intraseasonal oscillations *Proc. Natl. Acad.*, 106 (21), **8417-8422**.

Chen, S. and Stechmann, S. N. Nonlinear traveling waves for the skeleton of the Madden-Julian Oscillation (*in preparation*).