Tropical cyclone boundary layer shocks and shock-like structures

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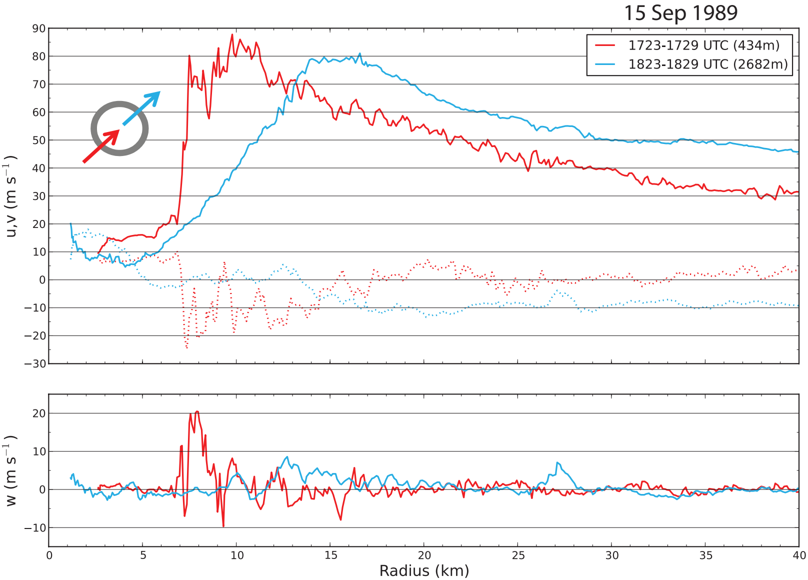
What is a shock?

- A shock is the spontaneous development of a discontinuity from a smooth initial condition
- In the tropical cyclone boundary layer, the shock develops from the advection of the radial momentum by the radial momentum, $u(\partial u/\partial r)$

Hurricane Hugo (1989) – Radial Profile



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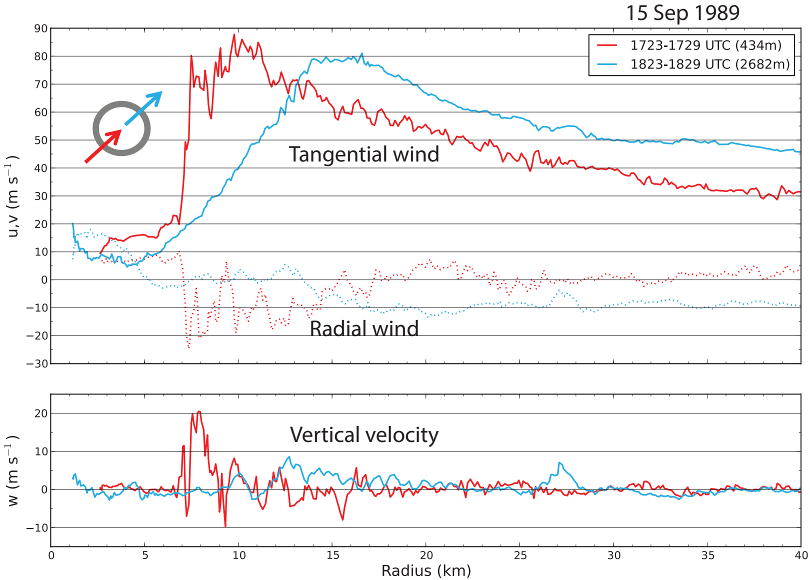

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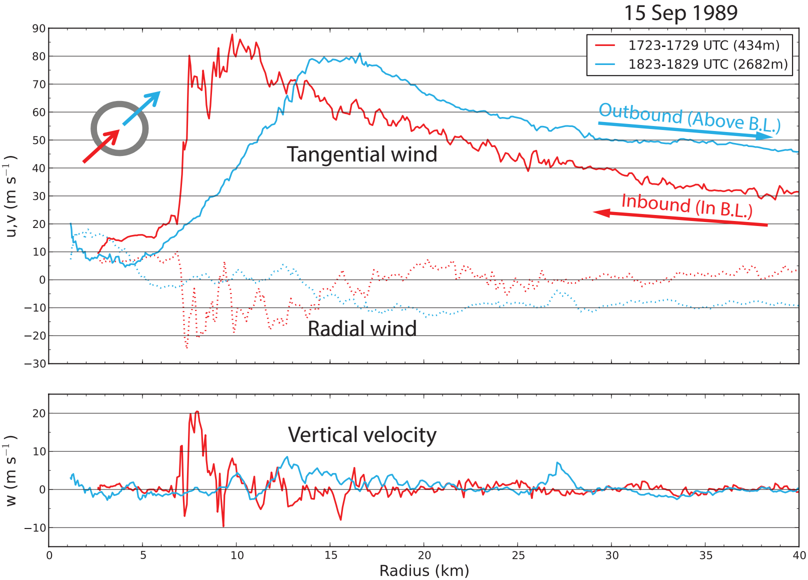

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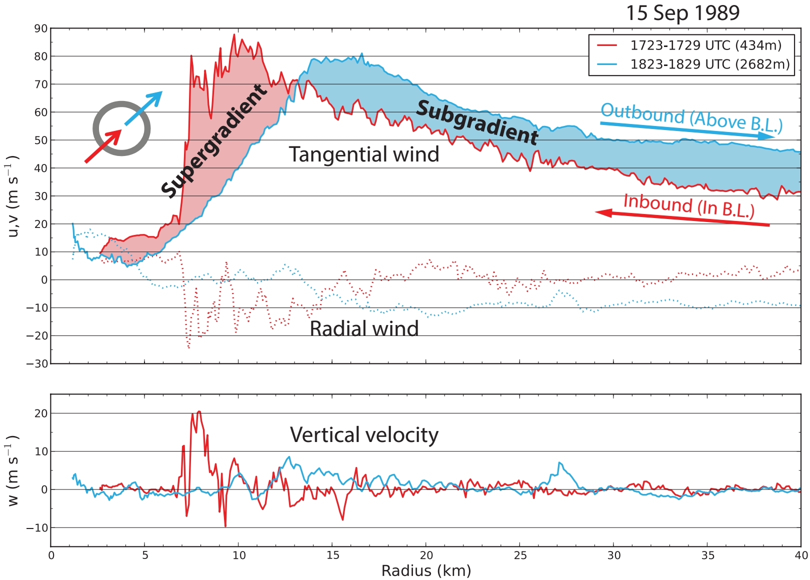

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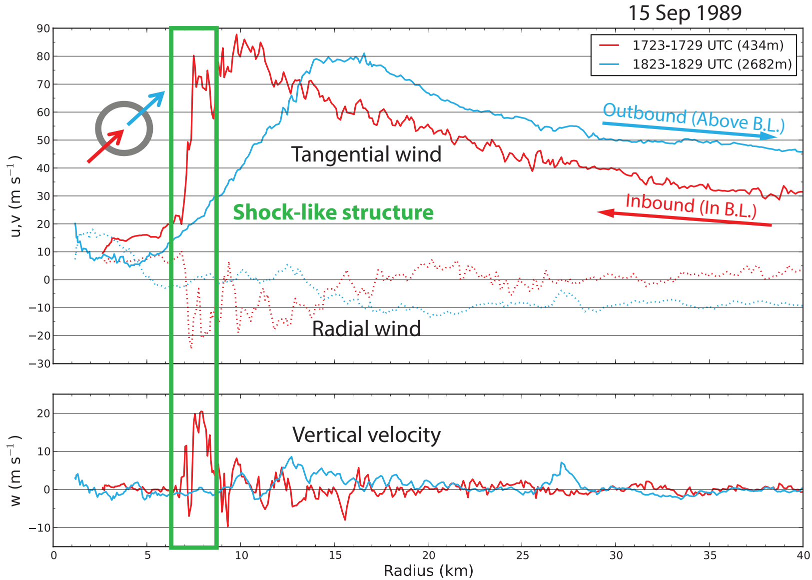
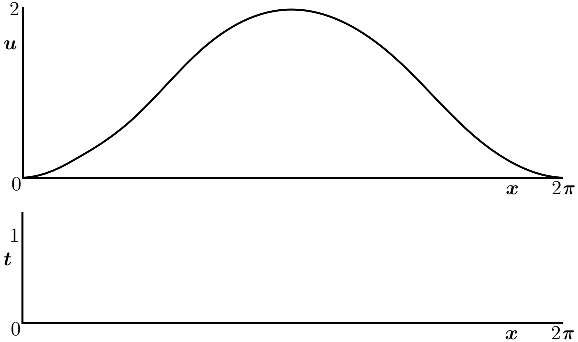


Inviscid Burgers' Equation

- Model for nonlinear wave propagation: $\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$
- Example initial condition: $u(x, 0) = 1 - \cos(x)$



[Schematic adapted from von Dommelen (2011)]

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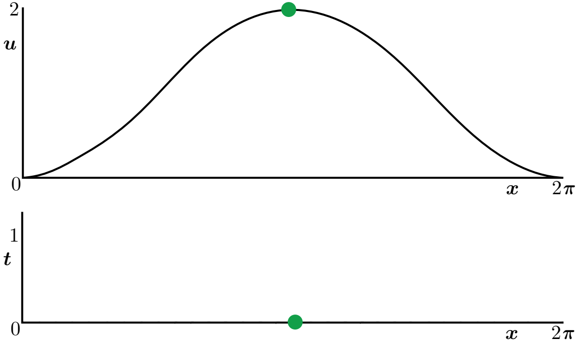

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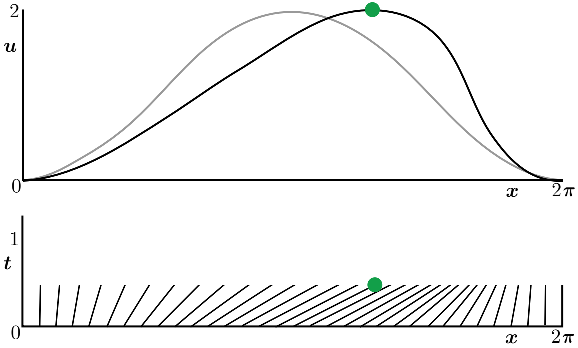

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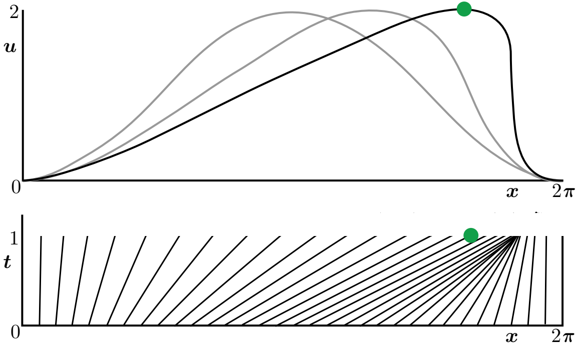

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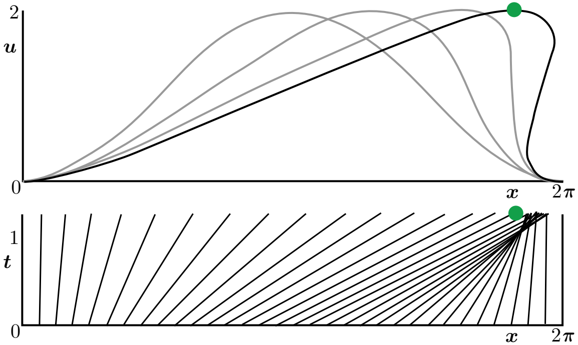

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Inviscid Burgers' Equation

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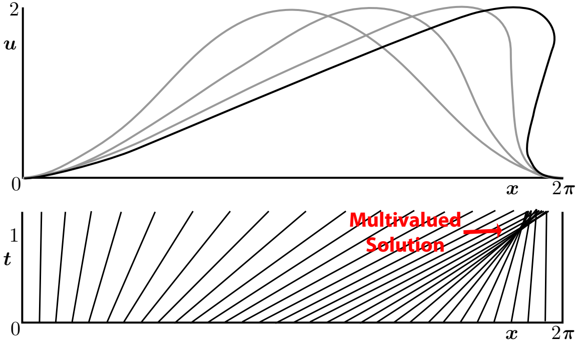
- Example initial condition: $u(x, 0) = 1 - \cos(x)$

- Characteristics intersect and cross
- $u(x, t)$ becomes multivalued
- Not physically meaningful

[Schematic adapted from von Dommelen (2011)]

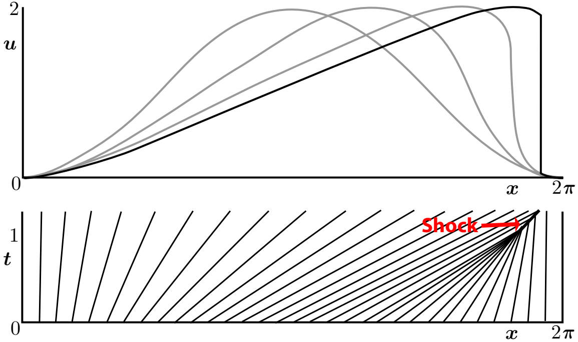


Viscous Burgers' Equation

- Now include a viscosity term: $\dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} = \varepsilon\dfrac{\partial^2 u}{\partial x^2}$
- Same initial condition: $u(x,0) = 1 - \cos(x)$

- A jump-discontinuity or "shock" develops
- Characterstics run into the shock and disappear

[Schematic adapted from von Dommelen (2011)]



Slab boundary layer model governing equations

Horizontal wind predictive equations

$$\frac{\partial u}{\partial t} = -u\frac{\partial u}{\partial r} - w^{-}\left(\frac{u}{h}\right) + \left(f + \frac{v + v_{\mathrm{gr}}}{r}\right)(v - v_{\mathrm{gr}}) - c_D U \frac{u}{h} + K\frac{\partial}{\partial r}\left(\frac{\partial (ru)}{r\partial r}\right)$$

$$\frac{\partial v}{\partial t} = w^{-}\left(\frac{v_{\mathrm{gr}} - v}{h}\right) - \left(f + \frac{\partial (rv)}{r\partial r}\right)u - c_D U \frac{v}{h} + K\frac{\partial}{\partial r}\left(\frac{\partial (rv)}{r\partial r}\right)$$

Vertical velocity diagnostic equations

$w = -h\frac{\partial(ru)}{r\partial r}$ and $w^{-} = \frac{1}{2}(|w| - w)$ ← rectified Ekman suction

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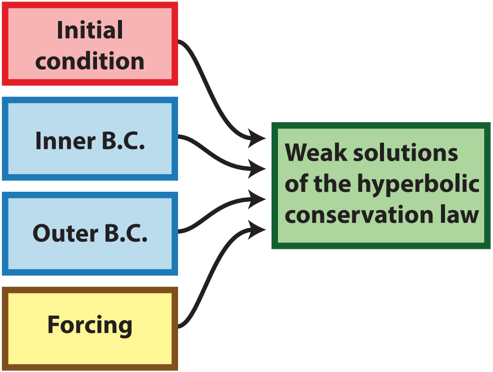
- embedded Burgers'
- horizontal diffusion, Ekman suction, agradient forcing
- surface drag terms (linearize later)

Slab boundary layer model as a hyperbolic system

- In the absence of diffusion, the system is hyperbolic not parabolic *(ODEs not PDEs)*



Slab boundary layer model as a hyperbolic system

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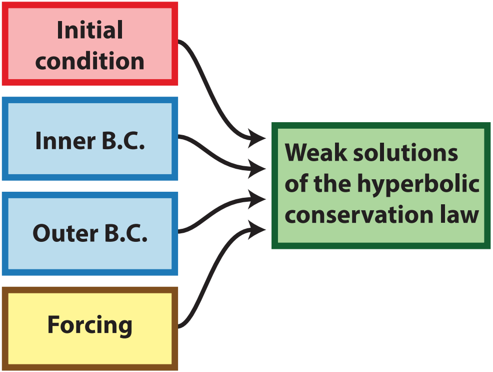
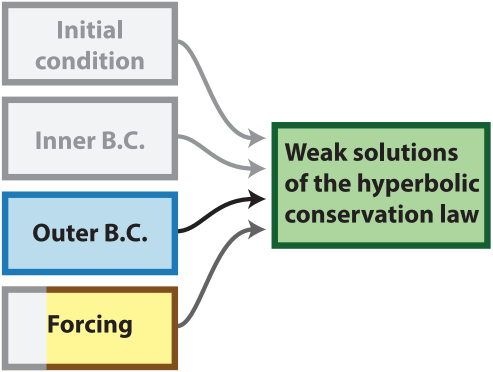
- Allows system to be solved using the characteristic form



Slab boundary layer model as a hyperbolic system

- In the absence of diffusion, the system is hyperbolic not parabolic *(ODEs not PDEs)*
- Allows system to be solved using the characteristic form
- Issues solving as a steady-state problem



Slab boundary layer model in characteristic form

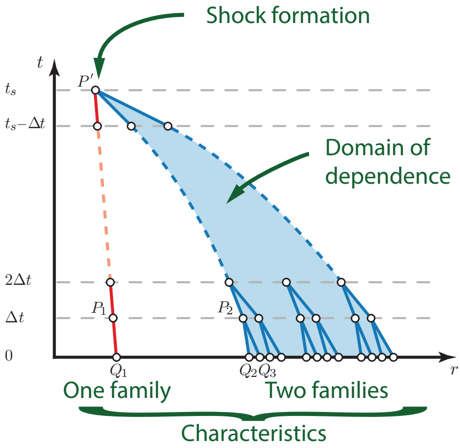
- On $\frac{dr}{dt} = (1+\alpha)u$

$$\frac{du}{dt} = F_1$$

- On $\frac{dr}{dt} = u$

$$(m - m_{gr})\frac{du}{dt} - u\frac{dm}{dt} = F_2$$

where m is angular momentum and F_1 and F_2 are linear and nonlinear terms



Idealized argument (neglecting w^- and diffusion terms)

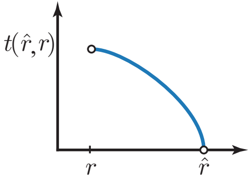
$$\left.\begin{aligned} \frac{d}{dt}\left[\frac{1}{2}\left(u^2+v^2\right)+\frac{p(r)}{\rho}\right] &= -\frac{u}{\tau} \\ \frac{d\left(rv+\frac{1}{2}fr^2\right)}{dt} &= -\frac{rv}{\tau} \end{aligned}\right\} \text{ on } \frac{dr}{dt}=u$$

where $\tau = h/(c_D U)$

If we neglect the τ terms, the Bernoulli function and the absolute angular momentum are Riemann invariants:

$$\frac{1}{2}\left[u^2(\hat{r},r)+v^2(\hat{r},r)\right]+\frac{p(r)}{\rho}=\frac{1}{2}\left[u_0^2(\hat{r})+v_0^2(\hat{r})\right]+\frac{p(\hat{r})}{\rho}$$

$$rv(\hat{r},r)+\frac{1}{2}fr^2=\hat{r}v_0(\hat{r})+\frac{1}{2}f\hat{r}^2$$



Solving for $u(\hat{r}, r)$

$$r^2 u^2(\hat{r}, r) = \left(u_0^2(\hat{r}) + v_0^2(\hat{r}) + \frac{2}{\rho}\left[p(\hat{r}) - p(r)\right]\right) r^2 - \left(\hat{r} v_0(\hat{r}) + \frac{1}{2} f \left[\hat{r}^2 - r^2\right]\right)^2$$

To solve for the slope of the characteristic, integrate $dt = dr/u$

$$t(\hat{r}, r) = \int_{\hat{r}}^{r} \frac{dr'}{u(\hat{r}, r')}$$

Can be done analytically for some cases:

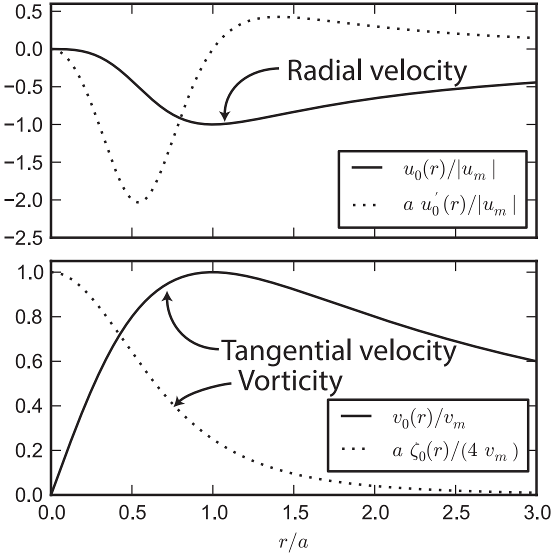
- When only $u_0(\hat{r})$ is retained above,

$$r = \hat{r} + t u_0(\hat{r})$$

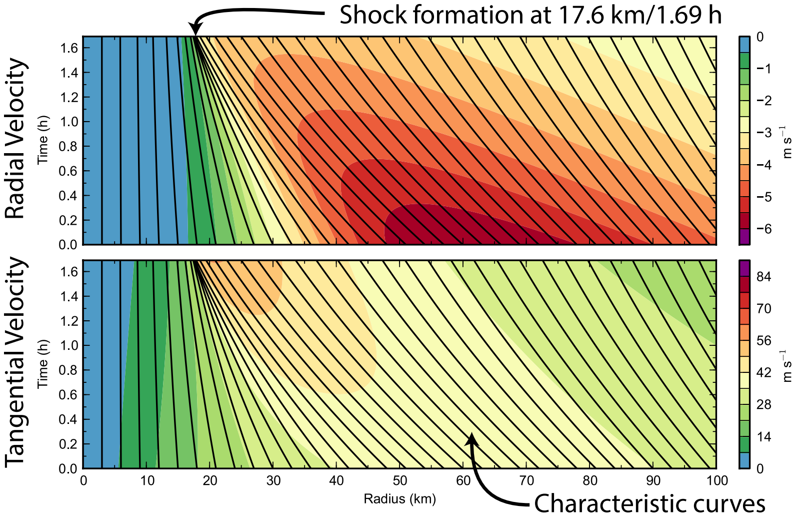
- With a fixed value for surface drag,

$$r = \hat{r} + \hat{t} u_0(\hat{r}) \quad \text{and} \quad \hat{t} = \tau\left(1 - e^{-t/\tau}\right)$$

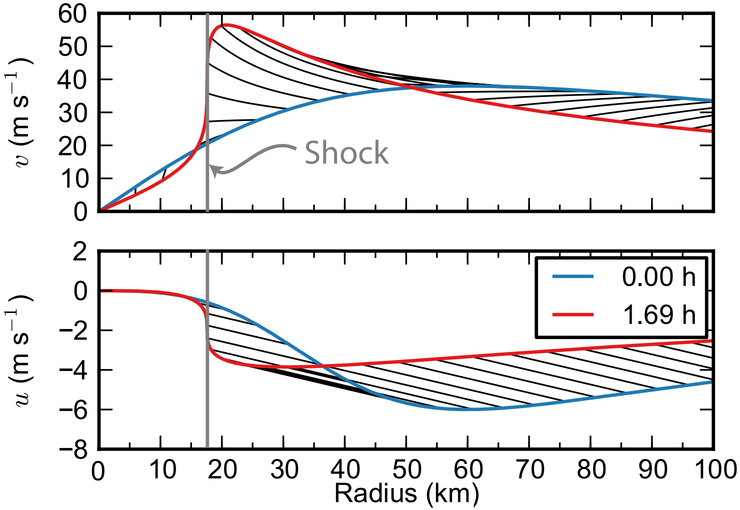
Initial conditions



Results


Shock formation at 17.6 km/1.69 h
Radial Velocity
Time (h)
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0.0
0
−1
−2
−3
−4
−5
−6
m s^{-1}
Tangential Velocity
Time (h)
84
70
56
42
28
14
0
0
10
20
30
40
50
60
70
80
90
100
Radius (km)
Characteristic curves

Results



60
50
40
30
20
10
0
v (m s^{-1})
Shock
2
0
−2
−4
−6
−8
u (m s^{-1})
0.00 h
1.69 h
0
10
20
30
40
50
60
70
80
90
100
Radius (km)

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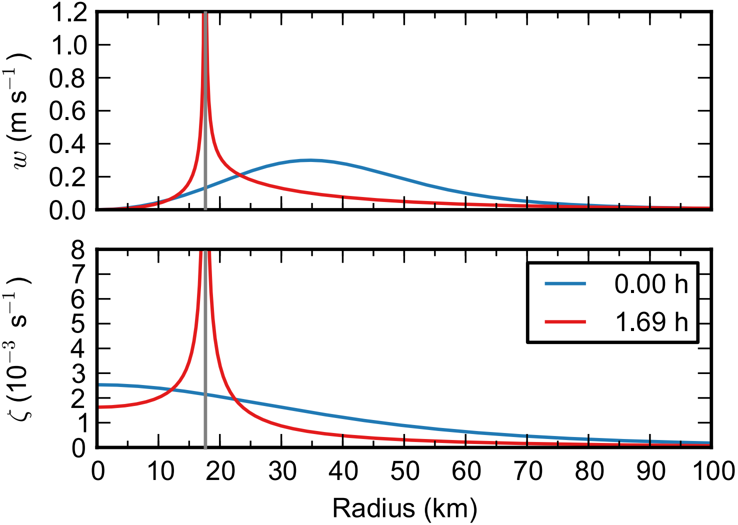

1.2
1.0
0.8
0.6
0.4
0.2
0.0
w (m s^{-1})
8
7
6
5
4
3
2
1
0
ζ (10^{-3} s^{-1})
0.00 h
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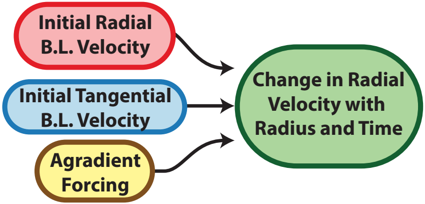
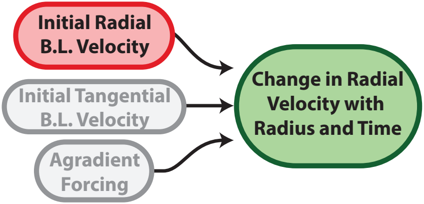
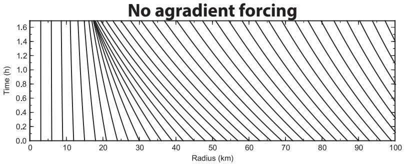
Radius (km)

Is neglecting the agradient forcing in $\partial u/\partial t$ unrealistic?



Is neglecting the agradient forcing in $\partial u/\partial t$ unrealistic?





- The idealized model presented here neglects the agradient forcing and initial tangential velocity

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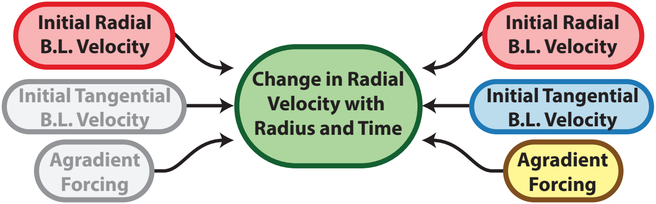

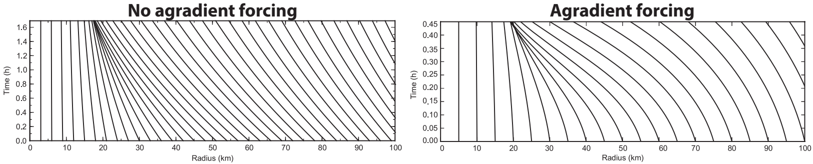

- The idealized model presented here neglects the agradient forcing and initial tangential velocity
- Quantitatively unrealistic; however, still conceptually useful

Conclusions

- Hurricane Hugo's wind profile can be explained by shocks
- The system is effectively hyperbolic except near the shock
  - Allows system to be solved in the characteristic form
  - Explains steady-state solution properties discovered by Smith (2003), Smith and Vogl (2008), and Kepert (2010a,b)
- Shocks develop with and without the agradient forcing, but without forcing effects, radius and time of shock formation are inaccurate
- Shocks control the size of the hurricane eye and the location of maximum boundary layer pumping

Why don't we see shocks or shock-like structures in our current 3D models?

- Coarse radial resolution and high numerical diffusion
- Moist physics may limit the sharpness of such features
- Lack of a shock-capturing or -tracking numerical method

Future work

- Repeat the numerical modeling work of Williams et al. (2013) with a shock-capturing procedure (e.g., Essentially Non-Oscillatory scheme)
- Develop a multilevel model using a shock-capturing procedure

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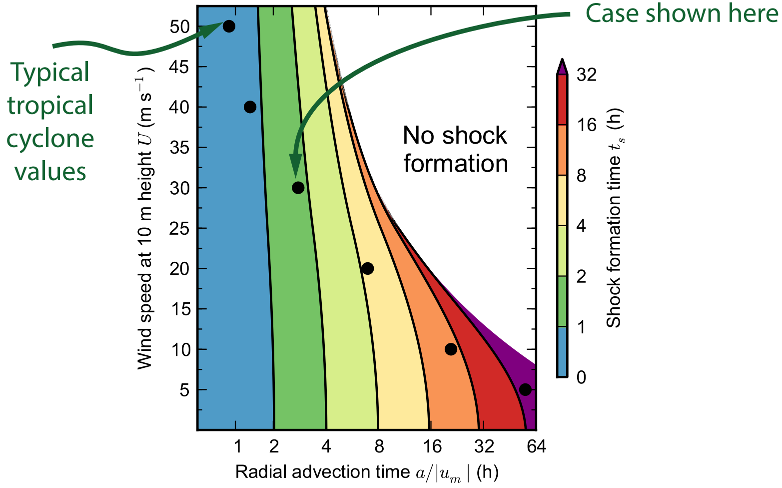
Time of shock formation



Shock disruption offers a mechanism for spin-down

Why shocks and not fronts?

- Frontogenesis is described by semigeostrophic theory
- Semigeostrophic theory neglects the $u(\partial u/\partial r)$ term
- Fronts are not true mathematical discontinuities