1. INTRODUCTION

Williams et al. (2013) interpreted the structure of the wind field within the boundary layer of Hurricane Hugo (1989) using an axisymmetric slab boundary layer model. This work shows that the dry dynamics form a “shock-like” structure in the boundary layer inflow that resembles the Hurricane Hugo (1989) observations. In applied mathematics, a shock is the spontaneous development of a discontinuity in a smooth initial condition through the advection of momentum by momentum (LeVeque 2002). For weak solutions of hyperbolic conservation laws, this feature of fluid dynamics is referred to as a Burgers’ shock (Whitham 1974).

Previous work on the tropical cyclone boundary layer model has not included procedures in the numerics to handle shock development for weak solutions of hyperbolic conservation laws (Rosenthal 1962; Ooyama 1969a,b; Chow 1971; Shapiro 1983; Kepert 2010a,b; Smith 2003; Smith and Montgomery 2008; Smith et al. 2009; Smith and Montgomery 2010; Abarca and Montgomery 2013). This is true in the work of Williams et al. (2013). The authors acknowledge this limitation and introduce the terminology “shock-like.” In the slab boundary layer model used to interpret the structure of Hurricane Hugo (1989), the finite-differencing and horizontal diffusion terms prevent the formation of true discontinuities. Since the numerics do not include a shock-capturing or -tracking method (Whitham 1974; Shu 1998; LeVeque 2002; Durran 2010), they dismiss the importance of the horizontal diffusion terms in relation to the stability constraint and preventing multivalued solutions from developing.

The purpose of this work is to begin to understand how removing the aforementioned restrictions by solving an overly simplified form of the slab boundary layer model equations through the method of characteristics would impact the results of Williams et al. (2013). We will begin by outlining the slab boundary layer model used by Williams et al. (2013), and then we will present an idealized analytical argument in characteristic form.

2. SLAB BOUNDARY LAYER MODEL

The model considers axisymmetric, boundary layer motions of an incompressible fluid on an f-plane with a constant depth $h$. The radial and azimuthal velocities are independent of height. The vertical velocity $w(r,t)$ is defined at height $h$. In the overlying layer the radial velocity is assumed to be negligible and the azimuthal velocity $v_{0}(r)$ is assumed to be in gradient balance and to be a specified function of radius. The boundary layer flow is driven by the same radial pressure gradient force that occurs in the overlying fluid, so that, in the radial equation of boundary layer motion, the pressure gradient force can be expressed as the specified function $[f + (v_{0}/r)]v_{0}$.

The governing system of differential equations for the boundary layer variables $u(r,t)$, $v(r,t)$, and $w(r,t)$ then takes the form

$$
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial r} + w - \left( \frac{u}{h} \right) = \left( f + \frac{v + v_{0}}{r} \right) \left( v - v_{0} \right) - c_{D}U \frac{u}{h} + K \frac{\partial}{\partial r} \left( \frac{\partial (ru)}{r \partial r} \right),
$$

(1)

$$
\frac{\partial v}{\partial t} + u \left( f + \frac{\partial v}{\partial r} + \frac{v}{r} \right) + w - \left( \frac{v - v_{0}}{r} \right) = -c_{D}U \frac{v}{h} + K \frac{\partial}{\partial r} \left( \frac{\partial (rv)}{r \partial r} \right),
$$

(2)

$$
w = -\frac{h}{r} \frac{\partial (ru)}{r \partial r} \quad \text{and} \quad w = \frac{1}{2} \left( |w| - w \right),
$$

(3)

where

$$
U = 0.78 \left( a^{2} + c^{2} \right)^{1/2}
$$

(4)

is the wind speed at 10 m height, $f$ the constant Coriolis parameter, and $K$ the constant horizontal diffusivity. The drag coefficient $c_{D}$ is assumed to depend on the 10 m wind speed according to

$$
c_{D} = 10^{-3} \begin{cases} 
2.70/U + 0.142 + 0.0764U & \text{if } U \leq 25 \\
2.16 + 0.5406 \left( 1 - \exp(-U - 25)/7.5 \right) & \text{if } U \geq 25, 
\end{cases}
$$

(5)

where the 10 m wind speed $U$ is expressed in m s$^{-1}$. Appendix A of Williams et al. (2013) explains the above derivation of the slab boundary model in greater detail.

The boundary conditions are

$$
\begin{align*}
&u = 0 & \text{at } r = 0, \\
&v = 0 & \text{at } r = b,
\end{align*}
$$

(6)

where $b$ is the radius of the outer boundary. The initial conditions are

$$
u(r,0) = u_{0}(r) \quad \text{and} \quad v(r,0) = v_{0}(r),
$$

(7)

where $u_{0}(r)$ and $v_{0}(r)$ are specified functions.

3. IDEALIZED ANALYTICAL ARGUMENT

The formation of shocks in the $u$ and $v$ fields in the hurricane boundary layer depends on the $u(\partial u/\partial r)$ and $v(f + (\partial v/\partial r) + (v/r))$ terms in (1) and (2), with the term...
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forcing mechanism for
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The analytical solutions $u(r,t)$ and $v(r,t)$ (color contours), as well as the characteristic curves (black curves on each panel). These solutions are for the particular initial conditions (17) and (18) for the case when $a = 60$ km, $u_m = -6$ m s$^{-1}$, $v_m = 38$ m s$^{-1}$. The plots cover the time interval $0 \leq t \leq t_s$, where $t_s = 1.69$ h is the shock formation time for this particular initial condition. [Figure from Slocum et al. (2014).]

As a simple example, consider the initial conditions

$$u_0(r) = u_m \left( \frac{4(r/a)^3}{1 + 3(r/a)^4} \right),$$

(17)

$$v_0(r) = v_m \left( \frac{2(r/a)}{1 + (r/a)^2} \right),$$

(18)

where the constants $a$, $u_m$, and $v_m$ specify the radial extent and strength of the initial radial and tangential flow. The derivative of (17) is

$$u_0'(r) = \frac{12u_m}{a} \left( \frac{(r/a)^2[1 - (r/a)^4]}{[1 + 3(r/a)^4]^2} \right),$$

(19)

while the initial relative vorticity, obtained by differentiation of (18), is

$$\zeta_0(r) = \frac{4v_m}{a[1 + (r/a)^2]^2}.$$  

(20)

The dimensionless forms of (17)–(20) are shown in Fig. 1.

For the initial conditions (17)–(20), the solutions take the form

$$u(r,t) = u_m \left( \frac{d(r/a)^3 e^{-t/r}}{1 + 3(r/a)^4} \right),$$

(21)
are defined by there. [Figure from Slocum et al. (2014).]

\[ \hat{v}(t) = \hat{v}_{\text{in}} \left( \frac{2(\hat{r}/a)}{1 + (\hat{r}/a)^2} \right) + \int_0^t \left( \hat{r} + (r - \hat{r}) \left( \frac{r(t - \hat{t})}{t \hat{t}} \right) \right) e^{-t/\tau} \, dt, \]

where the characteristic curves (along which \( \hat{r} \) is fixed)
are defined by

\[ r = \hat{r} + u_m \hat{t} \left( \frac{4(\hat{r}/a)^3}{1 + 3(\hat{r}/a)^2} \right). \]

The solutions for \( u(r, t), v(r, t), \hat{r}(r, t) \), as given by
(21)–(23), are plotted in the two panels of Fig. 2 for the
case when \( a = 60 \text{ km}, u_m = -6 \text{ m s}^{-1}, v_m = 38 \text{ m s}^{-1} \).
The plots cover the radial interval \( 0 \leq r \leq 100 \text{ km} \) and the
time interval \( 0 \leq t \leq t_s \), where \( t_s = 1.69 \text{ h} \) is the shock
formation time for this particular initial condition. Another
view of this analytical solution is given in Fig. 3, with the
four panels displaying the radial profiles (at \( t = 0 \) in blue
and at \( t = t_s \) in red) of \( u, v, w, \zeta \). Also shown by the black
curves in the top two panels are fluid particle displacements
for particles that are equally spaced at the initial
time. At \( t = t_s \), the \( u \) and \( v \) fields become discontinuous
at \( r = 17.6 \text{ km} \), while the \( w \) and \( \zeta \) fields become singular
there. [Figure from Slocum et al. (2014).]

\[ \tau = \frac{h}{c_D U} \],

(15) can also be regarded as giving the shock formation
time \( t_s \) as a function of the radial advection time \( a/|u_m| \)
and the 10 m wind speed \( U \). Contours of \( t_s \) are shown in
Fig. 4. The ordinate is the boundary layer wind speed that
can be used to determine the strength of the tropical cyclone,
\( U \). The abscissa is the radial advection time, which
is defined by the radius of maximum wind, \( a \), divided by
the maximum radial velocity \( |u_m| \). More intense storms
are located near the upper left corner of the figure; weak
tropical storms are in the lower right corner of the plot.
The six dots correspond to typical values of \( a \) and \( U \) for
various strength vortices as noted in Table 1.

4. DISCUSSION AND CONCLUDING REMARKS

While the idealized analytical arguments presented here
neglect the horizontal diffusion terms, the \( w^- \) terms,
and the \( (\nu - \nu_w) \) terms, (8) and (9) do provide valuable
insight into how to interpret the dynamics involved
in boundary layer shocks. Figs. 2 and 3 show that information
originating from a smooth initial condition will develop
discontinuities in \( u \) and \( v \). The corresponding singularities
in \( w \) and \( \zeta \) provide insight into what determines
the size of an eye and how potential vorticity rings are
produced through nonlinear processes. While the singularities may seem unrealistic, moist dynamics associated with the eyewall would act to lessen $\zeta$ to values seen in Williams et al. (2013). The idealized analytical argument also is limited by the fact that the shock originates solely from the smooth initial condition for the radial velocity defined by (17) in the absence of the $(v - v_{gr})$ forcing term. Returning to (1) and (2), the gradient tangential forcing term. Returning to (1) and (2), the gradient tangential flow will act as a forcing mechanism for $(\partial u/\partial t)$. However, it is encouraging to see similar structures develop in the Williams et al. (2013) solutions.

Fig. 4 shows that hurricane strength vorticies will quickly generate and maintain a shock. The figure suggests that shocks will develop over long periods of time in tropical storms, but as storms intensify, the shocks develop rapidly. However, the threshold for shock development is small based on the radial advection time associated with strong tropical cyclones. This potentially offers insight into how a process like landfall can cause a vortex to weaken. If landfall increases the radial advection time of a storm, a shock may not redevelop and the mechanism, producing the potential vorticity ring and maintaining the eye, would dissipate.

In comparing the idealized analytical argument to previous work, the characteristic form of (1)–(7) without the diffusion terms is a weak solution of a hyperbolic system. As such, the system cannot be solved as a steady-state problem. Even though Williams et al. (2013) includes diffusion, the numerical solutions should be considered as reaching quasi-steady-state. As shown by Fig. 2, characteristics containing information from the initial condition and the boundary conditions are still contributing and changing the solution. When the $(v - v_{gr})$ forcing term is included, the characteristics accrue information associated with the forcing. The fact that a hyperbolic system requires knowledge from all four information sources means the system cannot be regarded as steady-state. Trying to integrate inward from the outer boundary condition at $r = 0$ to find the steady-state solution will result in $u$ and $v$ becoming singular or in spurious oscillations as seen in Smith (2003) and Kepert (2010a,b). In order to solve the hyperbolic time dependent system of equations numerically, a shock-capturing or -tracking procedure must be included to compute solutions past $t_s$. These procedures and methods are briefly outlined in (Whitham 1974; Shu 1998; LeVeque 2002; Durran 2010).

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REFERENCES


