9A.3 Model Selection and Inference for Atmospheric Vortices

Lynn Raburn Greenleaf *
University of Oklahoma, Norman, OK

1 Introduction

Intense atmospheric vortices occur in dust devils, waterspouts, tornadoes, mesocyclones and tropical cyclones. The tangential wind profiles of these atmospheric vortices are often approximated by continuous functions that are zero at the vortex center, increase to a maximum at some radius and then decrease asymptotically to zero. A number of models, such as the idealized, inviscid Rankine (1882) [4], the viscous Burgers (1948) [2] -Rott(1958) [5] and Sullivan (1959) [7] analytical vortex models have been used to approximate observed profiles of tangential winds. New models have been proposed by Wood-White (2011) [10] that use a rational function to model an inner core of solid-body rotation and an outer profile that decays to zero at infinity. The various models for tangential winds have parameters that contain useful information about the physical structure of a vortex. Also, it is possible that these models can be used to determine vortex structure when there is incomplete data and predict other physical quantities, such as radial and vertical winds and the vortex pressure field. Therefore, It is of interest to select the most appropriate tangential wind model that provides the best possible estimate for each task.

The focus of this paper is to make valid inferences from meteorological vortex data when the analysis depends on a model of the information in the data. Candidate models with parameters are selected that model the tangential velocity of the vortex. There are no assumption that the "true" model of vortex tangential velocity is in this candidate list. The goal is to select the "best" model using information and likelihood theory, (Burnham 2002 [3]). The "best" model may be defined as the model that loses the minimum amount of information that the data contains on tangential velocity. Furthermore, the "best" model may be defined as the model that gives the best estimation of radial and vertical wind components along with the pressure in a vortex. There is an assumption that the data contains information about the true vortex winds which are inherently unknown. Therefore, an approximating model must be used to estimate these vortex winds. Data arise from full reality and can be used to make formal inferences back to this truth. The model selected should provide precise predictions, as well as fit the data without over-fitting. This means selecting a model with the smallest possible number of parameters for adequate representation of the data. There are three aspects of valid statistical inference when the analysis depends on a model. These are:

1. Model specification. A list of candidate models is selected based on scientific principles.

2. Estimation of model parameters. Precise unbiased estimators are desired. Let Θ be a vector of parameters. An unbiased estimator of Θ would have the property that the expected value of Θ would be some Θ_0 which corresponds to the parameters that produce the "best" model. Since this "best" model is unknown and must be estimated, then the Maximum Likelihood Estimation (MLE) method is used to estimate Θ_0. After this estimation of Θ_0, the "best" model is estimated.

3. Estimation of precision. The probability that the correct model was selected.

2 Information and likelihood theory

2.1 Maximum Likelihood Theory

Let Θ be the vector of parameters in an approximating model. This vector may differ from model to model. Some models may have a subset of parameters from other models and some models may

*Corresponding author address: Lynn R. Greenleaf, Univ. of Oklahoma, Dept. of Mathematics, Norman, OK 73019-3101; e-mail: lynngreenleaf@ou.edu

have no parameters in common with other models. Assume that there is only one observation at each distance from the center of the vortex and that there are n distances. Let $\{x(r_i) | i = 1, \ldots, n\}$ be the observations. For each $x(r_i)$, assume $\epsilon(r_i) = (v(r_i) - x(r_i))^2$ is normally distributed with mean zero and unknown variance σ^2, i.e.,

$$g(\epsilon(r_i) | \theta) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\left(\frac{\epsilon(r_i)}{\sqrt{2}\sigma}\right)^2\right\}$$

where $v(r)$ is calculated using a model from the set of candidate models. Assume the $\{\epsilon(r_i) | i = 1, \ldots, n\}$ are independent. In this case, the likelihood function becomes

$$L(\theta | \text{ observations}) = \prod_1^n g(\epsilon(r_i) | \theta) =$$
$$\left(\frac{1}{\sqrt{2\pi}\sigma}\right)^n \exp\left\{-\sum_1^n \left(\frac{\epsilon(r_i)}{\sqrt{2}\sigma}\right)^2\right\}$$

There is an assumption that $\epsilon(r_i)$ has mean zero with unknown variance σ^2 that must be estimated from the data. It is easier to work with the natural logarithm of the likelihood function because

$$\log(L(\theta | \text{ observations})) =$$
$$-n\log\sqrt{2\pi} - n\log\sigma - \frac{1}{2\sigma^2}\sum_1^n (\epsilon(r_i))^2$$

The following computation will lead to an estimation of σ^2:

$$\frac{\partial}{\partial\sigma^2}\log(L(\theta | \text{ observations})) =$$
$$-\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4}\sum_1^n (\epsilon(r_i))^2 = 0$$
$$\text{when } \hat{\sigma^2} = \frac{1}{n}\sum_1^n (\epsilon(r_i))^2$$

The model parameters are now estimated by minimizing $\sum_1^n (\epsilon(r_i))^2$.

2.2 K-L information and Different Model Selection Approaches

The Kullback-Leibler (K-L) information between models f and g is defined for continuous functions as the (usually multi-dimensional) integral. This gives the information lost when g is used to approximate f.

$$I(f, g) = \int f(x) \log\left(\frac{f(x)}{g(x|\theta)}\right) dx,$$

where log denotes the natural logarithm.

Note that the data is integrated out, therefore $I(f, g)$ does not depend on the data if θ is known. Normally, θ is unknown and must be estimated by the Maximum Likelihood Estimator (MLE).

There are different model selection approaches.

1. Statistical hypothesis testing tests the null hypothesis. This provides little information of scientific interest whether rejected or not (Burnham 2002 [3]). In particular, hypothesis testing for model selection is often poor (Akaike 1981 [?]) There is no statistical theory that supports the notion that hypothesis testing with a fixed α level is a basis for model selection. Furthermore, under Monte-Carlo simulations and other data analysis scenarios, they have performed poorly in selecting an appropriate model for inference and prediction (Burnham 2002 [3]).

2. Akaike's Information Criterion (AIC) estimates the relative Kullback-Liebler distance between truth $f(x)$ and the approximating model. This gives a method to determine which model captures the greatest amount of information in the data without over-fitting the data.

3. Bayesian Information Criterion (BIC) was developed based on the assumptions that an exactly "true model" exists, that it is one of the candidate models being considered, and that the model selection goal is to select the "true" model. Implicit is the assumption that "truth" is of fairly low dimension and that this dimension and the data-generating (true) model has a fixed sample size.

2.3 AIC and BIC

Akaike's Information Criterion (AIC) uses the Kullback-Leibler (K-L) information as a fundamental basis for model selection. This paper will approach model selection in the special case of least squares parameter estimation with normally distributed errors with constant variance. In this special case, AIC can be computed as

AIC $= n\log(\hat{\sigma}^2) + 2K$

where

$\hat{\sigma}^2 = \frac{\sum \hat{\epsilon}_i^2}{n}$ (the Maximum Likelihood Estimate (MLE) of σ^2)

$\hat{\epsilon}_i^2$ are the estimated residuals for a particular candidate model

n is the sample size

K is the number of parameters including $\hat{\sigma}^2$

AIC may perform poorly if there are too many parameters in relation to the size of the sample. Burnham [3] advocates the use of AICc when the ratio $n/K < 40$. Sugiura (1978) [6] derived a second order variant of AIC given by

$$\text{AICc} = \text{AIC} + \frac{2K(K+1)}{n-K-1}$$

The Baysian Information Criterion (BIC) can be computed as

$$\text{BIC} = n\log(\hat{\sigma}^2) + (\log n)\, K$$

3 Underlying scientific models

The momentum and conservation of mass equations in cylindrical coordinates (r, θ, z) are given by

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + \frac{v}{r}\frac{\partial u}{\partial \theta} - \frac{v^2}{r} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial r}$$
$$+\kappa\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) - \frac{u}{r^2} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} - \frac{2}{r^2}\frac{\partial v}{\partial \theta} + \frac{\partial^2 u}{\partial z^2}\right)$$
$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial r} + \frac{v}{r}\frac{\partial v}{\partial \theta} + \frac{uv}{r} + w\frac{\partial v}{\partial z} =$$
$$+\kappa\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right) - \frac{v}{r^2} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} + \frac{2}{r^2}\frac{\partial u}{\partial \theta} + \frac{\partial^2 v}{\partial z^2}\right)$$
$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial r} + \frac{v}{r}\frac{\partial w}{\partial \theta} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z}$$
$$+\kappa\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} + \frac{\partial^2 w}{\partial z^2}\right) + gb(r,z)$$
$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z} = 0$$

where ρ is the density of air and g is gravity.

Under the assumptions of time independence ($t = 0$), axisymmetry ($\theta = 0$), pressure dependence only on r and z, and body force b(r, z) due to buoyancy alone, the equations in cylindrical coordinates (r, z) become:

$$u\frac{\partial u}{\partial r} - \frac{v^2}{r} + w\frac{\partial u}{\partial z} =$$
$$-\frac{1}{\rho}\frac{\partial p}{\partial r} + \kappa\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) - \frac{u}{r^2} + \frac{\partial^2 u}{\partial z^2}\right)$$
$$u\frac{\partial v}{\partial r} + \frac{uv}{r} + w\frac{\partial v}{\partial z} = \kappa\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right) - \frac{v}{r^2} + \frac{\partial^2 v}{\partial z^2}\right)$$
$$u\frac{\partial w}{\partial r} + w\frac{\partial w}{\partial z} =$$
$$-\frac{1}{\rho}\frac{\partial p}{\partial z} + \kappa\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) + \frac{\partial^2 w}{\partial z^2}\right) + gb(r,z)$$
$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z} = 0$$

For the purposes of this study, the tangential component of the vortex is assumed known.

4 The Cylindrical Case

The tangential velocity is assumed known and a function of r only. In this cylindrical case, the vertical vorticity is given by

$$\zeta = \frac{\partial v}{\partial r} + \frac{v}{r}$$

Therefore, the vertical vorticity ζ is a function of r only. The tangential momentum equation becomes

$$u\frac{\partial v}{\partial r} + \frac{uv}{r} = \kappa\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv)\right)\right)$$
$$u\zeta = \kappa\left(\frac{\partial \zeta}{\partial r}\right), r > 0$$

and this gives a solution for u given by

$$u = \kappa\left(\frac{\frac{\partial \zeta}{\partial r}}{\zeta}\right), \zeta \neq 0$$

Therefore, the radial velocity u is a function of r only. Using the conservation of mass equation, the vertical velocity w is found by

$$\frac{\partial w}{\partial z} = -\left(\frac{\partial u}{\partial r} + \frac{u}{r}\right)$$
$$w = -\left(\frac{\partial u}{\partial r} + \frac{u}{r}\right) z$$

Under the assumption that pressure is a function of r alone, the radial momentum equation can be used to compute pressure.

Let $\Lambda = \frac{\partial u}{\partial r} + \frac{u}{r}$ so that $w = -z\Lambda$, since

$$w = -\left(\frac{\partial u}{\partial r} + \frac{u}{r}\right) z$$

$$u\frac{\partial u}{\partial r} - \frac{v^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \kappa\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(ru)\right)\right)$$

which can be seen to be

$$u\frac{\partial u}{\partial r} - \frac{v^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \kappa\left(\frac{\partial \Lambda}{\partial r}\right)$$

so that

$$\frac{\partial p}{\partial r} = \rho\left(-u\frac{\partial u}{\partial r} + \frac{v^2}{r} + \kappa\left(\frac{\partial \Lambda}{\partial r}\right)\right)$$

Solving this differential equation gives

$$p(r) = \rho\left(-u^2/2 + \int_0^r \frac{v^2}{s} ds + \kappa\Lambda\right) + p(0)$$

Now use the vertical momentum equation to solve for the buoyancy $b(r, z)$

$$u\frac{\partial w}{\partial r} + w\frac{\partial w}{\partial z} = \kappa\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) + \frac{\partial^2 w}{\partial z^2}\right) + gb(r, z)$$

$$gb(r, z) = u\frac{\partial w}{\partial r} + w\frac{\partial w}{\partial z} - \kappa\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) + \frac{\partial^2 w}{\partial z^2}\right)$$

$$gb(r, z) = -uz\frac{\partial \Lambda}{\partial r} + z\Lambda^2 + \kappa z\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \Lambda}{\partial r}\right)\right)$$

5 Vortex candidate models

5.1 The Wood-White tangential velocity vortex models

The radial, vertical velocity, vorticity, pressure and buoyancy terms are calculated as in the previous section for the Wood-White vortex models. For all the Wood-White models, k mainly controls the inner core of the vortex with values that allow for both a one- and two-cell structure. The parameter n mainly governs the decay of the vortex at radial distances beyond the location of maximum tangential winds. The parameter λ controls the curvature of the vortex. The form of the tangential velocity profile $\phi(r)$ is given below for each model.

1. The Wood-White vortex 1.

$$\phi(r) = \frac{r}{1 + r^n}$$

2. The Wood-White vortex 2.

$$\phi(r) = \frac{r^k}{1 + r^n}$$

3. The Wood-White vortex 3.

$$\phi(r) = \frac{r}{\left(1 + r^{\frac{n}{\lambda}}\right)^\lambda}$$

4. The Wood-White vortex 4.

$$\phi(r) = \frac{r^k}{\left(1 + r^{\frac{n}{\lambda}}\right)^\lambda}$$

5. The Wood-White vortex 5.

$$\phi(r) = \frac{r^k + r}{1 + r^n}$$

6. The Wood-White vortex 6.

$$\phi(r) = \frac{r^k + r}{\left(1 + r^{\frac{n}{\lambda}}\right)^\lambda}$$

5.2 The Rankine combined vortex

$$u(r, \theta, z) = 0$$

$$v(r, \theta, z) = \begin{cases} \frac{V_x r}{R_x} & \text{if } r \leq R_x \\ \frac{V_x R_x}{r} & \text{if } r > R_x \end{cases}$$

$$w(r, \theta, z) = 0$$

where V_x is the maximum tangential velocity magnitude and R_x is the radius of the vortex core. $\frac{V_x}{R_x}$ is the angular velocity of the solid body rotation.

Since the angular momentum at infinity, Γ_∞ is

$$\lim_{r\to\infty} 2\pi r v = \Gamma_\infty,$$

then these equations can be written as

$$u(r, \theta, z) = 0$$

$$v(r, \theta, z) = \begin{cases} \frac{\Gamma_\infty r}{2\pi R_x^2} & \text{if } r \leq R_x \\ \frac{\Gamma_\infty}{2\pi r} & \text{if } r > R_x \end{cases}$$

$$w(r, \theta, z) = 0$$

In this model the vertical vorticity is given by

$$\zeta = \frac{\partial v}{\partial r} + \frac{v}{r} = \begin{cases} \frac{2V_x}{R_x} & \text{if } r \leq R_x \\ 0 & \text{if } r > R_x \end{cases}$$

Under the assumption of cyclostrophic balance, for $r \leq R_x$ the pressure term can be calculated by

$$\frac{\partial p}{\partial r} = \rho\frac{v^2}{r} = \rho\frac{V_x^2 r}{R_x^2}$$

Therefore, p is a function of r alone.

For $r > R_x$ the pressure term can be calculated by

$$\frac{\partial p}{\partial r} = \rho\frac{v^2}{r} = \rho\frac{V_x^2}{R_x^2 r^3}$$

Therefore, p is a function of r alone.

In summary, for the combined Rankine vortex, pressure is given by

$$p(r) = \begin{cases} p(0) + \frac{\rho V_x^2}{R_x^2}\frac{r^2}{2} & \text{if } r \leq R_x \\ p(0) + \frac{2\rho V_x^2}{R_x^2}\frac{r^2}{2} - \frac{\rho V_x^2 R_x^2}{2r^2} & \text{if } r > R_x \end{cases}$$

The parameters in this vortex are R_x and V_x.

5.3 The Burgers-Rott vortex

$$u(r,\theta,z) = -ar$$
$$v(r,\theta,z) = \frac{\Gamma_\infty}{2\pi r}\left(1 - e^{-\frac{ar^2}{2\kappa}}\right)$$
$$w(r,\theta,z) = 2az$$

where a is the strength of the suction, and Γ_∞ is

$$\lim_{r\to\infty} 2\pi r v = \Gamma_\infty,$$

Setting $\frac{\partial v}{\partial r} = 0$ gives maximum tangential winds when $\frac{1}{2\pi}\left(-\frac{1}{r^2} + \frac{1}{r^2}e^{-\frac{ar^2}{2\kappa}} + \frac{a}{\kappa}e^{-\frac{ar^2}{2\kappa}}\right) = 0$

This can be used to add parameters V_x and R_x.

$$p(r,z) = p_0 + \rho\int_0^r \frac{v^2}{s}ds - \rho\frac{a^2}{2}\left(r^2 + 4z^2\right)$$

The parameters in this vortex are R_x and V_x.

5.4 Vatistas vortex

The radial, vertical velocity, vorticity, pressure and buoyancy terms are calculated as in the previous section for the Vatistas vortex model. The form of the tangential velocity profile $\phi(r)$ is given below.

$$\phi(r) = \frac{r}{(1 + r^{2q})^{\frac{1}{q}}}$$

5.5 The Sullivan vortex

$$u(r) = -ar + \frac{6\kappa}{r}\left(1 - e^{-\frac{ar^2}{2\kappa}}\right)$$
$$v(r) = \frac{\Gamma_\infty}{2\pi H(\infty) r} H\left(\frac{ar^2}{2\kappa}\right)$$
$$w(r) = 2az\left(1 - 3e^{-\frac{ar^2}{2\kappa}}\right)$$

where $\Gamma_\infty = \lim_{R\to\infty} v(R)\cdot 2\pi R$ and

$$H(x) = \int_0^x exp\left[-\beta + 3\int_0^\beta (1 - e^{-s})\frac{1}{s}ds\right] d\beta$$

The vertical vorticity is computed as follows

$$\frac{\partial H(r)}{\partial r} = exp\left(-r + 3\int_0^r \frac{1 - e^{-s}}{s}ds\right)$$
$$\frac{\partial}{\partial r}\left(H\left(\frac{ar^2}{2\kappa}\right)\right) =$$
$$\frac{ar}{\kappa} exp\left(-\frac{ar^2}{2\kappa} + 3\int_0^{\frac{ar^2}{2\kappa}} \frac{1 - e^{-s}}{s}ds\right)$$
$$\zeta = \frac{\Gamma_\infty}{2\pi r H(\infty)}\frac{\partial}{\partial r}\left(H\left(\frac{ar^2}{2\kappa}\right)\right)$$
$$\zeta = \frac{a\Gamma_\infty}{2\pi\kappa H(\infty)} exp\left(-\frac{ar^2}{2\kappa} + 3\int_0^{\frac{ar^2}{2\kappa}} \frac{1 - e^{-s}}{s}ds\right)$$

The azimuthal vorticity for is given by the equation

$$\omega_\theta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial r} = \frac{-6a^2 rz}{\kappa}e^{-\frac{ar^2}{2\kappa}}$$

The pressure computations are as follows:

$$u\frac{\partial\Lambda}{\partial r} - \Lambda^2 - \kappa\left(\frac{\partial^2\Lambda}{\partial r^2} + \frac{1}{r}\frac{\partial\Lambda}{\partial r}\right) = -4a^2$$
$$p(r,z) = p_0 + \rho\int_0^r \frac{v^2}{s}ds - \rho\frac{a^2}{2}\left(r^2 + 4z^2\right) - \frac{18\rho\kappa^2}{r^2}\left(1 - e^{-\frac{ar^2}{2\kappa}}\right)^2$$

The parameters in this vortex are R_x and V_x.

6 Normalized vortex candidate models

These candidate models have been normalized so that the maximum tangential velocity is equal to one at the distance $\rho = 1$ from the center of the vortex.

1. The Wood-White vortex 1

$$\Phi(\rho) = \frac{n\rho}{n-1+\rho^n}$$

2. The Wood-White vortex 2

$$\Phi(\rho) = \frac{n\rho^k}{n-k+k\rho^n}$$

3. The Wood-White vortex 3

$$\Phi(\rho) = \frac{\rho}{\left(1+\frac{1}{n}\left(\rho^{\frac{n}{\lambda}}-1\right)\right)^{\lambda}}$$

4. The Wood-White vortex 4

$$\Phi(\rho) = \frac{\rho^k}{\left(1+\frac{k}{n}\left(\rho^{\frac{n}{\lambda}}-1\right)\right)^{\lambda}}$$

5. The Wood-White vortex 5

$$\Phi(r) = \frac{(rr_\star)^k + rr_\star}{1+(rr_\star)^n} \Big/ \frac{(r_\star)^k + r_\star}{1+(r_\star)^n}$$

   where $r_\star$ is the radius where $\phi(r)$ is maximum. Notice that all of the Wood-White tangential vortex models have a maximum value as long as k is less than n.

6. The Wood-White vortex 6

$$\Phi(r) = \frac{(rr_\star)^k + rr_\star}{\left(1+(rr_\star)^{\frac{n}{\lambda}}\right)^{\lambda}} \Big/ \frac{(r_\star)^k + r_\star}{\left(1+(r_\star)^{\frac{n}{\lambda}}\right)^{\lambda}}$$

   where $r_\star$ is the radius where $\phi(r)$ is maximum.

6.1 The Rankine combined vortex

$$v(r,\theta,z) = \begin{cases} ar & \text{if } r \leq 1 \\ \frac{a}{r} & \text{if } r > 1 \end{cases}$$

6.2 The Burgers-Rott vortex

The following equation with parameters a and b is normalized to have a maximum value of 1 at $r = 1$:

$$v(r,\theta,z) = \frac{b}{r}\left(1 - e^{-\frac{ar^2}{2\kappa}}\right)$$

6.3 Vatistas vortex

The normalized form of $\phi(r)$ is given by:

$$\phi(r) = \frac{2^{\frac{1}{q}} r}{(1+r^{2q})^{\frac{1}{q}}}$$

6.4 The Sullivan vortex

The following equation with parameters a and b is normalized to have a maximum value of 1 at $r = 1$:

$$v(r) = \frac{b}{r} H\left(\frac{ar^2}{2\kappa}\right)$$
$$H(x) = \int_0^x exp\left[-\beta + 3\int_0^\beta (1-e^{-s})\frac{1}{s}ds\right] d\beta$$

7 Analysis and Results

The data sets used in the analysis were provided by Vincent T. Wood of NOAA/OAR/National Severe Storms Laboratory. These data include data from the 2008 hurricane, Ike, a numerical model from Trapp (1999) [8] and a numerical model from Davies-Jones (1997) [9]. A description of the analysis performed on each vortex data set is given below. A table summarizes the analysis of each data set. The table contains the model number as described in the previous section, the number of parameters K, AIC, AICc, ΔAIC, ΔAICc, and the Akaike weights, w_i for each model. ΔAIC and ΔAICc are computed over all candidate models in the set as follows:

$$\Delta \text{AIC}_i = \text{AIC}_i - \text{AIC}_{min}$$
$$\Delta \text{AICc}_i = \text{AICc}_i - \text{AICc}_{min}$$

Models with ΔAIC > 19 (or ΔAICc > 10) have either essentially no support, and might be eliminated from further consideration, or at least those models fail to explain some substantial explainable variation in the data (Burnham). It is not the absolute size of the AIC value, it is the relative values that are important (Burnham). The Akaike weights w_i are used to measure the relative likelihood of a model, given the data and the candidate set of R models. The Akaike weights are calculated as follows:

$$w_i = \frac{\exp\left(-\frac{1}{2}\Delta_i\right)}{\sum_{r=1}^R \exp\left(-\frac{1}{2}\Delta_r\right)}.$$

Each model's parameters were estimated using a least squares fit to the data. In order to put all the models in the table, the variable a was used for k, the variable b was used for n and the variable c was used for λ in the Wood-White models. Also, the variables a and b were used for the Rankine, Burgers-Rott, Vatistas and the Sullivan models. The following tables summarize the AIC values computed for all the vortex models under consideration. In the first table the fitted parameters are given for the normalized models. The range of values for the Wood-White models were [0.5, 5,5] for k, [0.6, 8] for n with $n > k$ and [0.2, 2.7] for λ. In the second table, AIC is used to select the best approximating model (or models).

7.1 Ike040 hurricane data

Figure 1 contains all the normalized models under consideration fitted to data from the hurricane Ike040 for comparative purposes.

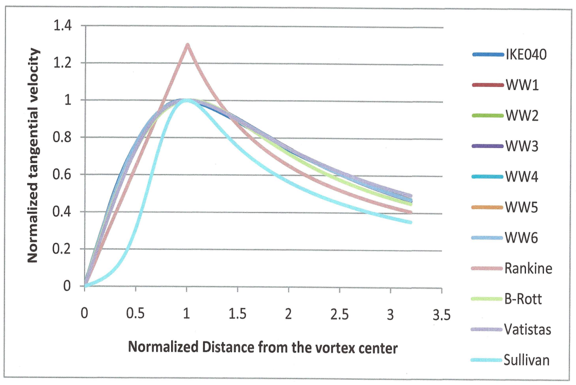

Figure 1: All models fitted to Ike040.

Model	a	b	c	σ^2
Wood-White1	NA	2.28	NA	0.00007
Wood-White2	0.98	2.30	NA	0.00007
Wood-White3	NA	2.32	1.04	0.00007
Wood-White4	0.96	2.25	0.93	0.00006
Wood-White5	0.97	2.30	NA	0.00005
Wood-White6	3.10	4.30	2.03	0.00005
Rankine	1.30	NA	NA	0.01078
Burgers-Rott	5.08	1.72	NA	0.00061
Vatistas	1.43	NA	NA	0.00031
Sullivan	2.40	1.17	NA	0.04327

AIC was used for the model selection because there are 301 data points. Results are given in the table below.

Model	K	ΔAIC	w
Wood-White1	2	76.3	0.000000
Wood-White2	3	55.9	0.000000
Wood-White3	3	68.4	0.000000
Wood-White4	4	48.5	0.000000
Wood-White5	3	0.0	0.810027
Wood-White6	4	2.8	0.189973
Rankine	1	1592.2	0.000000
Burgers-Rott	3	731.0	0.000000
Vatistas	2	525.1	0.000000
Sullivan	3	2014.5	0.000000

The Wood-White1, Wood-White2, Wood-White3, Wood-White4, Rankine, Burgers-Rott, Vatistas and Sullivan models all have ΔAIC values greater than 19 and could be eliminated from further consideration. The Wood-White5 model has a relative likelihood of 86% that this model is the best model for this set of data from this candidate list of models. The Wood-White6 model's ΔAIC is small enough to be considered for model averaging with the Wood-White5 model for this data set.

7.2 Ike180 hurricane data

Figure 2 contains all the normalized models under consideration fitted to data from the hurricane Ike180 for comparative purposes.

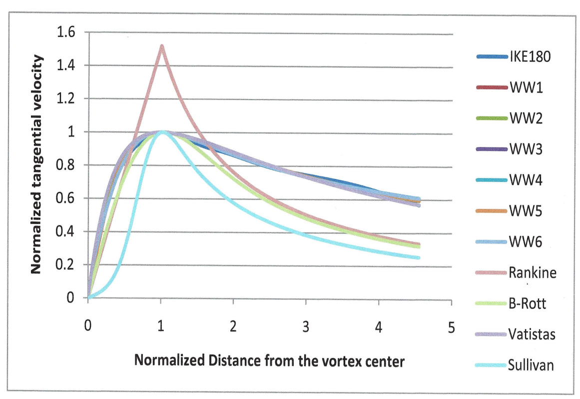

Figure 2: All models fitted to Ike180.

Model	a	b	c	σ^2
Wood-White1	NA	1.63	NA	0.00024
Wood-White2	1.03	1.62	NA	0.00024
Wood-White3	NA	1.58	0.88	0.00022
Wood-White4	0.81	1.30	0.54	0.00016
Wood-White5	1.11	1.56	NA	0.00024
Wood-White6	0.67	1.37	0.58	0.00016
Rankine	1.52	NA	NA	0.04947
Burgers-Rott	4.91	1.66	NA	0.03846
Vatistas	0.54	NA	NA	0.00085
Sullivan	5.04	2.69	NA	0.10515

AIC was used for the model selection because there are 301 data points. Results are given in the table below.

Model	K	ΔAIC	w
Wood-White1	2	129.7	0.000000
Wood-White2	3	121.1	0.000000
Wood-White3	3	92.9	0.000000
Wood-White4	4	0.0	0.808755
Wood-White5	3	120.5	0.000000
Wood-White6	4	2.9	0.191245
Rankine	1	1726.3	0.000000
Burgers-Rott	3	1654.1	0.000000
Vatistas	2	504.3	0.000000
Sullivan	3	1956.8	0.000000

In this case, all of the models could be eliminated from further consideration except for Wood-White4 and the modified Wood-White2. These two models are very similar in form. The Wood-White4 model has a relative likelihood of 81% that this model is the best model for this set of data from this candidate list of models. The modified Wood-White2 model's ΔAIC is small enough to be considered for model averaging with the Wood-White4 model for this data set.

7.3 Ike270 hurricane data

Figure 3 contains all the normalized models under consideration fitted to data from the hurricane Ike270 for comparative purposes.

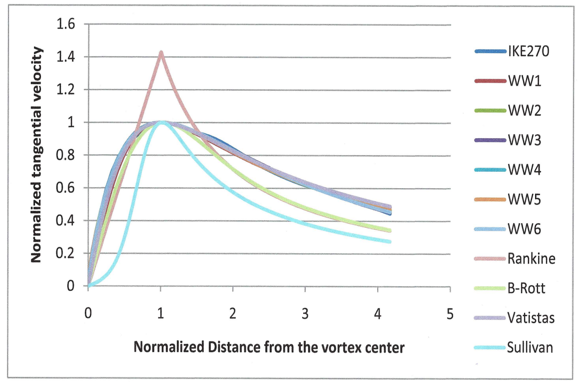

Figure 3: All models fitted to Ike270.

Model	a	b	c	σ^2
Wood-White1	NA	1.92	NA	0.00108
Wood-White2	0.71	1.83	NA	0.00010
Wood-White3	NA	1.98	2.75	0.00012
Wood-White4	0.80	2.24	1.55	0.00008
Wood-White5	0.57	1.90	NA	0.00014
Wood-White6	0.70	2.40	1.70	0.00007
Rankine	1.43	NA	NA	0.02567
Burgers-Rott	3.80	1.97	NA	0.01300
Vatistas	0.85	NA	NA	0.00079
Sullivan	4.57	0.87	NA	0.07255

AIC was used for the model selection because there are 301 data points. Results are given in the table below.

Model	K	ΔAIC	w
Wood-White1	2	810.4	0.000000
Wood-White2	3	104.8	0.000000
Wood-White3	3	141.8	0.000000
Wood-White4	4	16.04	0.000329
Wood-White5	3	185.2	0.000000
Wood-White6	4	0.0	0.999671
Rankine	1	1761.1	0.000000
Burgers-Rott	3	1560.3	0.000000
Vatistas	2	714.1	0.000000
Sullivan	3	2077.9	0.000000

In this case, all of the models could be eliminated from further consideration except for Wood-White4 and the modified Wood-White2. These two models are very similar in form. The modified Wood-White2 model has a relative likelihood of approximately 100% that this model is the best model for this set of data from this candidate list of models. The Wood-White4 model's ΔAIC is borderline and probably would not be selected for model averaging with the modified Wood-White2 model for this data set.

7.4 Trapp numerical data

Figure 4 contains all the normalized models under consideration fitted to data from the Trapp numerical model for comparative purposes.

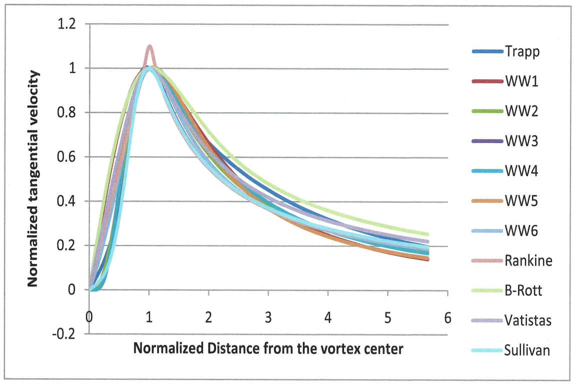

Figure 4: All models fitted to Trapp.

Model	a	b	c	σ^2
Wood-White1	NA	2.69	NA	0.01101
Wood-White2	2.58	3.80	NA	0.00214
Wood-White3	NA	2.03	0.18	0.00645
Wood-White4	4.96	6.38	2.75	0.00185
Wood-White5	2.04	3.33	NA	0.00486
Wood-White6	1.85	2.71	0.44	0.00456
Rankine	1.10	NA	NA	0.00664
Burgers-Rott	3.90	0.81	NA	0.01640
Vatistas	3.05	NA	NA	0.00808
Sullivan	4.90	2.42	NA	0.00431

The small sample size AIC_c was used because there are 35 data points. Results are given in the table below.

Model	K	ΔAICc	w
Wood-White1	2	104.4	0.000000
Wood-White2	3	6.8	0.032195
Wood-White3	3	74.0	0.000000
Wood-White4	4	0.0	0.967805
Wood-White5	3	56.7	0.000000
Wood-White6	4	55.2	0.000000
Rankine	1	71.4	0.000000
Burgers-Rott	3	131.0	0.000000
Vatistas	2	85.6	0.000000
Sullivan	3	49.4	0.000000

In this case, the Wood-White models 2 and 4 were the best models. Since the Wood-White4's relative likelihood of 95% that this model is the best model for this set of data from this candidate list of models and this model is a refinement of Wood-White2, it is likely that Wood-White4 would be selected as the best model for this data set.

7.5 Davies-Jones numerical data

Figure 5 contains all the normalized models under consideration fitted to data from the Davies-Jones numerical model for comparative purposes.

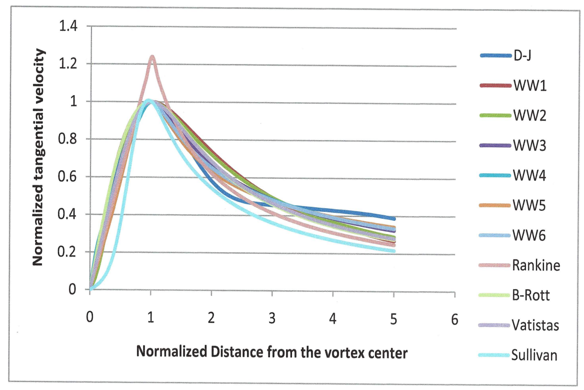

Figure 5: All models fitted to Davies-Jones.

Model	a	b	c	σ^2
Wood-White1	NA	2.32	NA	0.00743
Wood-White2	1.49	2.60	NA	0.00592
Wood-White3	NA	1.79	0.25	0.00302
Wood-White4	0.72	1.45	0.11	0.00211
Wood-White5	4.61	5.24	NA	0.00241
Wood-White6	5.00	5.71	1.16	0.00217
Rankine	1.24	NA	NA	0.00803
Burgers-Rott	4.54	1.80	NA	0.00501
Vatistas	2.04	NA	NA	0.00431
Sullivan	2.70	0.71	NA	0.02152

The small sample size AIC_c was used because there are 51 data points. Results are given in the table below.

Model	K	ΔAIC	w
Wood-White1	2	59.7	0.000000
Wood-White2	3	50.3	0.000000
Wood-White3	3	16.1	0.000201
Wood-White4	4	0.0	0.631349
Wood-White5	3	4.4	0.068340
Wood-White6	4	1.5	0.300110
Rankine	1	61.5	0.000000
Burgers-Rott	3	41.8	0.000000
Vatistas	2	31.9	0.000000
Sullivan	3	116.2	0.000000

In this case, three models have a low AIC_c, however the modified Wood-White2 is a refinement of the modified Wood-White1, so only the Wood-White4 and the modified Wood-White2 would be considered for model averaging for this data set.

8 Conclusions and Future Work

AIC consistently selects a Wood-White model as the model that best captures the information in the data in all the cases that were examined. Future work will involve a comparison of these predicted profiles with simulated or real data.

References

[1] Akaike, H., 1981: Likelihood of a model and information criteria. *Journal of Econometrics*, **16**, 3-14.

[2] Burgers, J. M., 1948: A mathematical model illustrating the theory of turbulence. *Adv. Appl. Mech.*, **1**, 197-199.

[3] Burnham, K. P. and Anderson, D. R., 2002: *Model Selection and Multimodel Inference: A Practical Information-Theoretic Approach*, (2nd edition), Springer-Verlag, New York.

[4] Rankine, W. J. M., 1882: *A Manual of Applied Physics*, (10th edition), Charles Griff and Company, London.

[5] Rott, N., 1958: On the viscous core of a line vortex. *Z. Angew Math. Physik*, **9**, 543-553.

[6] Sugiura, N., 1978: Further analysis of the data by Akaike's information criterion and the finite corrections. *Communications in Statistics, Theory and Methods*, **A7**, 13-26.

[7] Sullivan, R. D., 1959: A two-cell vortex solution of the Navier-Stokes equations. *J. Aerospace Sci.*, **26**, 767-768.

[8] Trapp, R. Jeffrey, 1999: A clarification of vortex breakdown and tornadogenesis. *Monthly Weather Review*, **128**, 888-895.

[9] Trapp, R. Jeffrey and Davies-Jones, R. P., 1997: Tornadogenesis with and without a dynamic pipe effect. *J. Atmos. Sci.*, **54**, 113-133.

[10] Wood, Vincent T. and White, Luther W., 2011: A new parametric model of vortex tangential-wind profiles: development, testing and verification. *J. Atmos. Sci.*, **68**, 990-1006.