Microwave scattering by hexagonal ice crystals, and the implications for interpreting dual-polarisation radar measurements

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1. Summary of key findings:

- Want to accurately predict dual polarisation radar parameters for hexagonal ice crystals
- Discrete dipole approximation calculations show that modelling hexagonal crystals as spheroids can give errors up to 1.5dB in differential reflectivity Z_{DR}
- Empirical modification of Gans theory allows very accurate prediction of scattering from hexagonal crystals using simple analytical formulae
- Complex branched and dendritic crystals can be captured using the same formula and a reduced permittivity

2. Background

- Dual polarisation radar is a powerful tool to probe oriented ice crystals
- Need an accurate scattering model for quantitative interpretation.
- In the Rayleigh limit (crystal size ≪ wavelength) we just need to calculate the "polarisability tensor" **X** which relates the dipole moment **p** induced in the ice crystal to the applied electric field $\mathbf{E_0}$: $\mathbf{p} = 4\pi \mathbf{X}\mathbf{E_0}$
- Then the co-polar radar cross section is then simply

$$\sigma_{co} = 4\pi k^4 \left|(\mathbf{X}\hat{\mathbf{E}}_0)\cdot\hat{\mathbf{E}}_0\right|^2$$

where $\hat{\mathbf{E}}_0$ represents the polarisation of the radar pulse, and k is the wavenumber

- The problem is we don't know what **X** is for ice crystals – usually we approximate them as spheroids instead. For a spheroid Gans (1912) determined the polarisability tensor exactly
- If we choose our coordinate system so that the principal axes are parallel to x, y, z then **X** is diagonal with elements $X_{ii} = \frac{V}{4\pi} \times \frac{\varepsilon - 1}{L_i(\varepsilon - 1) + 1}$ — V is the volume of the particle, ε is the permittivity of ice
- The shape functions L_i are dimensionless and depend on aspect ratio:

$$L_z = \frac{1-e^2}{e^2}\left(-1 + \frac{1}{2e}\ln\frac{1+e}{1-e}\right)$$ for prolate particles

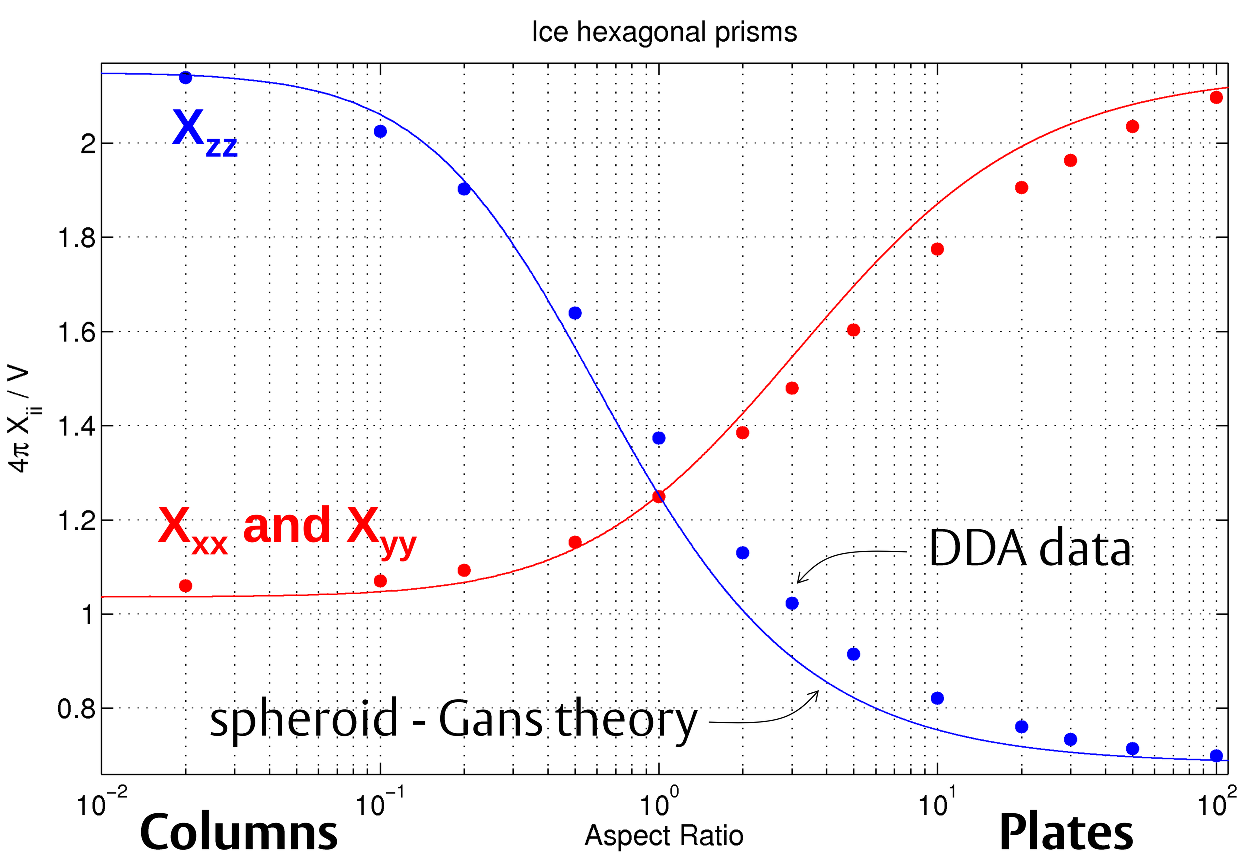
$$L_z = \frac{1+e^2}{e^2}\left(1 - \frac{1}{e}\tan^{-1} e\right)$$ for oblates

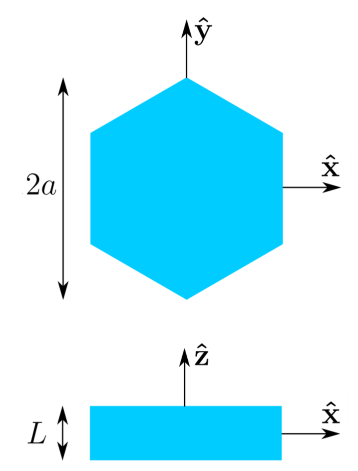
$$e = \sqrt{1 - \text{aspect ratio}^2} \qquad L_x = L_y = \tfrac{1}{2}(1 - L_z)$$

- We frequently apply these formulae to hexagonal crystals – but is that a good approximation?

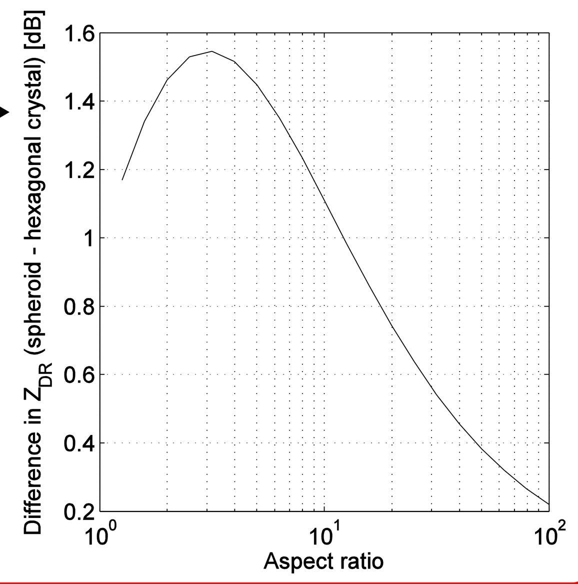
3. DDA Calculations

- I used the discrete dipole approximation (DDA) to calculate **X** for hexagonal crystals, and compared the results to the exact Gans solution for spheroids:





- Good *qualitative* agreement. What about *quantitatively*?
- Example for 3:1 plate - dipole moment is 12% too weak along minor (z) axis and 5% too strong on major (x,y) axes – sounds acceptable.
- But radar cross-section σ_{co} is proportional to $\mathbf{X}^2$. So for horizontal orientation σ_{HH} is 25% too small, σ_{VV} is 10% too big, **so Z_{DR} is overestimated by 40% (1.5dB)**. ***Not very accurate!***
  - Consider horizontally oriented plate crystals. Let's plot the error in Z_{DR} from approximating them as a spheroid *vs* aspect ratio →
  - Max error ≈ 1.5dB at aspect ratio of 3
  - Error > 1dB for aspect ratios < 10
  - Agreement is better as aspect ratio becomes more extreme
  - Equivalent calculation for columns shows that spheroids work better for columns than for plates: differences are < 0.5dB
  - Differential phase shift $K_{DP} \propto (X_{xx} - X_{zz})$ ~ max error of 17%



4. Empirical modification of Gans theory

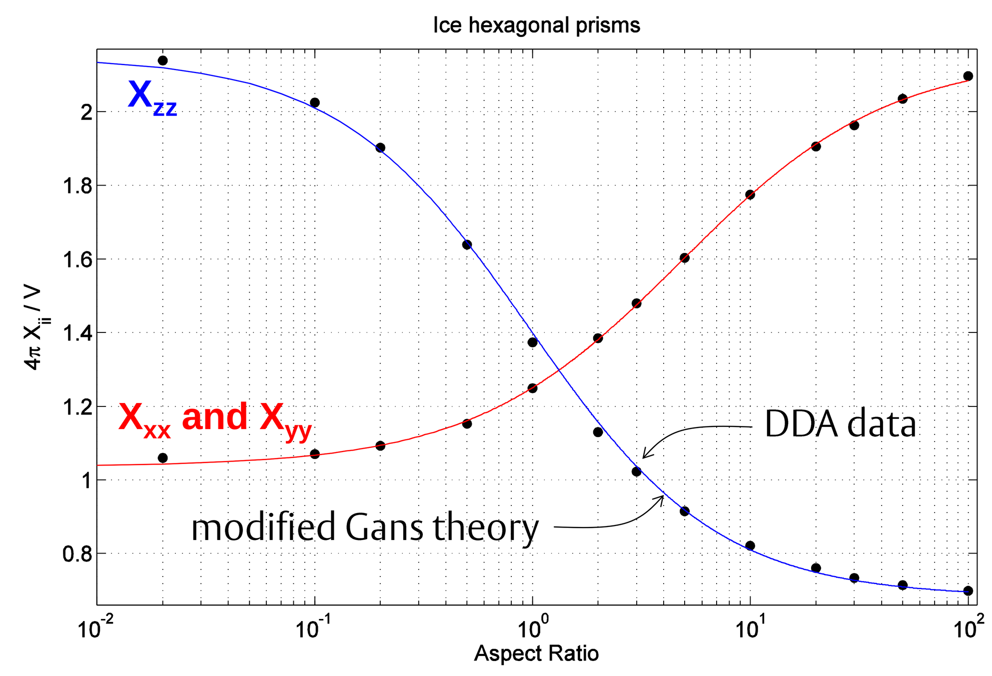
- Results for spheroids were *qualitatively* OK, but *quantitatively* inaccurate
- Suggests it might be possible to modify Gans approximation. Let's assume that the polarisability tensor has the same form $X_{ii} = \frac{V}{4\pi} \times \frac{\varepsilon - 1}{L_i(\varepsilon - 1) + 1}$ but with different shape functions L_i
- Invert this equation and use DDA data to determine L_i , and fit a simple function to the data. Assume asymptotic limits are same as spheroid for thin plates and long needles. $(L_x \to 0, L_z \to 1)$ $(L_x \to \frac{1}{2}, L_z \to 0)$
- Simple fit:

$$L_z = \frac{1}{2}\left(\frac{1 - 3/w}{1 + 3/w} - 1\right)$$

$$L_x = L_y = \left(\frac{1 - 0.5w^{0.9}}{1 + 0.5w^{0.9}} - 1\right)$$

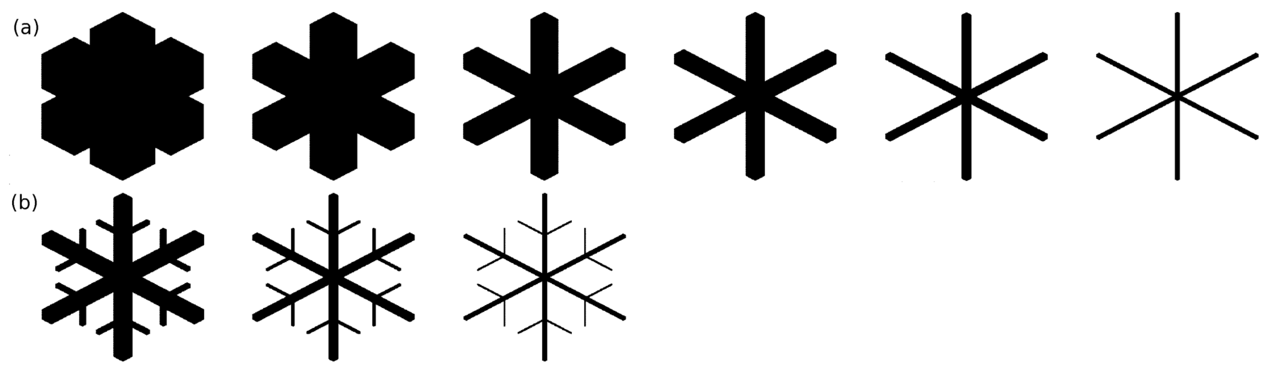
where w = aspect ratio

- Plug these functions into the Gans formula and compare to DDA data points
- RMS differences < 1% = **very accurate approximation**
- Simple modification, easily implemented in existing programs
- No numerical problems at w=1

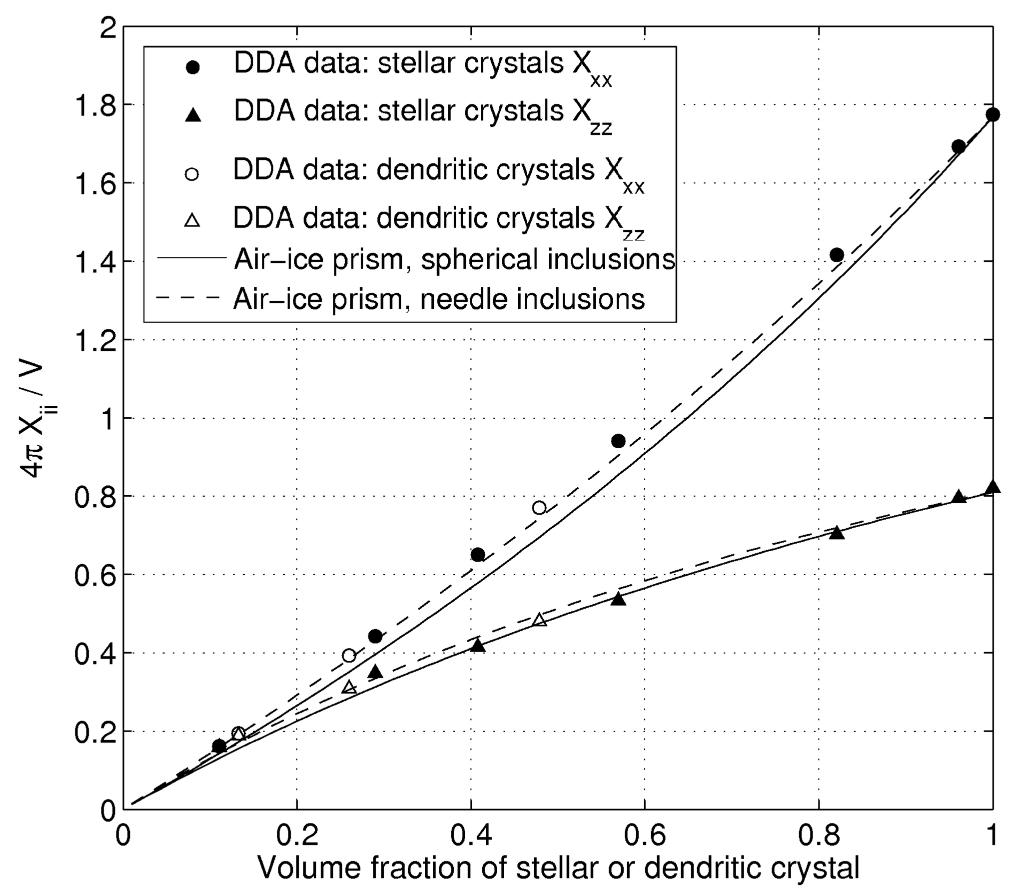


5. More complex branched crystals

- We can now calculate radar parameters for simple hexagonal prisms – but real crystals are often more complicated than this. Logical next step is branched / dendritic crystals
  - New set of DDA calculations for these shapes
  - aspect ratio of 10:1



- Question: could we approximate these complex particles as enclosing hexagonal prisms with a reduced permittivity?
- Use standard Maxwell-Garnett mixing theory to obtain the effective permittivity – depends on volume fraction of ice in the enclosing hexagon



- Plot shows polarisability tensor from DDA data (markers) and from the modified Gans theory with reduced permittivity (solid lines)
- **Agreement is excellent !**
- Can tweak parameter in the Maxwell-Garnett mixing theory to get slightly more accurate results (needle shaped ice inclusions vs spherical ones) = dashed line.
- Note $X_{xx} \approx X_{zz}$ when particles are very dilute, ie $Z_{DR} \to$ 0dB for the most tenuous crystals.

6. Conclusions

- Modelling hexagonal crystals as spheroids can lead to significant errors in dual-pol radar parameters, especially Z_{DR}
- A simple modification of Gans theory allows hexagonal prism crystal scattering to be accurately capured. Polarisability tensor approach allows you to determine scattering for any incident polarisation, co/cross-polar, in any scattering direction easily
- Branched planar crystals and dendrites can be accurately simulated by an enclosing hexagonal prism with a reduced permittivity via Maxwell-Garnett mixture theory

7. For more details...

preprint: tinyurl.com/hexradar

CD Westbrook 'Rayleigh scattering by hexagonal ice crystals and the interpretation of dual-polarisation radar measurements' *Q. J. R. Meteorol. Soc.*