314 DETERMININING THE GEOMETRY FOR THE COMMON COMPARISON OF RADAR SPACE

Zlatko R. Vukovic[1], Jim M.C. Young[1], and Norman Donaldson[2]
[1] Meteorological Service of Canada, Environment Canada
[2] Science and Technology Branch, Environment Canada

1. INTRODUCTION

The Minister of Environment Canada announced in January 2012 that the Meteorological Service of Canada's (MSC) weather radar network will receive $45.2 million for improved performance and upgrading to next-generation technology. With upgrades to the existing radar network, it is required to determine the consistency of our adjacent measurements and spatial coverage. A condition for the successful inter-radar comparison between two radars is the accurate time-space synchronization in the middle region where the comparison is the most effective.

The current Canadian weather radar network has an average inter-radar spacing of approximately 300 km. The long distances imply that an approach is needed that fully represents the Earth's geoid.

A review of the literature has indicated a number of statistical approaches to comparing data in the common radar volume. Here, we present a description of the theoretical basis for the geometrical evaluation of a common inter-radar space (CIS) utilizing a common spatial reference to offer a more accurate determination of the CIS.

2. THE REFERENCE FRAME FOR CIS

Since the radar locates the target volume in reference to the local reference system, it is necessary to convert local coordinates to a common "fixed" frame of reference. A geocentric coordinate system (GCS), in which the Earth is modeled as an oblate spheroid, was chosen as the prime framework with zero coordinates at the center of the Earth.

Schematic representation of the geometry of the two radars that measure the location of a common target at point C, is shown in Figure 1. Geometric position vectors of point C from the two radars relative to the locations of radars are the pink colored vectors **(** $\overrightarrow{A_hC}$ and $\overrightarrow{B_hC}$). Note that the pink vectors do not represent the curving rays of an actual radar beam. The local altitudes above the mean sea level are represented by green colored vectors ($\overrightarrow{AA_h}$ and $\overrightarrow{BB_h}$). The geodetic position vectors given in terms of geographic longitude, latitude, and height, of two surface points (A and B) of the geoid are shown as red colored vectors ($\overrightarrow{O_AA}$ and $\overrightarrow{O_BB}$) which are perpendicular to the local horizons. The origins of the vectors (O_A and O_B **)** vary with latitude, but a more practical approach uses the geocentric position vectors of two surface points which are given as blue colored vectors ($\overrightarrow{OA}$ and $\overrightarrow{OB}$).

The position vector of a common target point of two radars relative to the GCS, the bold black vector ($\overrightarrow{OC}$) in Figure 1, is the composition of the three vectors: a position vector from the geocentric origin to the surface of the oblate geoid at the geographical latitude and longitude of a radar location, an altitude vector that is normal to the oblate spheroid surface at the point of the radar location, which represents the height of the radar antenna above mean sea level, and a third vector that is a local position vector of the common target point measured from the radar. The location of the same target point from two radars in reference to the GCS is given as a composition of two sets of the corresponding three vectors:

$$\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{AA_h} + \overrightarrow{A_hC}$$
$$\overrightarrow{OC} = \overrightarrow{OB} + \overrightarrow{BB_h} + \overrightarrow{B_hC}$$

[1] *Corresponding author address:* Dr. Zlatko R. Vukovic, Meteorological Service of Canada, Environment Canada, 4905 Dufferin Street, Toronto, ON, CANADA M3H 5T4; email: Zlatko.Vukovic@ec.gc.ca

Since the two sets of three vectors describe the position of the same target with respect to the Earth centre, equating them results in one vector equation:

$$\overrightarrow{OA}+\overrightarrow{AA_h}+\overrightarrow{A_hC}=\overrightarrow{OB}+\overrightarrow{BB_h}+\overrightarrow{B_hC} \quad (1)$$

Equation 1 results in three scalar equations that we will use to define a target point that is equidistant from both radars.

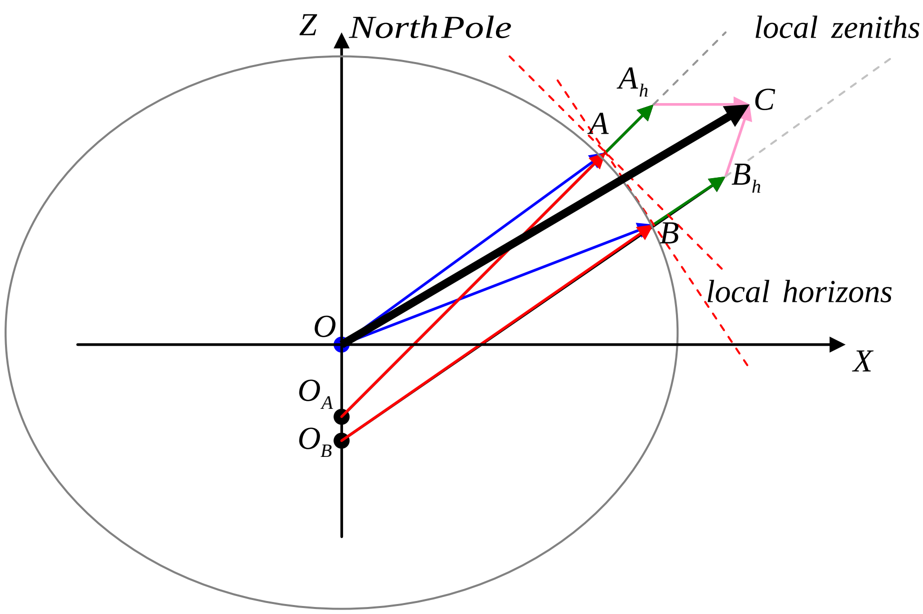

Figure 1. The geometric construction of the geocentric position vector of point C (bold black line) from the geodetic (red color), geocentric (blue color), and local point vectors (pink color) from two locations and altitudes (green color).

To obtain the explicit form of vector $\overrightarrow{OC}$ as a function of radar polar coordinates (radar measured azimuth, elevation, and distance of point C) and geographic coordinates, we need to first convert the local position vector $\overrightarrow{A_hC}$ and $\overrightarrow{B_hC}$ from a local spherical coordinate to a rectangular coordinate. Second, we need to rotate the local frame to have its axes parallel with the Earth-fixed coordinate frame, where the $x-y$ plane coincides with the Earth equatorial plane. The x axis is permanently fixed in the direction of the Greenwich meridian while the z axis extends through the North Pole.

General considerations about all transformations will be done first for a general point P and later will be applied to points A and B, Figure 2. For clarity and further reference to the vectors and its components, all transformations are presented graphically and the resulting formulas could be found in literature.

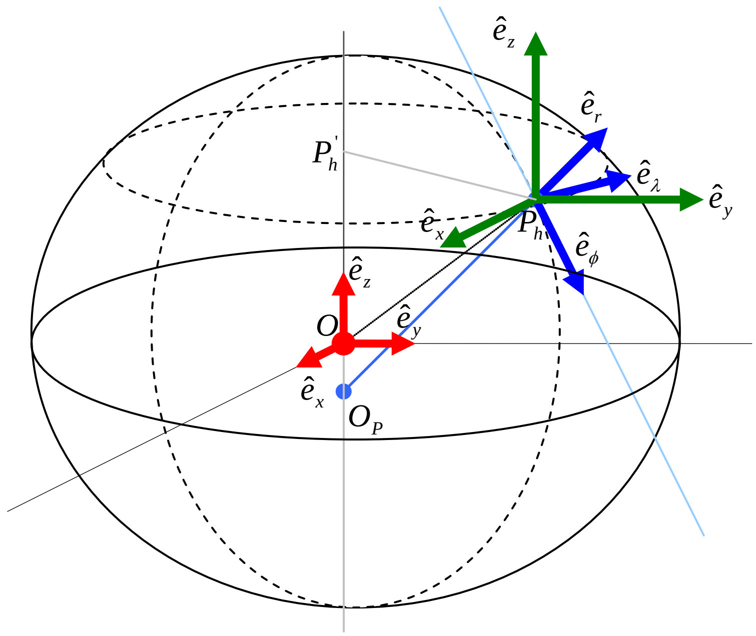

Figure 2. Space rotation of local Cartesian frame $\hat{e}_\phi, \hat{e}_\lambda, \hat{e}_r$ (blue color) around point P_h to get the frame $\hat{e}_x, \hat{e}_y, \hat{e}_z$ (green color) with axes parallel to the geocentric frame (red color).

2.1 *Converting spherical to rectangular coordinates*

Transformation of spherical coordinates of point C (azimuth α, elevation β and distance r), i.e. components of position vector $\overrightarrow{P_hC}$, to rectangular coordinate ($\hat{e}_\phi, \hat{e}_\lambda, \hat{e}_r$) has the form (Figure 3):

$$\overrightarrow{P_hC} = \overrightarrow{P_hC_\phi} + \overrightarrow{P_hC_\lambda} + \overrightarrow{P_hC_r} \quad (2)$$

$$\overrightarrow{P_hC_\phi} = r\cos\beta\cos\alpha\ \hat{e}_\phi \quad (2a)$$

$$\overrightarrow{P_hC_\lambda} = r\cos\beta\sin\alpha\ \hat{e}_\lambda \quad (2b)$$

$$\overrightarrow{P_hC_r} = r\sin\beta\ \hat{e}_r \quad (2c)$$

Where azimuth α increases from $\hat{e}_\phi$ to $\hat{e}_\lambda$ and elevation β increases from $\hat{e}_\phi, \hat{e}_\lambda$ plane to $\hat{e}_r$. Index h stands for a point at altitude h. For the moment, the antenna height will be neglected.

Next we need to rotate the local frame $\hat{e}_\phi, \hat{e}_\lambda, \hat{e}_r$ to get parallel axes with the geocentric $OXYZ$ frame.

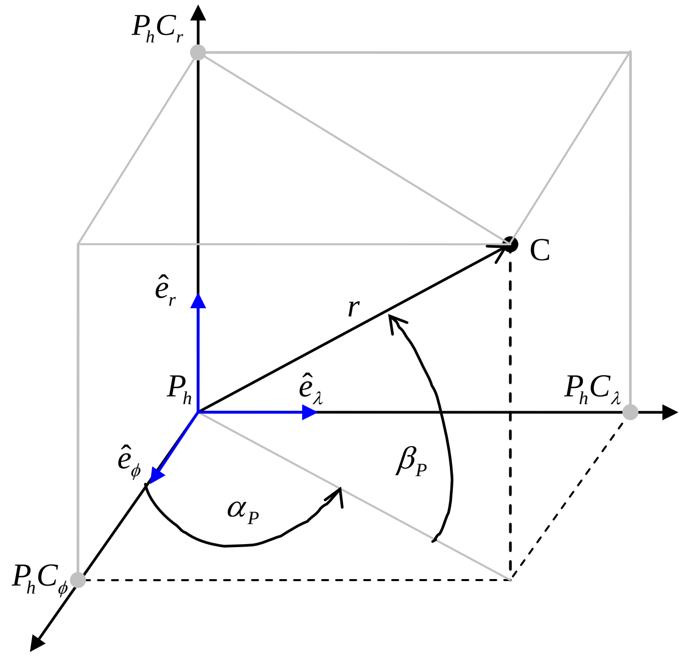

Figure 3. The geometry for computations rectangular coordinates of point C $(P_hC_\phi, P_hC_\lambda, P_hC_r)$ in ($\hat{e}_\phi, \hat{e}_\lambda, \hat{e}_r$) frame from azimuth α_P, elevation β_P and distance r.

2.2 *The space frame rotation*

Space rotation of local Cartesian frame $\hat{e}_\phi, \hat{e}_\lambda, \hat{e}_r$ (blue color) around point P_h to get the frame $\hat{e}_x, \hat{e}_y, \hat{e}_z$ with axes parallel to the geocentric frame $OXYZ$ is presented in Figure 2. Details for space rotation of local Cartesian frame around point P_h to get the frame with axes parallel to the geocentric frame are shown in Figure 4.

From right side of Figure 4 (a, b, c), unit vectors of local Cartesian frame $\hat{e}_\phi, \hat{e}_\lambda, \hat{e}_r$ have following components in frame parallel to the geocentric frame $\hat{e}_x, \hat{e}_y, \hat{e}_z$:

$$\hat{e}_\phi = \sin\phi_P \cos\lambda_P \hat{e}_x + \sin\phi_P \sin\lambda_P \hat{e}_y - \cos\phi_P \hat{e}_z \quad \ldots \quad (3a)$$

$$\hat{e}_\lambda = -\sin\lambda_P \hat{e}_x + \cos\lambda_P \hat{e}_y \quad (3b)$$

$$\hat{e}_r = \cos\phi_P \cos\lambda_P \hat{e}_x + \cos\phi_P \sin\lambda_P \hat{e}_y + \sin\phi_P \hat{e}_z \quad \ldots \quad (3c)$$

When we substitute above $\hat{e}_\phi, \hat{e}_\lambda, \hat{e}_r$ unit vectors in (2), we get position vector of point C in local frame with axes parallel to the geocentric frame:

$$\overrightarrow{P_hC_x} = r\left(\cos\beta_P \cos\alpha_P \sin\phi_P \cos\lambda_P - \cos\beta_P \sin\alpha_P \sin\lambda_P + \sin\beta_P \cos\phi_P \cos\lambda_P\right)\hat{e}_x \quad (4a)$$

$$\overrightarrow{P_hC_y} = r\left(\cos\beta_P \cos\alpha_P \sin\phi_P \sin\lambda_P + \cos\beta_P \sin\alpha_P \cos\lambda_P + \sin\beta_P \cos\phi_P \sin\lambda_P\right)\hat{e}_y \quad (4b)$$

$$\overrightarrow{P_hC_z} = r\left(-\cos\beta_P \cos\alpha_P \cos\phi_P + \sin\beta_P \sin\phi_P\right)\hat{e}_z \quad (4c)$$

2.3 *Altitude vector*

From geometry shown in Figure 4, we see that the local vertical line (zenith line) coincides with $\hat{e}_r$ direction and therefore $\hat{e}_r$ has a latitude angle ϕ_P in respect to the Earth equatorial plane ($x - y$ plane). The longitude angle λ_P is the same in both geodetic and geocentric system of references since the vertical axes are on the same line (see left side of Figure 4).

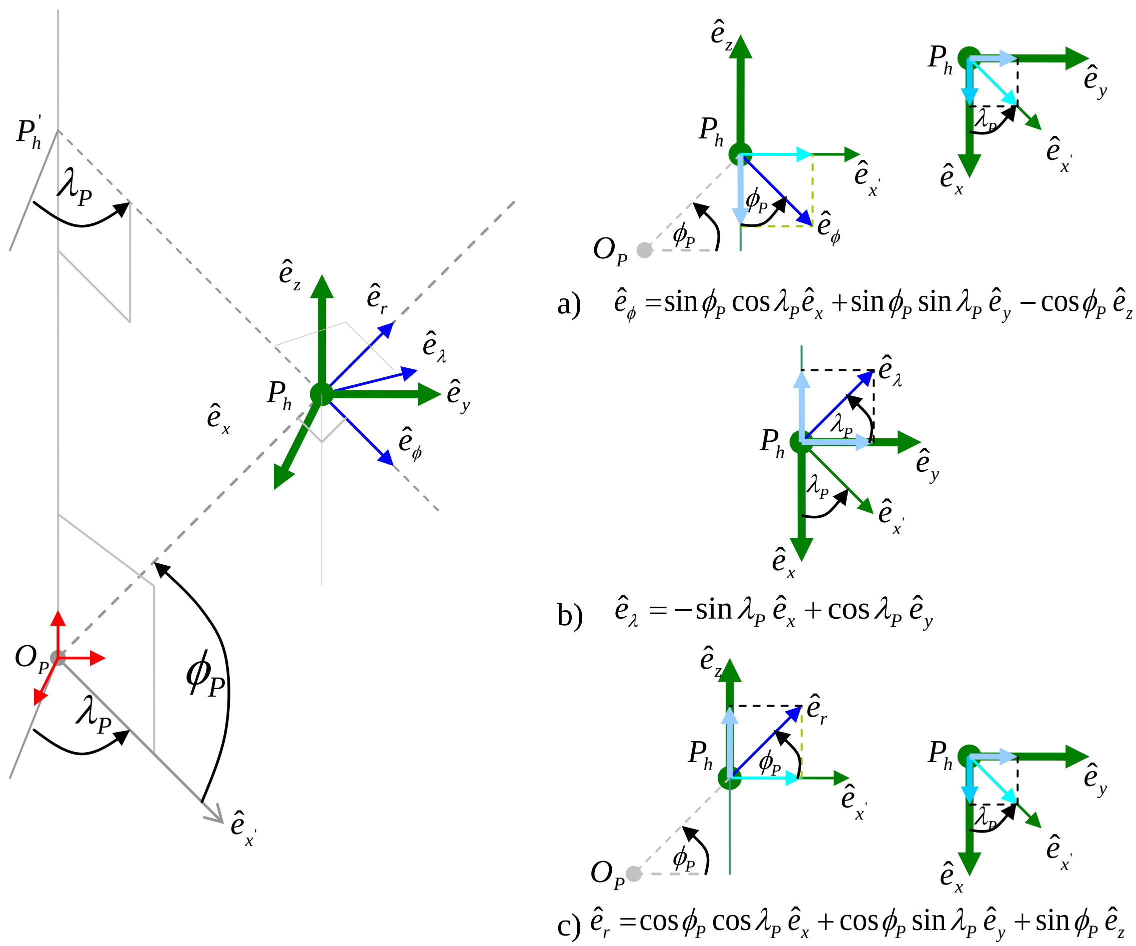

Figure 4. Details of space rotation of local Cartesian frame $\hat{e}_\beta, \hat{e}_\alpha, \hat{e}_r$ (blue color) around point P_h to get the frame $\hat{e}_x, \hat{e}_y, \hat{e}_z$ (green color) with axes parallel to the geocentric frame. Sub figures on the right side show the components of unit vectors $\hat{e}_\beta$ (a) , $\hat{e}_\alpha$ (b) , and $\hat{e}_r$ (c).

To get a coordinate of the altitude vector $\overrightarrow{PP_h}$ in local frame with axes parallel with the geocentric frame and the origin of axes in point. Since $\overrightarrow{PP_h}$ has the same direction as geodetic radius of point P and the intensity of vector $\overrightarrow{PP_h}$ is the height h_A (or altitude, or ellipsoidal height) of point P_h which is the local vertical distance between the P_h point and the reference ellipsoid.

$$\overrightarrow{PP_h} = h_A\left(\cos\phi\cos\lambda\hat{e}_x + \cos\phi\sin\lambda\hat{e}_y + \sin\phi\hat{e}_z\right)$$

Or in explicit vector form for each component when we add the height of the radar antenna at location P, H_P:

$$\overrightarrow{PP_h}_x = (h_P + H_P)\cos\phi\cos\lambda\hat{e}_x \qquad (5a)$$

$$\overrightarrow{PP_h}_y = (h_A + H_P)\cos\phi\sin\lambda\hat{e}_z \qquad (5b)$$

$$\overrightarrow{PP_h}_z = (h_A + H_P)\sin\phi\hat{e}_z \qquad (5c)$$

2.4 *Geocentric coordinates of ellipsoid*

To get geocentric coordinates of point P, blue vector $\overrightarrow{OP}$ in Figure 1, we have to notice that in general the radius vector from geocentric origin O will not be normal to the surface of the oblate spheroid (except at the poles and the equator). Therefore, we will have two latitudes, geodetic (angle ϕ_P) and geocentric latitude (angle $\phi_P^{'}$), see Figure 5.

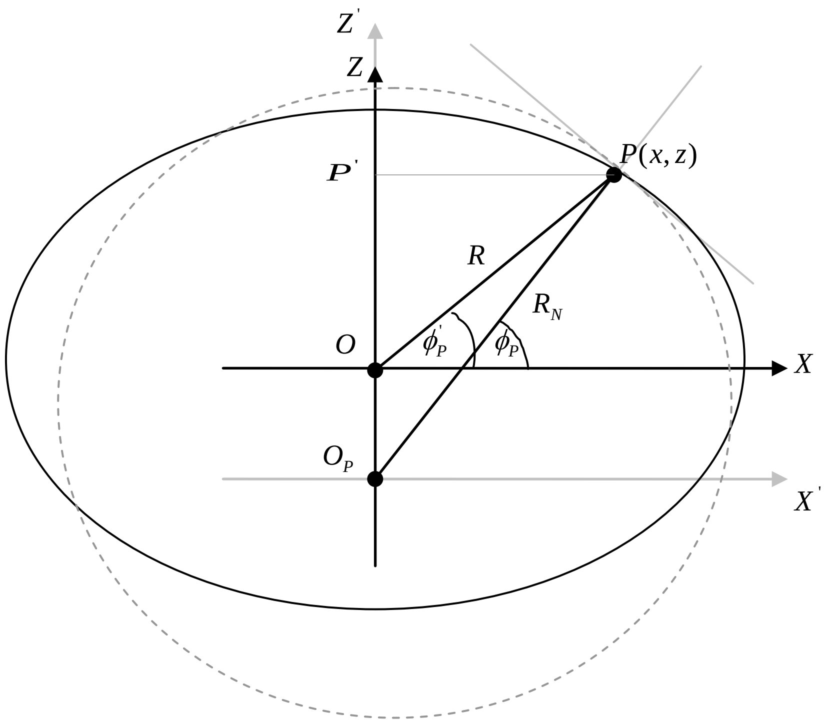

Figure 5. Geodetic ($X^{'}O_pZ^{'}$) and geocentric (XOZ) coordinate frames.

Appendix 1 gives details how to derive position vector $\overrightarrow{OP}$:

$$\overrightarrow{OP} = \frac{a}{\sqrt{1-e^2\sin^2\phi_P}}\left[\cos\phi_P\cos\lambda_P\hat{e}_x + \cos\phi_P\sin\lambda_P\hat{e}_y + \left(1-e^2\right)\sin\phi_P\hat{e}_z\right] \qquad (6)$$

Where a is the Earth equatorial radius and e is eccentricity (see Appendix 1).

2.5 *Position vector of a target point in geocentric coordinate frame*

From Figure 1, position vector of a target point C determined from local frame of reference in point P_h is given by:

$$\overrightarrow{OC} = \underbrace{\overrightarrow{OP} + \overrightarrow{PP_h}} + \overrightarrow{P_hC} = \overrightarrow{OP_h} + \overrightarrow{P_hC}$$

The first two vectors in the middle section of the equation are constant and depend on the geographical coordinate of surface point P_h. Only the last vector in the equation depends on the local coordinates of point C.

If we substitute all vector components from previous sections, (4a,b,c) , (5a,b,c), and (6), we are getting the explicit form of position vector and its components of target point C.

$$\overrightarrow{OC}_x = \left[OP_{hx} + r\left(\cos\beta_P \cos\alpha_P \sin\phi_P \cos\lambda_P - \cos\beta_P \sin\alpha_P \sin\lambda_P + \sin\beta_P \cos\phi_P \cos\lambda_P\right)\right]\hat{e}_x \quad \ldots \quad (7a)$$

$$\overrightarrow{OC}_y = \left[OP_{hy} + r\left(\cos\beta_P \cos\alpha_P \sin\phi_P \sin\lambda_P + \cos\beta_P \sin\alpha_P \cos\lambda_P + \sin\beta_P \cos\phi_P \sin\lambda_P\right)\right]\hat{e}_y \quad \ldots \quad (7b)$$

$$\overrightarrow{OC}_z = \left[OP_{hz} + r\left(-\cos\beta_P \cos\alpha_P \cos\phi_P + \sin\beta_P \sin\phi_P\right)\right]\hat{e}_z \quad (7c)$$

Where local parameters are given by:

$$OP_{hx} = \left(\frac{a}{\sqrt{1-e^2\sin^2\phi_P}} + h_P\right)\cos\phi_P \cos\lambda_P \quad \ldots \quad (8a)$$

$$OP_{hy} = \left(\frac{a}{\sqrt{1-e^2\sin^2\phi_P}} + h_P\right)\cos\phi_P \sin\lambda_P \quad \ldots \quad (8b)$$

$$OP_{hz} = \left(\frac{a\left(1-e^2\right)}{\sqrt{1-e^2\sin^2\phi_P}} + h_P\right)\sin\phi_P \quad (8c)$$

3. MATHEMATICAL EQUATIONS OF CIS

Equation 7 shows how vector components of position vector $\overrightarrow{OC}$ are composed from vectors that are related to point P. Identical equations may be derived for points A and B. Since the position of point C in geocentric frame is the same no matter from which starting point the position vector is constructed, it must satisfy the condition that vector components are equal. From that condition we get three scalar equations after substituting (7a,b,c) with points A and B in (1):

$$\begin{aligned}&OA_{hx} + r\left(\cos\alpha_A \cos\beta_A \sin\phi_A \cos\lambda_A - \sin\alpha_A \cos\beta_A \sin\lambda_A + \sin\beta_A \cos\phi_A \cos\lambda_A\right) = \\ &OB_{hx} + r\left(\cos\alpha_B \cos\beta_B \sin\phi_B \cos\lambda_B - \sin\alpha_B \cos\beta_B \sin\lambda_B + \sin\beta_B \cos\phi_B \cos\lambda_B\right)\end{aligned} \quad (9a)$$

$$\begin{aligned}&OA_{hy} + r\left(\cos\alpha_A \cos\beta_A \sin\phi_A \sin\lambda_A + \sin\alpha_A \cos\beta_A \cos\lambda_A + \sin\beta_A \cos\phi_A \sin\lambda_A\right) = \\ &OB_{hy} + r\left(\cos\alpha_B \cos\beta_B \sin\phi_B \sin\lambda_B + \sin\alpha_B \cos\beta_B \cos\lambda_B + \sin\beta_B \cos\phi_B \sin\lambda_B\right)\end{aligned} \quad (9b)$$

$$\begin{aligned}&OA_{hz} + r\left(\sin\beta_A \sin\phi_A - \cos\alpha_A \cos\beta_A \cos\phi_A\right) = \\ &OB_{hz} + r\left(\sin\beta_B \sin\phi_B - \cos\alpha_B \cos\beta_B \cos\phi_B\right)\end{aligned} \quad (9c)$$

All geographical parameters are assumed to be known; only 6 local spherical coordinates are unknown. And because they are coupled with three Equations (9a,b,c) we have only 3 independent variables, spherical coordinates from one of two local frames. Since we are analyzing a specific case when point C is at same distance from both local frames:

$$\left|\overrightarrow{A_h C}\right| = \left|\overrightarrow{B_h C}\right| = r \,,$$

we have an additional reduction of degrees of freedom, from 3 to 2.

Specifying the directions α_A, β_A from point A gives the remaining two equations. Solving Equations 9a,b,c we can find the other three dependent coordinates: r, α_B, β_B.

After evaluation of the above equations we get:

$$\alpha_B = \arccos\left[\pm\sqrt{\frac{b^2 - 2(c-a)(d-c) \pm |b|\sqrt{b^2 - 4(d-c)(d-a)}}{2\left[b^2 + (c-a)^2\right]}}\right] \tag{10}$$

$$\beta_B = \arctan\left[\frac{\cos\alpha_B (L1M3 - L3M1) - \sqrt{1-\cos^2\alpha_B}\,(J2L3 + I2M3)}{L3M2 - L2M3}\right] \tag{11}$$

$$r = \frac{OA_{hz} - OB_{hz}}{\cos\beta_B(\tan\beta_B K1 - \cos\alpha_B K2) - K3(\alpha_A, \beta_A)} \tag{12}$$

Explicit forms of constants and parameters for chosen locations A and B for α_A, β_A values are given in Appendix 2.

Since for each pair of α_A, β_A values, there is at most one ray from radar B (α_B, β_B) that meets it with equal distances (r) from both radars, multiple solutions in (10) because of $\pm$ signs are prohibited. Therefore, procedures for obtaining r, α_B, β_B from Equations (10, 11, and 12) must use filters, and restrictions, that will reject all solutions except the physical one.

4. ANALYZE

Theoretical CIS formulas (10, 11, 12) obtained in the previous section enclose three main CIS characteristics: the size, direction and slope of CIS. Visual presentation of these characteristic is given in the next three subsections.

4.1 CIS size

As an example of CIS calculation, the WKR (King City, lat=43.96, lon= -79.57, alt=360m) and WSO (Exeter, lat=43.37, lon= -81.38, alt=303m) radars were used.

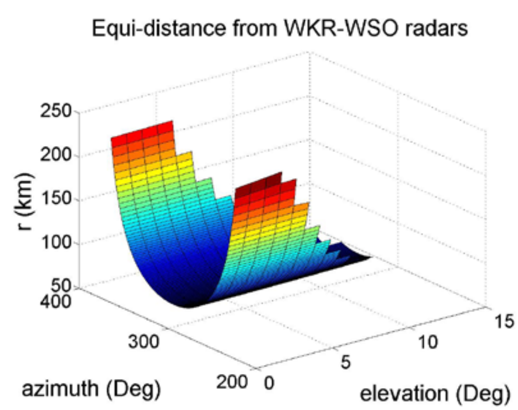

Figure 6. Graphical presentation of calculated equal-distance $r(\alpha_A, \beta_A)$ as a function of azimuth and elevation of WKR radar for an angle resolution of 1^o.

The graphical presentation of $r(\alpha_A, \beta_A)$ for WKR and WSO is shown in Figure 6 for 1^o angle resolutions.

As we expected the equal-distance is symmetrical in regards to the vertical plane in WKR-WSO direction and the number of equidistant points decreases with the elevation angle. The shape of $r(\alpha_A, \beta_A)$ doesn't change with angle resolutions, only the density of points is changing.

More practical presentation of CIS is in the local Descartes reference system as it is shown in Figure 7 for two angle resolutions, 1^o (top) and 0.1^o (bottom). The shape of the CIS is more like a wall as the angle resolution increases.

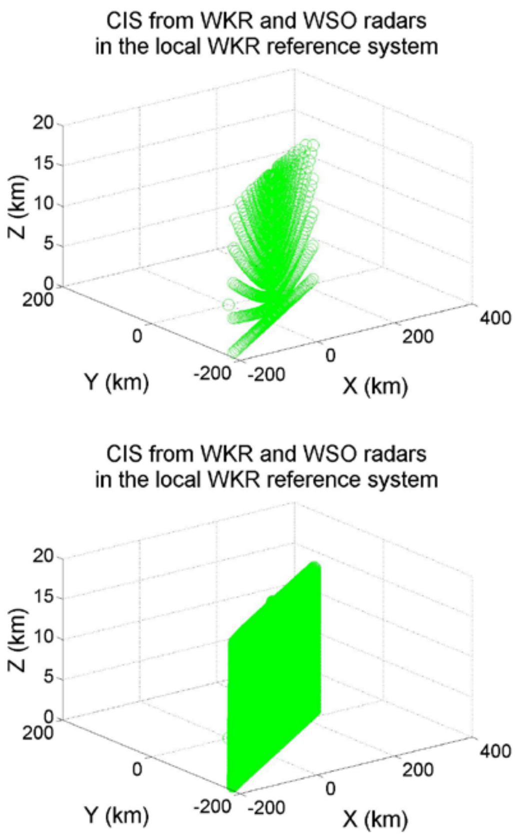

Figure 7. Graphical presentation of calculated CIS in the local WKR radar Descartes reference system $Z(x, y)$ for angle resolutions 1^o (top) and 0.1^o (bottom).

The size of CIS depends on the geographical positions of two radar locations, mostly of their relative distances as it was shown in Figure 8. Three colors (red, black, and blue) were marked three different radar distances (x_1, x_2, x_3). For the third case (blue) was sketched CIS length (L) and height (H) for a chosen maximal radar range (d) and troposphere height ($H_{\max}$) .

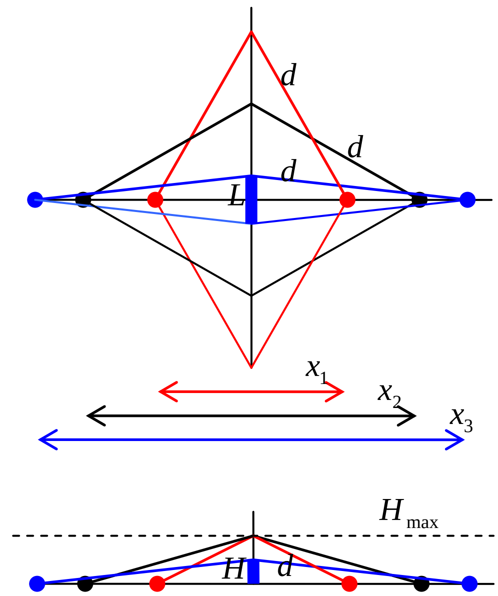

Figure 8. Sketch of the horizontal (top) and vertical (bottom) cross section of relative positions of radar locations and the CIS plane of symmetry for the maximal radar range d and troposphere height $H_{\max}$. The size of CIS, length L and height H, depends on relative distance x of pair of radars.

4.2 *CIS direction*

Figure 9 shows how equal-distances are changing with elevation in WKR-WSO direction for angle resolutions 1^o (top) and 0.1^o (bottom). With higher angle resolution there is a more precise determination of the WKR-WSO direction, 246^o versus 246.3^o.

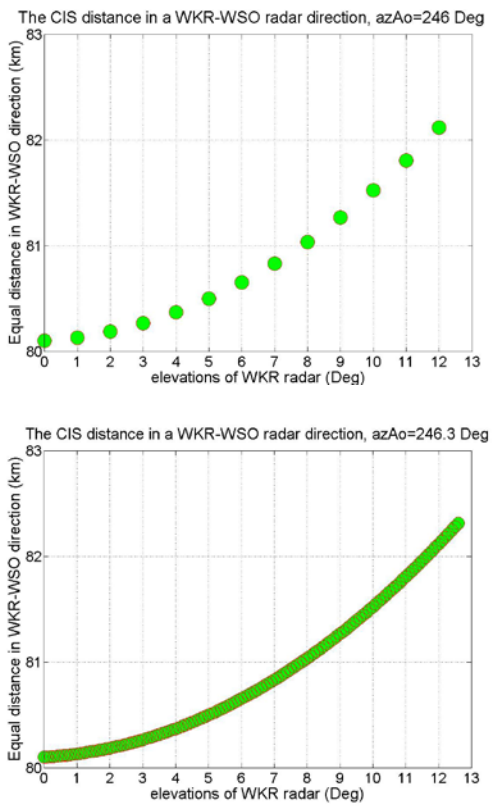

Figure 9. Graphical presentation of calculated CIS distance in WKR-WSO radar direction as a function of WKR elevation, $r_{\alpha_{A-B}}(\beta_A)$, for angle resolutions 1^o (top) and 0.1^o (bottom).

4.3 *CIS slope*

The collection of points at equal-distance from two radars belongs to the plane of symmetry as it were sketched in Figure 10a and 10b. In the local reference systems the plane of symmetry is tilted for δ_A and δ_B.

The tilted angles δ_A and δ_B are comparable when both radars are at similar altitudes (Figure 10a). As the difference in altitude between radars increases, the difference between tilted angles also increases (Figure 10b).

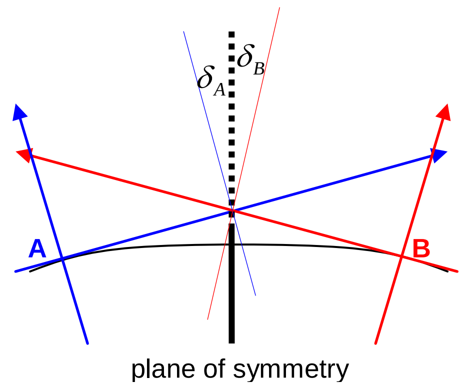

Figure 10a. The schematic cross section of position of the plane of symmetry that includes equal-distances between A and B radars that are at similar altitudes and the local verticals that are tilted for δ_A and δ_B angles.

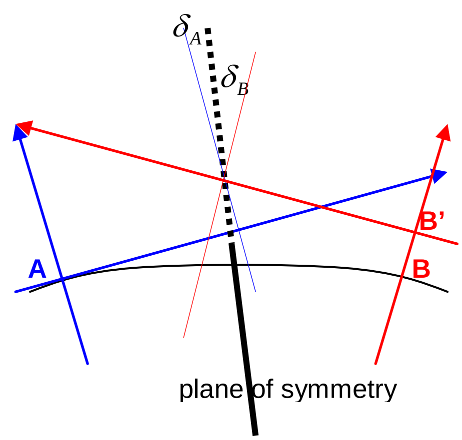

Figure 10b. Same as 10a but with a significant altitude difference.

The amount of CIS tilt is a function of radar's distance and altitude differences, i.e. geographical coordinates.

5. DISCUSSION AND SUMMARY

The presented study is an attempt to clarify the geometry of defining the common inter-radar space. The essence of the methodology is presumption that if we want to compare measurements from two radars we need to first establish accurate coordinates of a common target volume. Since the radar locates the target volume in reference to the local reference system, it is necessary to convert local coordinates into a common geocentric “fixed” frame of reference.

As it was shown, for two radars A and B, from vector equation of position components of a common target point C, we can calculate coordinates of points that are in equal distance from both radars. As a result, for each pair of independent coordinates α_A, β_A, of point C at equal distance from both radars, the radius distance r is determined by equation:

$$r = \frac{OA_{hz} - OB_{hz}}{\cos\beta_B(\tan\beta_B K1 - \cos\alpha_B K2) - K3(\alpha_A, \beta_A)}$$

Furthermore, for same pair of independent variables α_A, β_A , measured by radar at location A, there are coordinates α_B, β_B (r is the same) of radar at location B that point to the same target C given by:

$$\alpha_B = \arccos\left[\pm\sqrt{\frac{b^2 - 2(c-a)(d-c) \pm |b|\sqrt{b^2 - 4(d-c)(d-a)}}{2\left[b^2 + (c-a)^2\right]}}\right]$$

$$\beta_B = \arctan\left[\frac{\cos\alpha_B(L1M3 - L3M1) - \sqrt{1-\cos^2\alpha_B}(J2L3 + I2M3)}{L3M2 - L2M3}\right]$$

Where explicit forms of constants and parameters are given in Appendix 2.

In the presented work we have shown how to determine the CIS and from pure geometrical perspective the CIS is not a simple rectangle wall with constant dimensions. The size, direction and slope of the CIS vary with geographical locations of radars and also with maximal range of the radars. The derived formulas will be a complete and accurate method for comparison that can be used with the large inter-radar distances of the Canadian network. Obtained formulas are very accurate but not yet operationally suitable since they determine the mathematical points of equal distance not the common radar pulse volumes from operational discrete scanning angles. Therefore, the next steps should be the inclusion of the technical characteristics of the radars in the operational regime and the conversion between the geometric elevation and the antenna axis elevation.

REFERENCES:

Donaldson, N. 2010. Monitoring Canadian Weather Radars with Operational Observations, *6th European Conference on Radar Meteorology (ERAD), Sibu, Romania.* 239-243.

Gourley, J.J. et al. 2003. Evaluating the calibrations of radars : A software approach. *31st International Conference on Radar Meteorology,* Seattle, WA, USA, American Meteorological Society, 459-462.

Montopoli, M. et al. 2012. Radar inter-calibration analysis: potential use for weather radar networks, *7th European Conference on Radar Meteorology and Hydrology (ERAD), Toulouse, France.*

Appendix 1.

To get coordinates of $\overrightarrow{OP}$ vector in the geocentric frame we need first to find the line equation of line $f_P(x)$ that have O_P and P points, in the fixed XOZ reference system, Figure 5. We will do it using condition that the tangent on geodetic ellipse is perpendicular to the $f_P(x)$ line. Since value $O_P P$ is the same for any longitude (meridian), for simplicity we can chose zero longitude for our calculations (Grinch meridian).

$$f_P(x) = m_P x + k_P = \tan\phi_P x + OO_P \quad \text{(A1)}$$

To find k_P, i.e. OO_P, we will use condition that both equations (A1) and equation of ellipse (shown below in canonical form) must be satisfied.

$$\frac{x^2}{a^2} + \frac{z^2}{b^2} = 1 \quad \text{(A2)}$$

where a>b>0 and $x \le a$. Applying derivation on the equation of ellipse we get:

$$\frac{2x}{a^2} + \frac{2z}{b^2}\frac{dz}{dx} = 0$$

$$\frac{dz}{dx} = -\frac{xb^2}{za^2}$$

Which for point P has specific value:

$$\left.\frac{dz}{dx}\right|_P = -\frac{x_P b^2}{z_P a^2}$$

Using trigonometry definition of tan in Fig 5, we have relation between geocentric coordinate of point P and radial angle $\phi_P^{'}$:

$$\tan\phi_P^{'} = \frac{z_p}{x_p} \quad \text{(A3)}$$

Substituting above equation for first derivation of ellipsoid equation in point P:

$$\left.\frac{dz}{dx}\right|_P = -\frac{x_P b^2}{z_P a^2} = -\frac{b^2}{\frac{z_P}{x_P}a^2} = -\frac{b^2}{\tan\phi_P^{'} a^2}$$

This tangent is perpendicular to the ellipsoid and also to the circle that has radius in point P that crossing ordinate Z at point O_P:

$$\tan\phi_P = -\frac{1}{\left.\frac{dz}{dx}\right|_P} = -\frac{1}{-\frac{b^2}{\tan\phi_P^{'} a^2}} = \frac{a^2}{b^2}\tan\phi_P^{'}$$

Or:

$$\tan\phi_P^{'} = \frac{b^2}{a^2}\tan\phi_P$$

Instead using equatorial radius
$a = 6\,378\,137.0m$
and polar radius
$b = 6\,356\,752.3142m$
we can use eccentricity squared
$e^2 = 0.006\,694\,379\,990\,14$. From definition of eccentricity:

$$e^2 = 1 - \frac{b^2}{a^2},$$

we get

$$\frac{b^2}{a^2} = 1 - e^2$$

which substituted in previous $\tan\phi_P^{'}$ equation gives final relation between geodetic latitude ϕ_P and geocentric latitude $\phi_P^{'}$:

$$\tan\phi_P^{'} = (1-e^2)\tan\phi_P \quad \text{(A4)}$$

Four Equations (A1, A2, A3, A4) are sufficient to obtain four unknown ($x_P, z_P, \phi_P^{'}, OO_P$) as a function of geodetic latitude ϕ_P.

From (A3) and (A4):

$$z_p = x_p(1-e^2)\tan\phi_P \quad \text{(A5)}$$

From (A1) at P point:

$$z_P = x_P \tan\phi_P + OO_P \quad \text{(A6)}$$

From (A5) and (A6):

$$x_p(1-e^2)\tan\phi_P = x_P \tan\phi_P + OO_P$$

$$x_p = -\frac{OO_P}{e^2\tan\phi_P} \qquad \text{(A7)}$$

Substitute (A7) in (A5):

$$z_p = -OO_P\,\frac{1-e^2}{e^2} \qquad \text{(A8)}$$

Substitute (A7) and (A8 in (A2) and multiply by e^4:

$$OO_P{}^2\left[\frac{1}{a^2\tan^2\phi_P}+\frac{(1-e^2)^2}{b^2}\right]=e^4$$

Since $\frac{b^2}{a^2}=1-e^2$, after substitution in above equation and multiple by a^2:

$$OO_P{}^2\left(\frac{1}{\tan^2\phi_P}+\frac{b^2}{a^2}\right)=a^2e^4$$

Substitute $\frac{b^2}{a^2}=1-e^2$ value and for $\tan\phi_P=\frac{\sin\phi_P}{\cos\phi_P}$ in above equation:

$$OO_P{}^2\left(\frac{\cos^2\phi_P}{\sin^2\phi_P}+1-e^2\right)=a^2e^4$$

Which become:

$$OO_P{}^2\left(\frac{1-e^2\sin^2\phi_P}{\sin^2\phi_P}\right)=a^2e^4$$

$$OO_P = \mp a\frac{e^2|\sin\phi_P|}{\sqrt{1-e^2\sin^2\phi_P}} \qquad \text{(A9)}$$

Where negative sine is for north hemisphere and positive for south hemisphere. Or :

$$OO_P = -a\frac{e^2\sin\phi_P}{\sqrt{1-e^2\sin^2\phi_P}} \qquad \text{(A10)}$$

In the vector form we have:

$$\overrightarrow{OO_P} = -a\frac{e^2\sin\phi_P}{\sqrt{1-e^2\sin^2\phi_P}}\hat{e}_z \qquad \text{(A11)}$$

To find final expression for P coordinates we need to substitute (A10) in (A7) and (A8):

$$x_p = a\frac{\cos\phi_P}{\sqrt{1-e^2\sin^2\phi_P}} \qquad \text{(A12)}$$

$$z_p = a\frac{(1-e^2)\sin\phi_P}{\sqrt{1-e^2\sin^2\phi_P}} \qquad \text{(A13)}$$

For three dimensional case Z coordinate is the same while x_p coordinate should be decomposed to x and y components as x_p projection on x and y axes. If longitude angle for point P is λ_p (Figure 4, left side) than we have following equations:

$$x_p = a\frac{\cos\phi_P}{\sqrt{1-e^2\sin^2\phi_P}}\cos\lambda_P \qquad \text{(A14)}$$

$$y_p = a\frac{\cos\phi_P}{\sqrt{1-e^2\sin^2\phi_P}}\sin\lambda_P \qquad \text{(A15)}$$

Or in concise vector form:

$$\overrightarrow{OP} = \frac{a}{\sqrt{1-e^2\sin^2\phi_P}}\left[\cos\phi_P\cos\lambda_P\hat{e}_x+\cos\phi_P\sin\lambda_P\hat{e}_y+(1-e^2)\sin\phi_P\hat{e}_z\right] \qquad \text{(A16)}$$

Appendix 2:

Explicit forms of parameters and constants for chosen locations A and B for α_A, β_A values are given in logical order:

$$a = \left(L1^2 - I2^2\right)\left(L3M2 - L2M3\right)^2 + 2L1L2\left(L1M3 - L3M1\right)\left(L3M2 - L2M3\right)$$
$$- \left(L3^2 - L2^2\right)\left(L1M3 - L3M1\right)^2$$

$$b = 2\begin{Bmatrix} -I2L1\left(L3M2 - L2M3\right)^2 \\ -L2\left(L3M2 - L2M3\right)\left[L1\left(J2L3 + I2M3\right) + I2\left(L1M3 - L3M1\right)\right] \\ +\left(L3^2 - L2^2\right)\left(L1M3 - L3M1\right)\left(J2L3 + I2M3\right) \end{Bmatrix}$$

$$c = 2I2L2\left(L3M2 - L2M3\right)\left(J2L3 + I2M3\right) - \left(L3^2 - L2^2\right)\left(J2L3 + I2M3\right)^2$$

$$d = \left(L3^2 - I2^2\right)\left(L3M2 - L2M3\right)^2$$

$$L1 = I1 + \frac{OA_{hx} - OB_{hx}}{OA_{hz} - OB_{hz}}K1$$

$$M1 = J1 + \frac{OA_{hy} - OB_{hy}}{OA_{hz} - OB_{hz}}K1$$

$$L2 = I3 - \frac{OA_{hx} - OB_{hx}}{OA_{hz} - OB_{hz}}K2$$

$$M2 = J3 - \frac{OA_{hy} - OB_{hy}}{OA_{hz} - OB_{hz}}K2$$

$$L3 = I4 - \frac{OA_{hx} - OB_{hx}}{OA_{hz} - OB_{hz}}K3$$

$$M3 = J4 - \frac{OA_{hy} - OB_{hy}}{OA_{hz} - OB_{hz}}K3$$

$$I1 = \sin\phi_B \cos\lambda_B$$
$$I2 = \sin\lambda_B$$
$$I3 = \cos\phi_B \cos\lambda_B$$

$$I4 = \cos\alpha_A \cos\beta_A \sin\phi_A \cos\lambda_A - \sin\alpha_A \cos\beta_A \sin\lambda_A + \sin\beta_A \cos\phi_A \cos\lambda_A$$

$$J1 = \sin\phi_B \sin\lambda_B$$
$$J2 = \cos\lambda_B$$
$$J3 = \cos\phi_B \sin\lambda_B$$

$$J4 = \cos\alpha_A \cos\beta_A \sin\phi_A \sin\lambda_A + \sin\alpha_A \cos\beta_A \cos\lambda_A + \sin\beta_A \cos\phi_A \sin\lambda_A$$

$$K1 = \sin\phi_B$$
$$K2 = \cos\phi_B$$
$$K3\left(\alpha_A, \beta_A\right) = \sin\beta_A \sin\phi_A - \cos\alpha_A \cos\beta_A \cos\phi_A$$