Using the Kalman-Bucy filter in an ensemble framework

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Notation

State variables: $\underline{x} \in \Re^n$ Ensemble: $\overline{\overline{X}} = \left[\underline{x}_1 \mid \underline{x}_2 \mid \cdots \mid \underline{x}_m\right] \in \Re^{n \times m}$

Observations: $\underline{y} \in \Re^l$ Observation operator: $H \in \Re^{l \times n}$

Sample mean: $\overline{\underline{x}} = \frac{1}{m}\sum_{i=1}^{m} \underline{x}_i = \frac{1}{m}\overline{\overline{X}}\,\underline{1}$ $\qquad U = m^{-1} 1_{m \times m}$

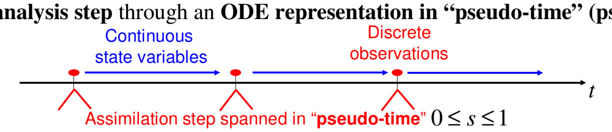
Ensemble of perturbations: $X = \left[\underline{x}_1 - \overline{\underline{x}} \mid \underline{x}_2 - \overline{\underline{x}} \mid \cdots \mid \underline{x}_m - \overline{\underline{x}}\right] = \overline{\overline{X}}[I - U]$

Sample covariance: $P = \frac{1}{m-1} X X^T = \frac{1}{m-1} \overline{\overline{X}}[I-U]\overline{\overline{X}}^T \in \Re^{n \times n}$

The superscript b indicates background and a indicates analysis.

Two new Bucy-based formulations for the analysis step

We review two new methods that extend the **Kalman-Bucy Filter** (Kalman and Bucy, 1961) to the ensemble framework. Both formulate the **analysis step** through an **ODE representation in "pseudo-time" (pst)**.



In **BGR09** (Bergemann *et al*, 2009) the analysis equation for the ensemble of **perturbations** is:

$$\frac{d}{ds} X = -\frac{1}{2(m-1)} X X^T H^T R^{-1} H X$$

$$X(0) = X^b \qquad \rightarrow \qquad X^a = X(1)$$

In **BR10** (Bergemman and Reich, 2010) the analysis step for the **full ensemble** comes from the **solution of a single ODE**.

$$\frac{d}{ds}\overline{\overline{X}} = -\frac{1}{m-1}\overline{\overline{X}}[I-U]\overline{\overline{X}}^T H^T R^{-1}\left[\frac{1}{2} H \overline{\overline{X}}[I+U] - \underline{y}\underline{1}^T\right]$$

$$\overline{\overline{X}}(0) = \overline{\overline{X}}^b \qquad \rightarrow \qquad \overline{\overline{X}}^a = \overline{\overline{X}}(1)$$

Transform-based alternatives

We propose a **transform-based formulations** in which the **operations are performed in the ensemble space**.

For **BGR09**: $X(s) = X^b W(s)$ $\qquad \frac{d}{ds} W = -\frac{1}{2(m-1)} W W^T Y^{b^T} R^{-1} Y^b W$

$W \in \Re^{m \times m}$ $\qquad W(0) = I \qquad \rightarrow \qquad W^a = W(1)$

For **BR10** we use the previous equation plus:

$$\overline{\underline{x}}(s) = \overline{\underline{x}}^b + X^b \underline{\overline{w}}(s) \qquad \frac{d}{ds}\underline{\overline{w}} = \frac{-1}{m-1} W W^T [I-U]\overline{\overline{X}}^{b^T} H^T R^{-1}\left(\frac{1}{m} H \overline{\overline{X}}^b \underline{1} + H \overline{\overline{X}}^b [I-U]\underline{\overline{w}}(s) - \underline{y}\right)$$

$$\underline{\overline{w}} \in \Re^{m \times 1} \qquad \underline{\overline{w}}(0) = \underline{0} \qquad \rightarrow \qquad \underline{\overline{w}}^a = \underline{\overline{w}}(1)$$

$$\overline{\overline{X}}^a = \overline{\overline{X}}^b\left[(I-U)\left[W^a + \underline{\overline{w}}^a \underline{1}^T\right] + U\right]$$

Advantages of the **transform-based** versions:

•**Operations in a lower dimensional space**.

•Once weights are found, **post-processing procedures** such as running in place or the quasi-outer loop (Kalnay and Yang 2010) can be **performed with little cost to improve performance**.

Experiments

We test the formulations in the Lorenz 1963 model observing directly the variables $H = I$, with 3 ensemble members and an observational error covariance $R = 2I$.

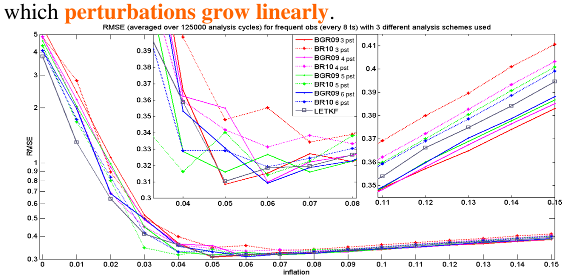
The performances of BGR09 and BR10 are compared to that of the local ensemble transform Kalman filter (LETKF, Hunt *et al* 2007).

We study the effect of two parameters:

	LETKF	Bucy-based
Multiplicative inflation $X^b \rightarrow (1+\rho)X^b$	✔	✔
Number of steps for the numerical solution of the analysis ODEs in pst (Euler-forward method used)		✔

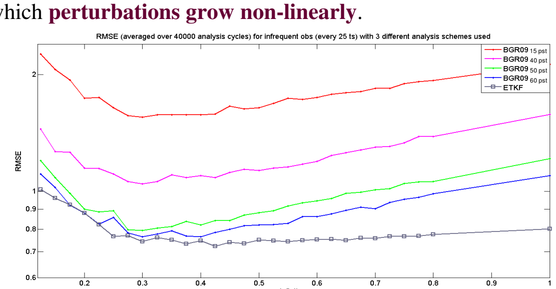
Two cases are considered:

•**Frequent observations** corresponding to **short assimilation windows** in which **perturbations grow linearly**.



	pst steps	inflation	RMSE
LETKF	----	0.05	0.3105
BGR09	3	0.05	0.3085
BR10	5	0.06	0.3142

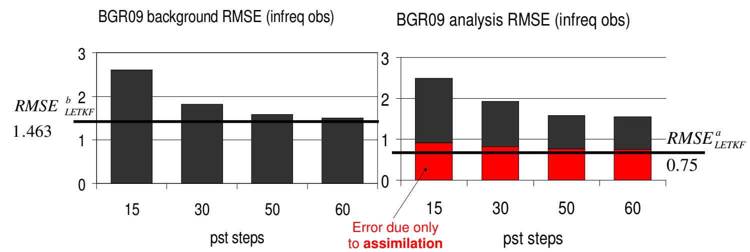
Infrequent observations corresponding to **long assimilation windows** in which **perturbations grow non-linearly**.



	pst steps	inflation	RMSE
LETKF	----	0.425	0.7231
BGR09	60	0.3	0.7653
BR10	the formulation becomes **unstable**		

Discussion

In the following figures we show the background RMSE and analysis RMSE for BGR09 in the infrequent observations case and $\delta = 0.35$ (in the optimal region). Of the total analysis RMSE, only the red portion corresponds to the error due to the **assimilation** step, the rest is due the growth of this error in the non-linear forecast window.

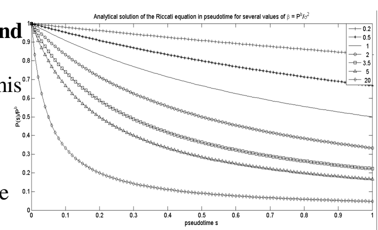


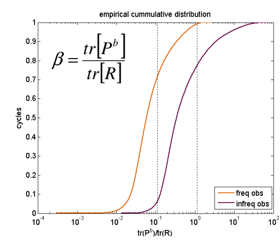
Both formulations, in particular BGR09, **are successful for short assimilation windows**, but **not** for **long assimilation windows** since BR10 becomes unstable and BGR09 requires a large number of steps. This is the result of the **stiffening of the ODE in pseudo-time**.

Consider the analytical solution of the Bucy equation for covariance (scalar version).

$$\frac{dP}{ds} = -PH^T R^{-1} HP \quad \longrightarrow \quad \frac{P(s)}{P^b} = \frac{1}{\beta s + 1}, \qquad \beta = \frac{P^b}{\sigma^2}$$

The **behavior of the solution is sensitive to the ratio of background variance over observational variance**. For small values of β this behavior is close to linear, few pst steps are needed for an accurate numerical solution. As β becomes larger the solution stiffens and more pst steps are required. This can be observed in the figure to the right.





The value of β depends on the frequency of observations and the length of the assimilation window. The figure to the left shows the difference in the typical values of this ratio for **frequent** and **infrequent** obs. For **frequent** obs practically in all cases (99%) $\beta < 1$, while for **infrequent** obs at least 25% of the cases have $\beta > 1$.

References

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