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A Three-Time-Level Explicit Economical (3TL-EEC) Scheme

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INTRODUCTION & MOTIVATION

The forward-backward (FB) scheme is a two-time-level scheme that allows a stable time step twice that of the standard three-time-level leapfrog (LF) scheme for the gravity wave propagation.

Since the FB scheme is a two-time-level scheme, it is convenient to use another two-time-level scheme such as the Adams-Bashforth (AB) scheme for the advection, as currently done in the NMMB model at NCEP.

Potential use of the LF scheme in place of the AB scheme for advection in NMMB, without losing the computational economy of the FB scheme for the gravity wave propagation, is our motivation behind this research.

Following Janjic's idea, we have developed a new three-time-level explicit economical (acronym 3TL-EEC) scheme that retains the computational economy of the FB scheme, and enables use of the LF scheme for advection.

1D Shallow-Water Gravity Waves

To review the LF and FB schemes, and later introduce the 3TL-EEC scheme, we employ the 1D shallow-water gravity-wave equations

$$\partial_t \eta + c\partial_x u = 0; \quad (1a) \qquad \partial_t u + c\partial_x \eta = 0, \quad (1b)$$

where $c \equiv \sqrt{gH}$ is the phase speed of the gravity waves and $\eta(x,t) = c\,h(x,t)/H$. H is the mean depth and h is the deviation.

LF Scheme for (1ab)

$$\frac{\eta^{n+1}-\eta^{n-1}}{2\Delta t} + c\partial_x u^n = 0; \quad (2a) \qquad \frac{u^{n+1}-u^{n-1}}{2\Delta t} + c\partial_x \eta^n = 0. \quad (2b)$$

FB Scheme for (1ab)

$$\frac{\eta^{n+1}-\eta^{n}}{\Delta t} + c\partial_x u^n = 0; \quad (3a) \qquad \frac{u^{n+1}-u^{n}}{\Delta t} + c\partial_x \eta^{n+1} = 0. \quad (3b)$$

For stability analysis, we pick a single wavenumber k. For the LF scheme, the amplification-factor λ satisfies a biquadratic equation

$$(\lambda^2-1)^2 + 4p^2\lambda^2 = 0; \quad p \equiv kc\Delta t, \qquad (4)$$

with two physical and two computational modes in time, and for stability $|p| \le 1$. For the FB scheme, λ satisfies a quadratic equation

$$\lambda^2 + (p^2-2)\lambda + 1 = 0, \qquad (5)$$

with two physical but no computational modes in time, and for stability $|p| \le 2$. Thus, the FB scheme is economical with a stable time step twice as large as the LF scheme.

3TL-EEC Scheme

As suggested by Janjic, to construct the 3TL-EEC scheme for the 1D system, we first apply the FB scheme from the time-level n-1 to n:

$$\frac{\eta^{n}-\eta^{n-1}}{\Delta t} + c\partial_x u^{n-1} = 0, \quad (6a) \qquad \frac{u^{n}-u^{n-1}}{\Delta t} + c\partial_x \eta^{n} = 0. \quad (6b)$$

Then, we reapply the FB scheme from the time-level n to n+1:

$$\frac{\eta^{n+1}-\eta^{n}}{\Delta t} + c\partial_x u^{n} = 0, \quad (7a) \qquad \frac{u^{n+1}-u^{n}}{\Delta t} + c\partial_x \eta^{n+1} = 0. \quad (7b)$$

Lastly, we take an average of the respective equations above [i.e., (6*a*+7*a*)/2 and (6*b*+7*b*)/2] to introduce the **3TL-EEC scheme**:

$$\frac{\eta^{n+1}-\eta^{n-1}}{2\Delta t} + \frac{c}{2}\partial_x (u^n + u^{n-1}) = 0, \qquad (8a)$$

$$\frac{u^{n+1}-u^{n-1}}{2\Delta t} + \frac{c}{2}\partial_x (\eta^{n+1} + \eta^{n}) = 0. \qquad (8b)$$

For the 3TL-EEC scheme, the amplification-factor λ satisfies two quadratic equations

$$(\lambda+1)^2 = 0; \quad \lambda^2 + (p^2-2)\lambda + 1 = 0. \qquad (9a, 9b)$$

Equation (9b) is identically same as (5) for the FB scheme. Thus, the 3TL-EEC scheme maintains the favorable economy and stability properties of the FB scheme. While (9a) shows that unlike the FB scheme, the 3TL-EEC scheme introduces two computational modes in time, but such modes are neutrally stable.

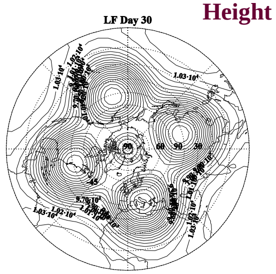
Application of the 3TL-EEC Scheme in a Global Shallow-Water model

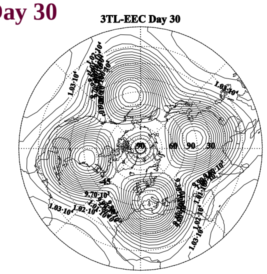
A global, shallow-water, grid-point model (Kar et al. 1994) is integrated in time, first, using the LF scheme and then using the 3TL-EEC+LF scheme. In 3TL-EEC+LF scheme, the 3TL-EEC scheme is used for gravity-wave terms and the LF scheme is used for advection and other terms. The horizontal finite-difference scheme of the model, following Arakawa and Lamb (1981), conserves total mass, energy and potential-enstrophy on the staggered C grid. A zonal polar filter can be employed at the high latitudes to alleviate the small time step restrictions.

A Rossby-Haurwitz wave-number 4 initial condition is used with the mean-depth of the shallow water, $H = 8 \times 10^3$ m. The model grid resolution is set to $\Delta\varphi = 4^\circ$ and $\Delta\lambda = 5^\circ$. The time steps for the LF and the 3TL-EEC+LF schemes are 40s and 80s, respectively, without polar filter; and 240s and 480s, respectively, with polar filter. The Robert-Asselin time-filter coefficient is set to 0.125. Time integrations were carried out for 210 days. The free-surface height on day 30, together with the computed global invariants of total mass, energy and potential-enstrophy are displayed.

LF (Δt = 240s) **3TL-EEC+LF (Δt = 480s)**

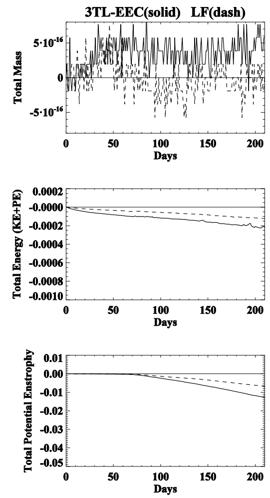
Height (m), Day 30



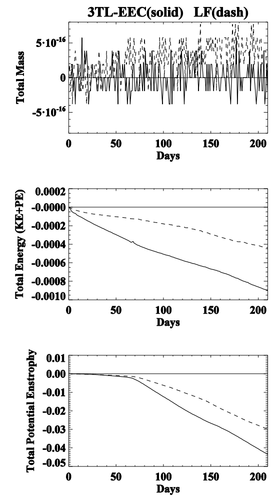


Global Invariants

No Polar Filter



With Polar Filter



CONCLUSIONS

A new three-time-level explicit economical (3TL-EEC) scheme has been developed that retains the computational economy of the forward-backward scheme and enables use of the leapfrog scheme for advection. The proposed scheme was implemented and tested in a global, grid-point shallow-water model using the Rossby-Haurwitz wave initial condition.

REFERENCES

Arakawa, A., and V. R. Lamb, 1981: A potential enstrophy and energy conserving scheme for the shallow water equations. *Mon. Wea. Rev*. **109**, 18-36.

Kar, S. K., R. P. Turco, C. R. Mechoso, and A. Arakawa, 1994: A locally one-dimensional semi-implicit scheme for global gridpoint shallow-water models. *Mon. Wea. Rev*., **122**, 205-222.