Closure of the Stochastic Dynamic Equations

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Lorenz equations for his strange attractor offer a significant test

$\dot{X} = P(Y - X)$ $\dot{Y} = -XZ + RX - Y$ $\dot{Z} = XY - BZ$

Use X(i) for i = 1 to 3 for the means of [X, Y, Z]

Use X(j) for j = 4 to 9 covariance's: σ(1,1), σ(1,2), σ(1,3), σ(2,2), … σ(3,3)

Use X(k) for k = 10 to 19 third moments: T(1,1,1), T(1,1,2), T(1,1,3), T(1,2,2), T(1,2,3), T(1,3,3), T(2,2,2), T(2,2,3), T(2,3,3), T(3,3,3)

For R < 24.74 there are fixed point solutions – however, even here there is initial explosive randomness

For R ≥ 24.74 there are chaotic trajectories the initial explosive randomness is more extreme

The closure methodology is slightly different for the stable case versus the chaos situation; one must address both the initial explosive randomness of the moments and the final solution for the moments in phase space

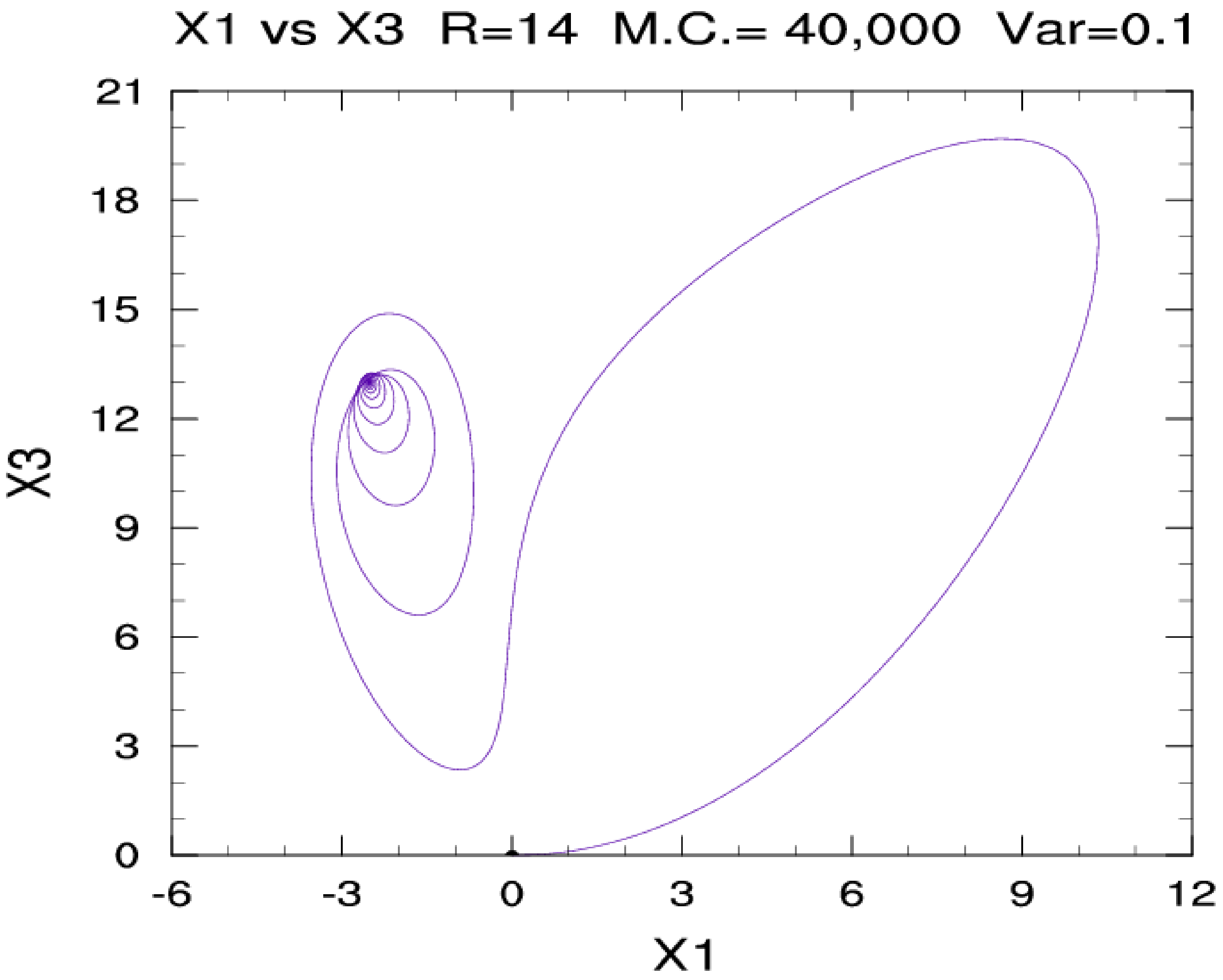


Fig.1 R = 14 in Lorenz: [X1, X2, X3] = [0, 1,0] and initial variance of X1 to X3 = 0.1 Sample size = 40,000. After initial wandering, X(3) = theoretical value R - 1 = 13

Since Fixed Point solutions, all z-deviates go to zero, so 2nd, 3rd, and 4th moments with one or more 3's as an index will → 0.0; thus: MC results are:
$\sigma(1,3) = \sigma(2,3) = \sigma(3,3) = 0$; and $\sigma(1,1) = \sigma(1,2) = \sigma(2,2) = 28.32$

3rd moments = 0, except $T(1,1,1) = T(1,1,2) = T(1,2,2) = T(2,2,2) = 142.723$

The MC results gave all 4th moments = 0 except
$\lambda(1,1,1,1) = \lambda(1,1,1,2) = \lambda(1,1,2,2) = \lambda(1,2,2,2), = \lambda(2,2,2,2) = 1521.1$

{ The full SD3 equations (with 4th moments & no assumptions on them) -- with the LHS of these equations = 0, produces the exact same results as the time averaged MC values (as shown in first talk) }

The 4th moments are not "normal"; if they were then: e.g., $\lambda(1,1,1,2) = 3\ \sigma(1,1)\ \sigma(1,2) = 3\ (28.32)^2 = 2406.1$; but $FF = \lambda_{calculated} / \lambda_{Normal}$

4th moments are platykurtic with FF = 1521.1 / 2406.1 = 0.632 < 1

The closure for the these fixed point solutions (e.g. R = 14) should be easy, and is; however the initial explosive randomness must be handled.

The main idea is to let the physics drive the closure.

The eddy-damped quasi-normal closure (sometimes used in the past and borrowed from turbulence theory) not needed here, and all 4th moments are platykurtic

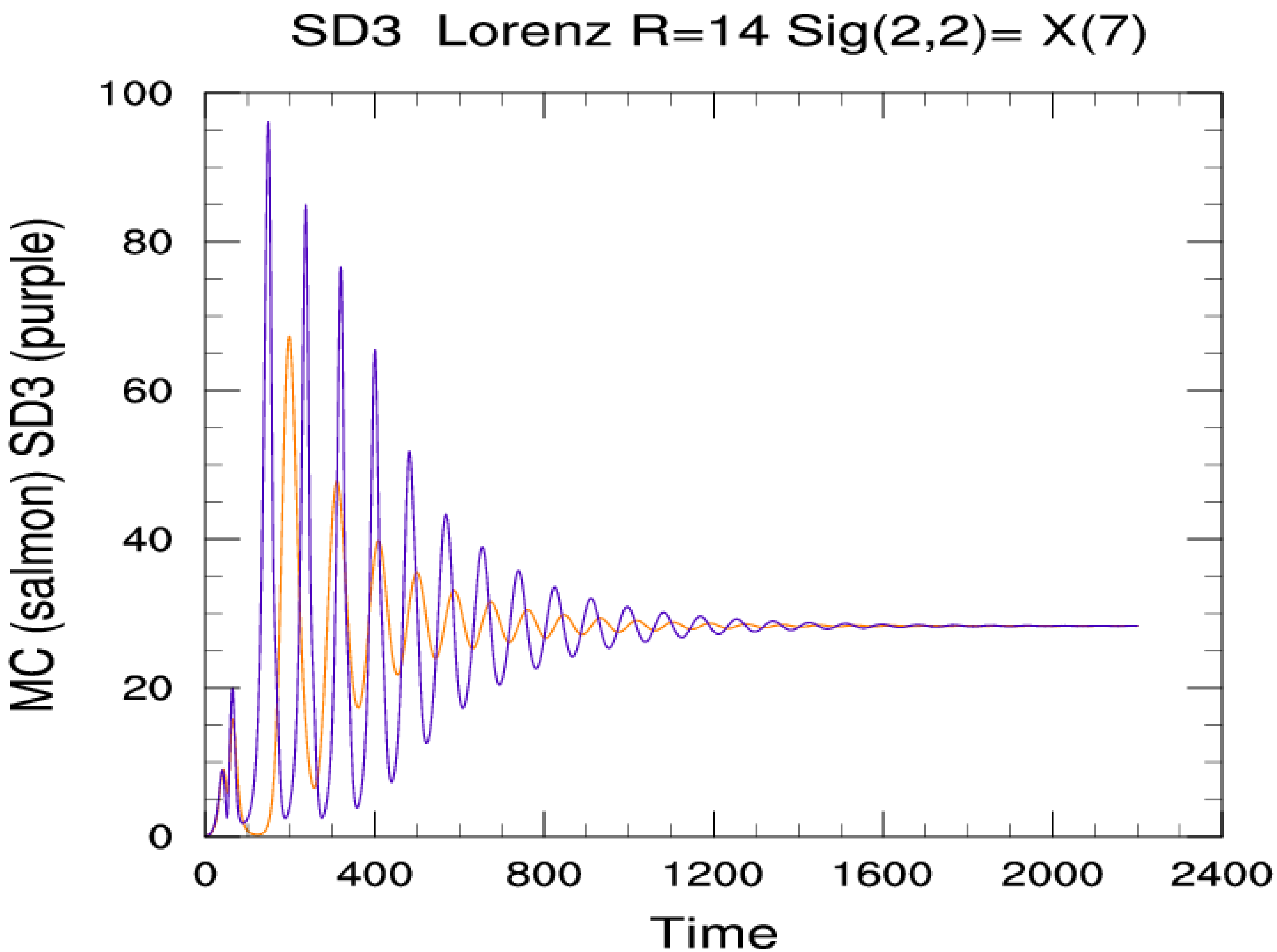
Since all the 4th moments → the same FF value of 0.632 or 0.0, the normal form of all 4th moments are multiplied by this FF forcing coefficient.

No damping factor is required on the 3rd moment equation containing the 4th moment → 0.0

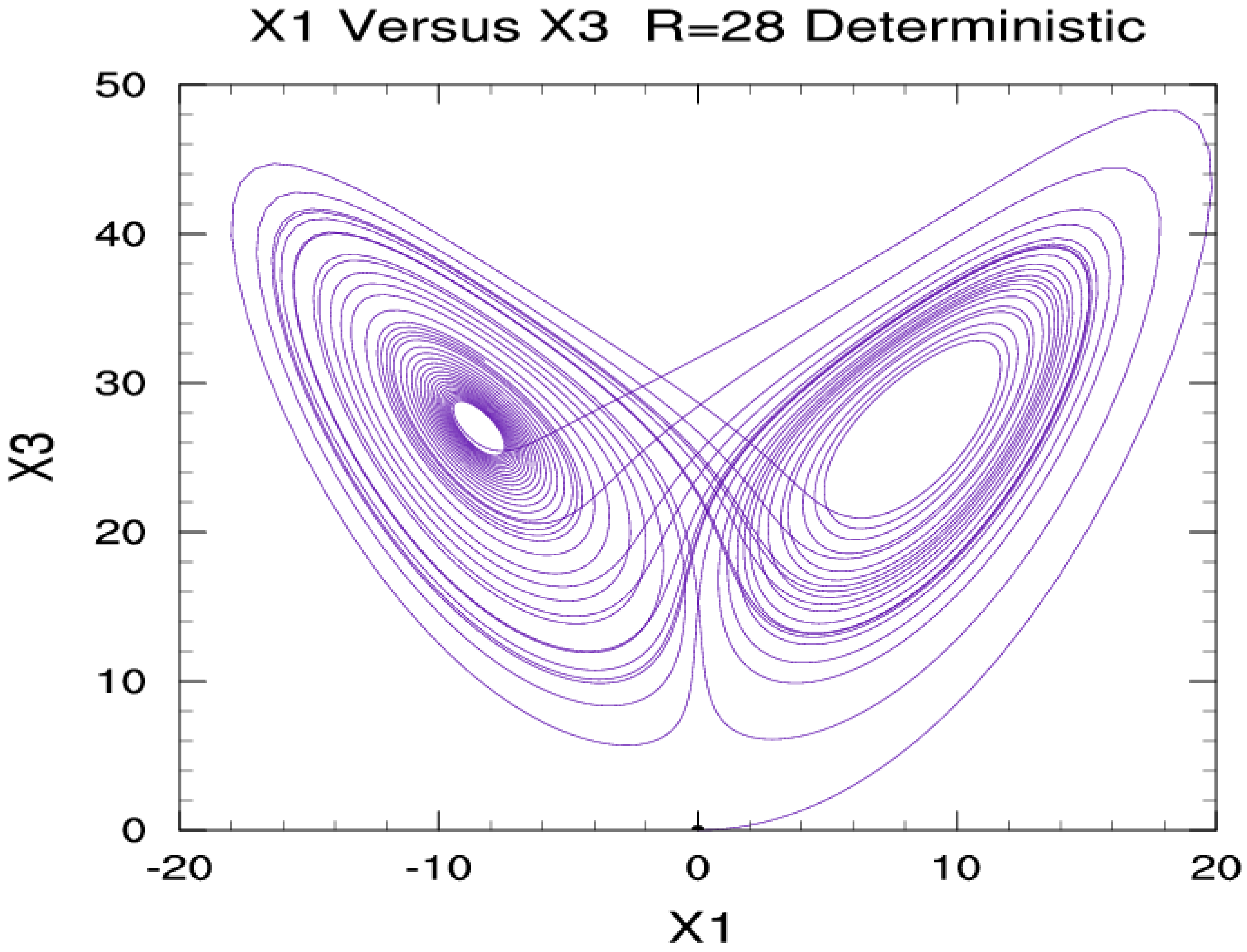
However, the same damping coefficient (DK) is used on all 3rd moment equations with a non-zero 4th moment to cope with the initial randomness encountered.

Values of (DK), used in $T(i, j, k)^{\cdot}$ = ... – (DK) T(i,j,k) , > 11.72 give correct answers.

The value used was DK = 12, but values as large as 5 times that value gave the same answers



MC (salmon) and SD3 (purple) agree, but even in this stable fixed point solution there is initial “explosive randomness”.



Significant changes from the fixed point solution – now Z ≠ R – 1 and Z - deviates are very important; and the probability distributions of X(1) and X(2) are symmetric

Because of X1 & X2 symmetry all moments with odd # of 1's and 2's → 0.0

Time averaged moments evolve to agree with the full SD3 equations

Moment	Variable #	MC Value	SD3 Value	Moment	Variable #	MC Value	SD3 Value
μ(1)	X(1)	– . 001	.000	T(1,1,1)	X(10)	.060	.000
μ(2)	X(2)	– .001	.000	T(1,1,2)	X(11)	.060	.000
μ(3)	X(3)	23.55	23.55	T(1,1,3)	X(12)	400.6	400.6
σ(1,1)	X(4)	62.80	62.80	T(1,2,2)	X(13)	.04	.000
σ(1,2)	X(5)	62.80	62.80	T(1,2,3)	X(14)	198.2	198.2
σ(1,3)	X(6)	– .005	– .000	T(1,3,3)	X(15)	– .08	–.000
σ(2,2)	X(7)	81.20	81.20	T(2,2,2)	X((16)	.009	.000
σ(2,3)	X(8)	.001	– .000	T(2,2,3)	X(17)	84.8	84.8
σ(3,3)	X(9)	74.34	74.34	T(2,3,3)	X(18)	– .06	– .000
				T(3,3,3)	X(19)	132.4	132.4

Table 1. Calculated MC values and computed SD3 values from full equations for R = 28

Table 2. The five 4th moment values in red / blue are active in four 3rd moment prediction equations.

Moment	Variable #	M C Value	SD3 Value	Moment	Variable #	M C Value	SD3 Value
λ(1,1,1,1)	X(20)	9060.1	9060.1	λ(1,1,3,3)	X(25)	6712.5	6713.0
λ(1,1,1,2)	X(21)	9060.2	9060.1	λ(1,2,2,2)	X(26)	13774	13774
λ(1,1,1,3)	X(22)	-0.14	0.0	λ(1,2,2,3)	X(27)	- 0.15	0.0
λ(1,1,2,2)	X(23)	10735	10735	λ(1,2,3,3)	X(28)	5021.5	5021.1
λ(1,1,2,3)	X(24)	-0.003	0.0	λ(1,3,3,3)	X(29)	- 1.2	0.0

The 4^{th} moments: X(20) and X(30) through X(34) are not in the SD3 equation set

The calculated MC values and computed SD3 values match extremely well

The number one rule in the closure exercise is to let the physics do the required damping where possible

By the statistical symmetry of X(1) and X(2), all 3rd and 4th moments with an even # of 1's and 2's will be non-zero and likely quite large (we examine these 4th mmts. in the next slide); 3rd and 4th moments with an odd # → 0.0

We do not damp the 3rd moment prediction equations where those 3rd moments ultimately → 0.0 [the 4th moments also → 0.0 in these eqs.]

However, we will need a small damping term (the same one for each such 3rd moment equation) to help control the degree of initial randomness

All 4th moments which → 0.0 are initially set to their "normal" form [$\lambda(i, j, k, l) = \sigma(i, j)\,\sigma(k, l) + \sigma(i, k)\,\sigma(j, l) + \sigma(i, l)\,\sigma(j, k)$]

MC time averages might suggest initially setting them to 0.0 but they only → 0.0

Still, these 4th moments can get quite large initially, then decline → 0.0 as dictated by the physics

Examine the 5 active 4^{th} moments which occur in 4 prediction equ.

$T(1,1,3)\cdot = X(12)^{\cdot}$ contains $\lambda(1,1,1,2)$ [FF < 1 platykurtic FF > 1 leptokurtic]

$\lambda^{Calc} = 9060.1$ from MC; $\lambda^{Normal} = 3\ \sigma(1,1)\ \sigma(1,2) = 11{,}831.5$ FF1 = 0.77

$T(1,2,3)\cdot = X(14)^{\cdot}$ contains $\lambda(1,1,2,2)$ and $\lambda(1,1,3,3)$

$\lambda^{Calc} = 10{,}737$ from MC; $\lambda^{Normal} = \sigma(1,1)\ \sigma(2,2) + 2\ \sigma(1,2)\ \sigma(1,2) = 12{,}987$ FF2 = 0.83

$\lambda^{Calc} = 6{,}713$ from MC; $\lambda^{Normal} = \sigma(1,1)\ \sigma(3,3) + 2\ \sigma(1,3)\ \sigma(1,3) = 4{,}668.6$ FF3 = 1.44

$T(2,2,3)\cdot = X(17)^{\cdot}$ contains $\lambda(1,2,2,2)$ and $\lambda(1,2,3,3)$

$\lambda^{Calc} = 5021.1$ from MC; $\lambda^{Normal} = \sigma(1,2)\ \sigma(3,3) + 2\ \sigma(1,3)\ \sigma(2,3) = 4668.6$ FF4 = 1.08

$\lambda^{Calc} = 13774$ from MC; $\lambda^{Normal} = 3\ \sigma(1,2)\ \sigma(2,2) = 15{,}298$ FF5 = 0.90

$T(3,3,3)\cdot = X(19)^{\cdot}$ contains $\lambda(1,2,3,3)$

$\lambda^{Calc} = 5021.1$ from MC; $\lambda^{Normal} = \sigma(1,2)\ \sigma(3,3) + 2\ \sigma(1,3)\ \sigma(2,3) = 4668.6$ FF4 = 1.08

Each of the above λ terms is replaced by FF(i) x (their normal form) which, of course, just makes them equal to their calculated value

Closing the SD3 equations for chaos requires two phases

 # 1: Compute a single damping term DK for those 3rd moments → 0 anyway to guide them through the explosive randomness phase; Use any small nominal value of damping for the active (non-zero) 3rd moments

Optimize DK by best results (i.e., all moments that → 0.0 are exactly 0.0) after 4000 iterations (value was 16.8) using DK1 through DK5 = 6.0

 # 2: Compute unique damping terms DK1 through DK4 for each of the four 3rd moment equations – computed from all terms in the equations (including 1.2*FF(i) for the five active 4th moments) to make the time tendency balance { LHS = 0 } (terms which → 0 are 0.0 by phase 1)

Example: time tendency of T(1,1,3) = X(12)

X(12)˙ = 0 with all the terms set to what the MC & SD3 equations provided.

= … - 9060 … + λ(1,1,1,2) = 0 ; because λ(1,1,1,2) = 9060

Fix by creating an imbalance!

Fix by increasing λ by 1.2 λ and add a damping term to RHS, then:

X(12)˙ = … - 9060 + (1.2) (9060) – DK1 X(12) = 0

Therefore, DK1 = (0.2) (9060) / X(12) = 1812. / 400.64 = ~ 4.523

Values of the damping coefficients at each stage of closure:

(using values X(1), X(12), X(14), X(17) and X(19) as examples at 4000 iterations)

PHASE	DK	DK1 X(12)	DK2 X(14)	DK3 X(17)	DK4 X(19)	X(1)	X(12)	X(14)	X(17)	X(19)
1.1	5.45	6.0	6.0	6.0	6.0	- 3.1	250.6	138.4	144.0	294.2
1.2	16.80	6.0	6.0	6.0	6.0	0.0	379.0	193.1	111.7	284.9
2.1	16.80	4.523	6.0	6.0	6.0	-6×10^{-10}	400.2	196.9	114.8	289.2
2.2	16.80	4.523	4.058	6.0	6.0	-3×10^{-9}	401.3	200.2	96.4	293.3
2.3	16.80	4.523	4.058	8.796	6.0	-2×10^{-10}	400.6	198.2	84.8	290.9
2.4	16.80	4.523	4.058	8.796	22.75	-2×10^{-11}	400.6	198.2	84.8	132.4

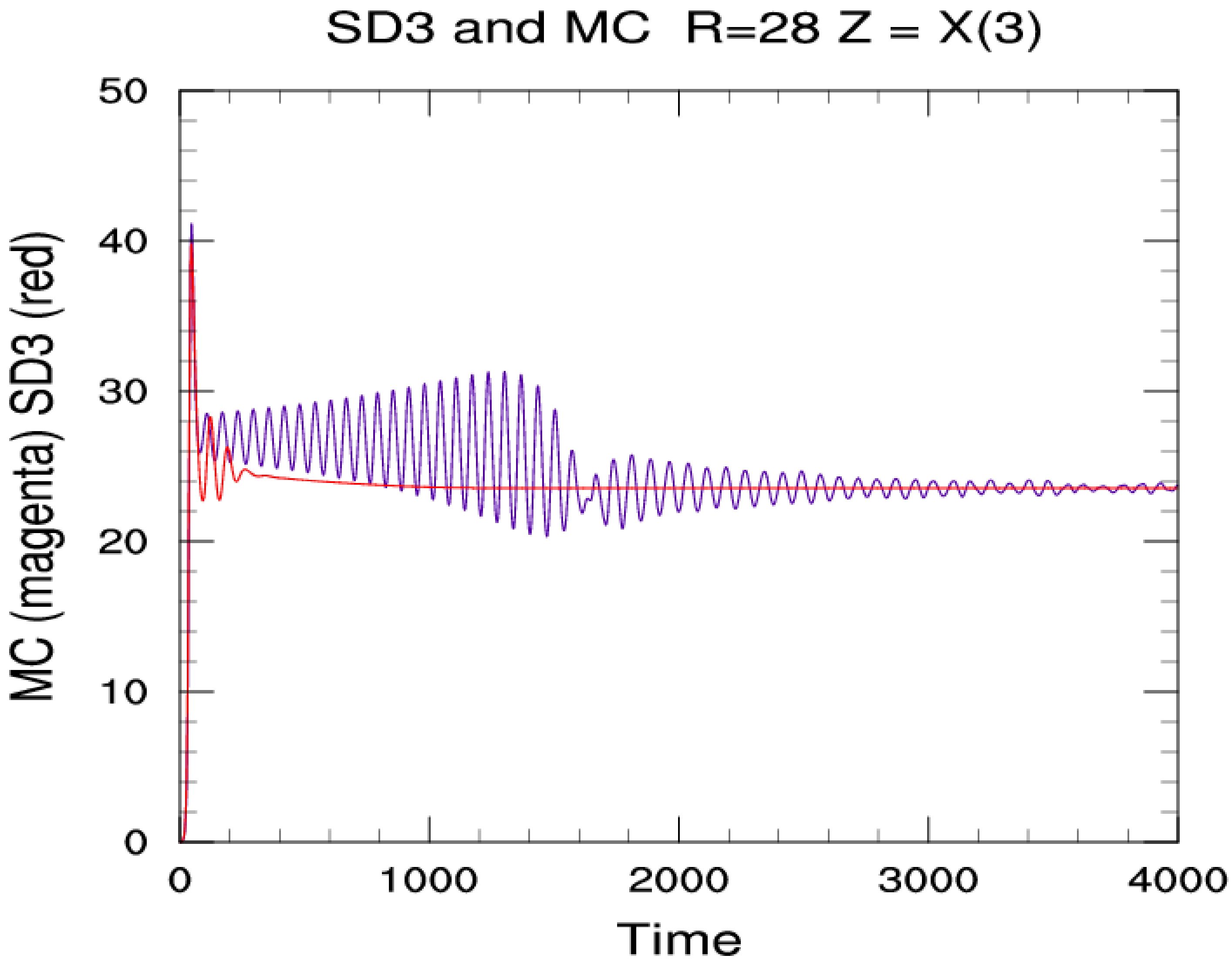


Figure 4. Both Monte Carlo and SD3 calculations have Z = X(3) = 23.55, and and the initial randomness has been captured

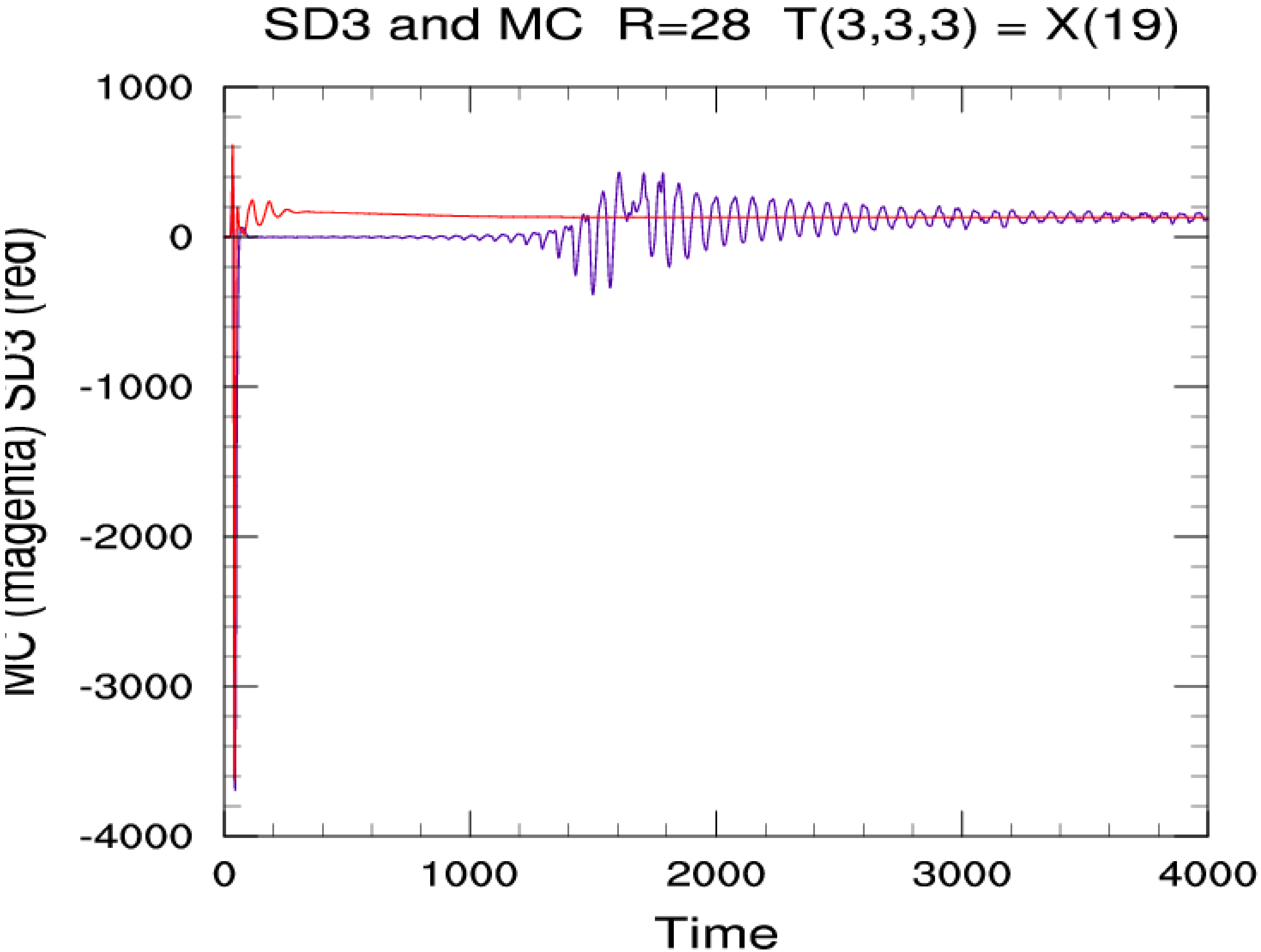
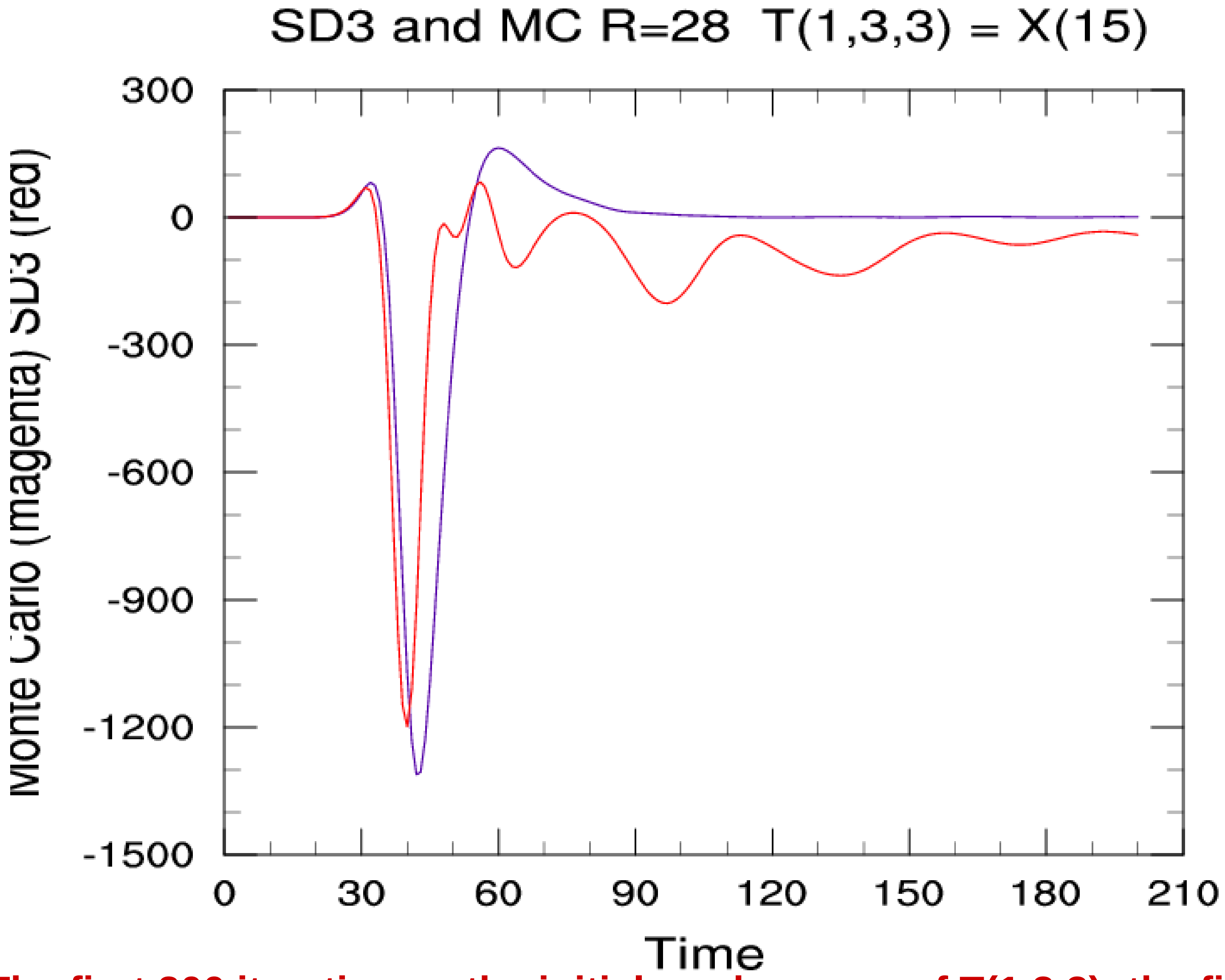


Figure 6. MC and SD3 reach correct value of T(3,3,3) of 132.4, the enormous explosive randomness is captured "virtually" perfectly



The first 200 iterations – the initial randomness of T(1,3,3); the final value is 0.0

Any bounded and dissipative system of equations, like the Lorenz set, can be closed with a similar procedure: flooding the attractor with a large sample size, then performing a suitable time averaging

Closure is more complicated with a system of nonlinear cubic equations

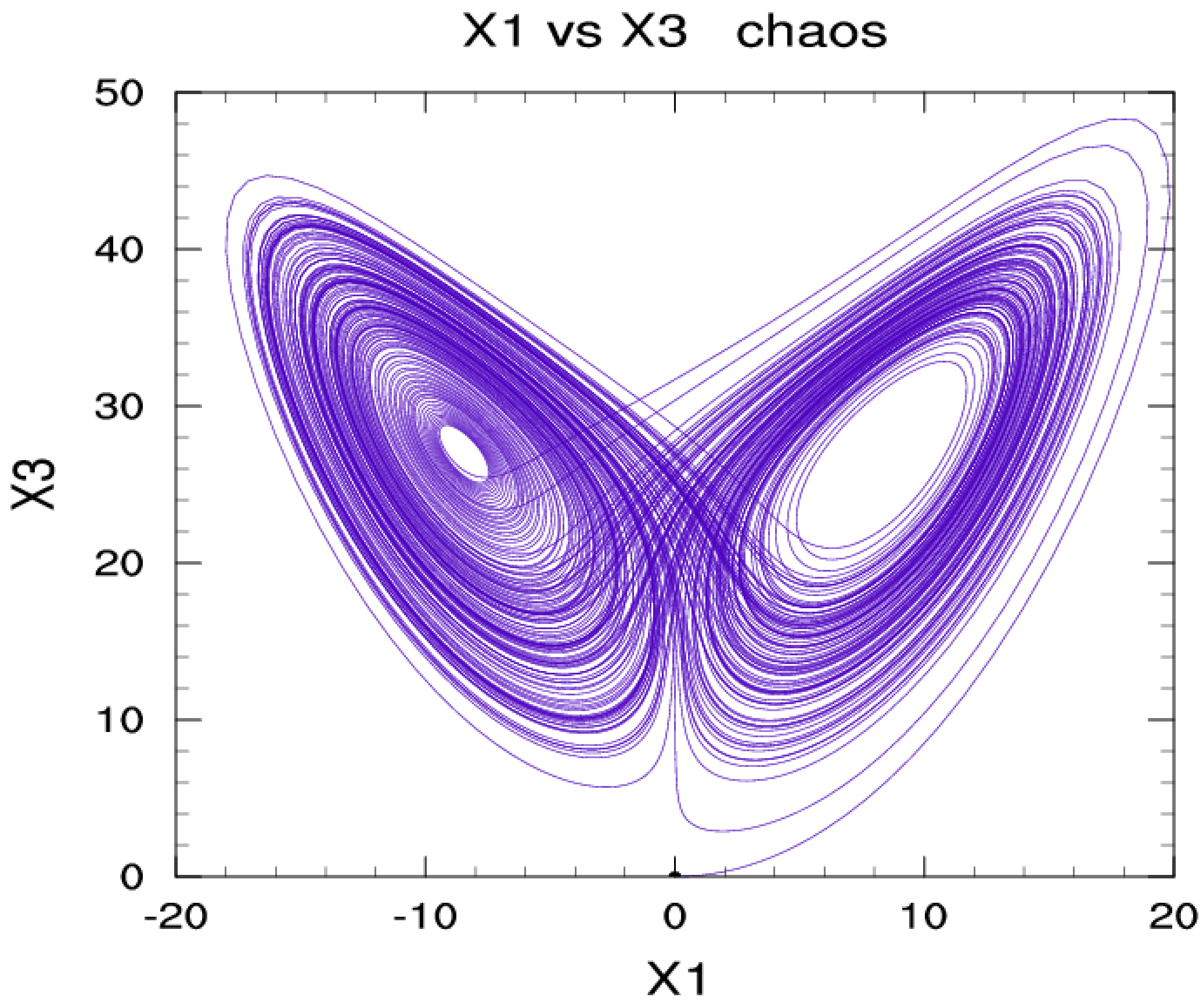
e.g. $\dot{X}_i = \ldots X_j X_k X_l \ldots$

Here the general form for the n-th moment about the mean is:

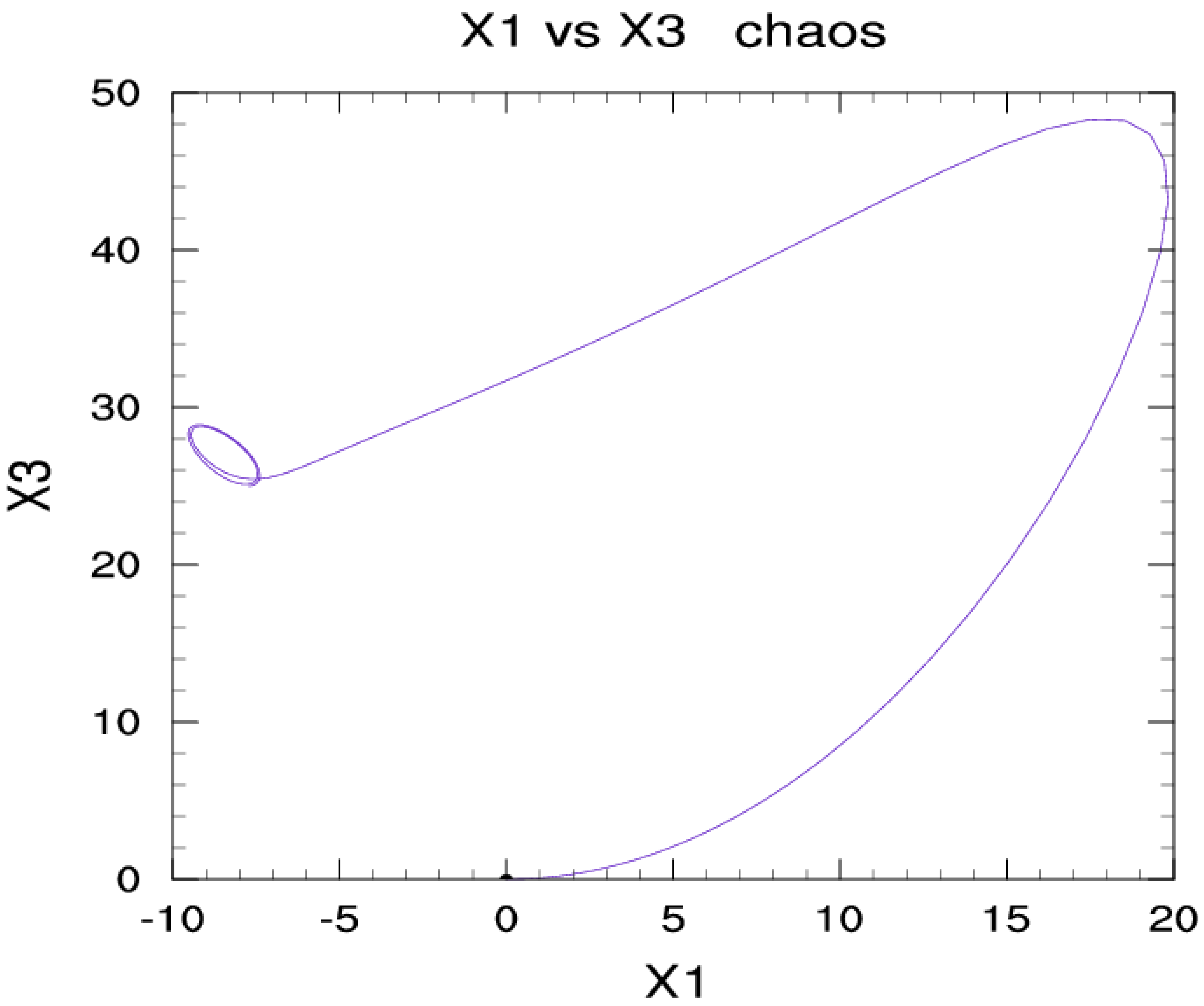
$$\dot{f}_n = n \, [\, 3 \mu f_{n+1} + 3 \mu^{(2)} f_n - 3 \mu f_2 f_{n-1} - f_3 f_{n-1} + f_{n+2} \,]$$

where the general form for the nonlinear quadratic case was simply:

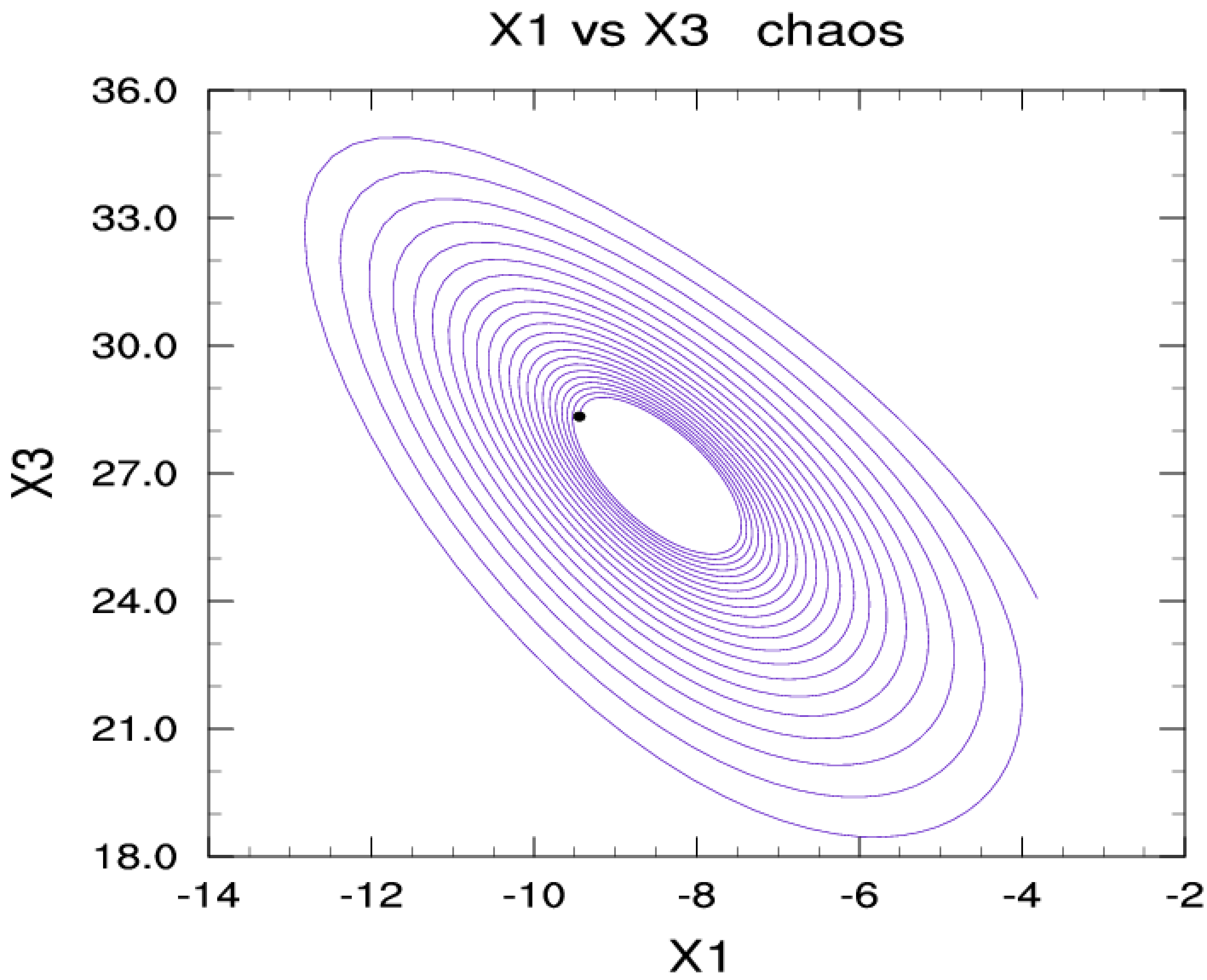
$$\dot{f}_n = n \, [\, 2 \mu f_n - f_2 f_{n-1} + f_{n+1} \,]$$



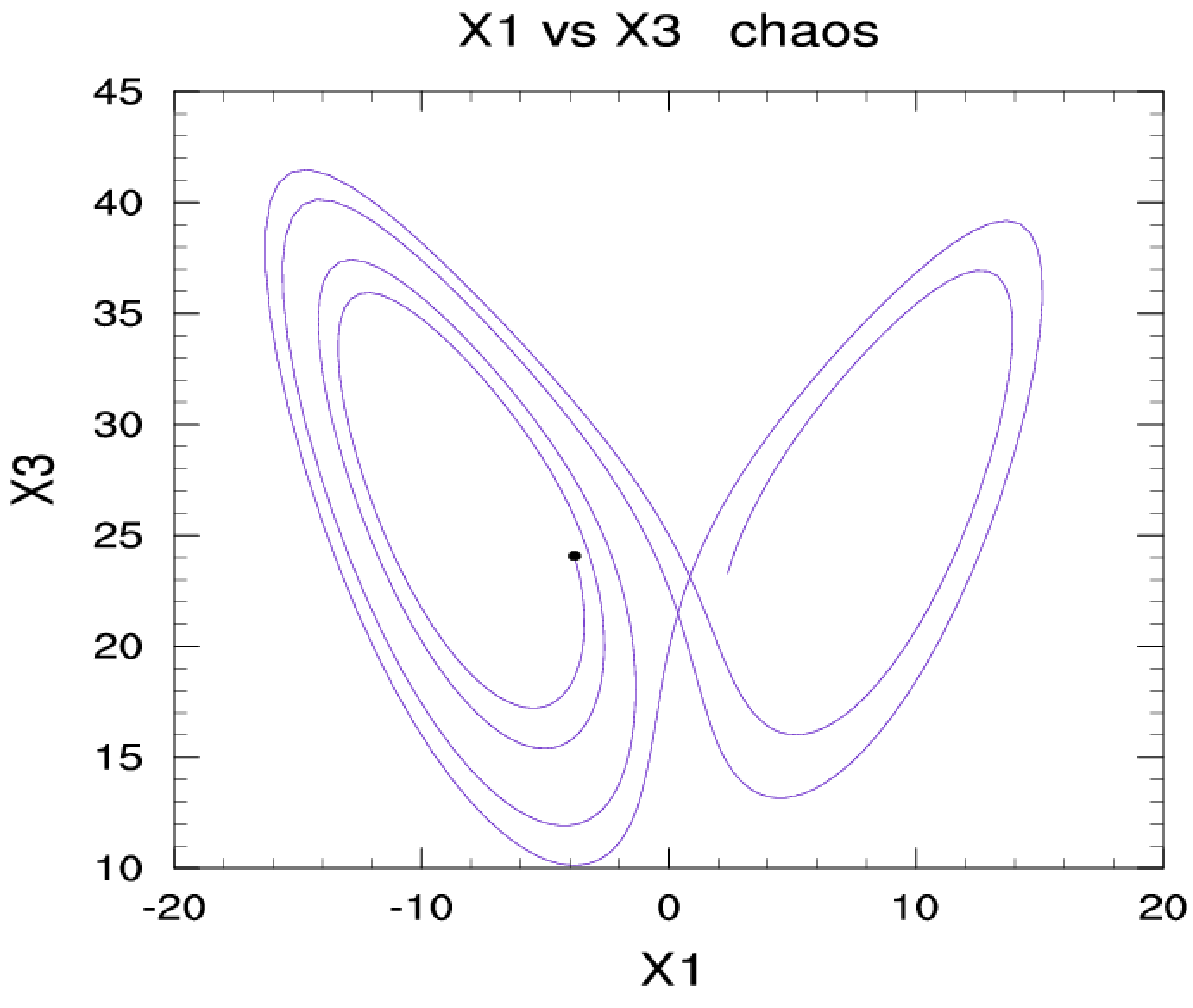
Lorenz set: R = 28; The full 16,000 iterations



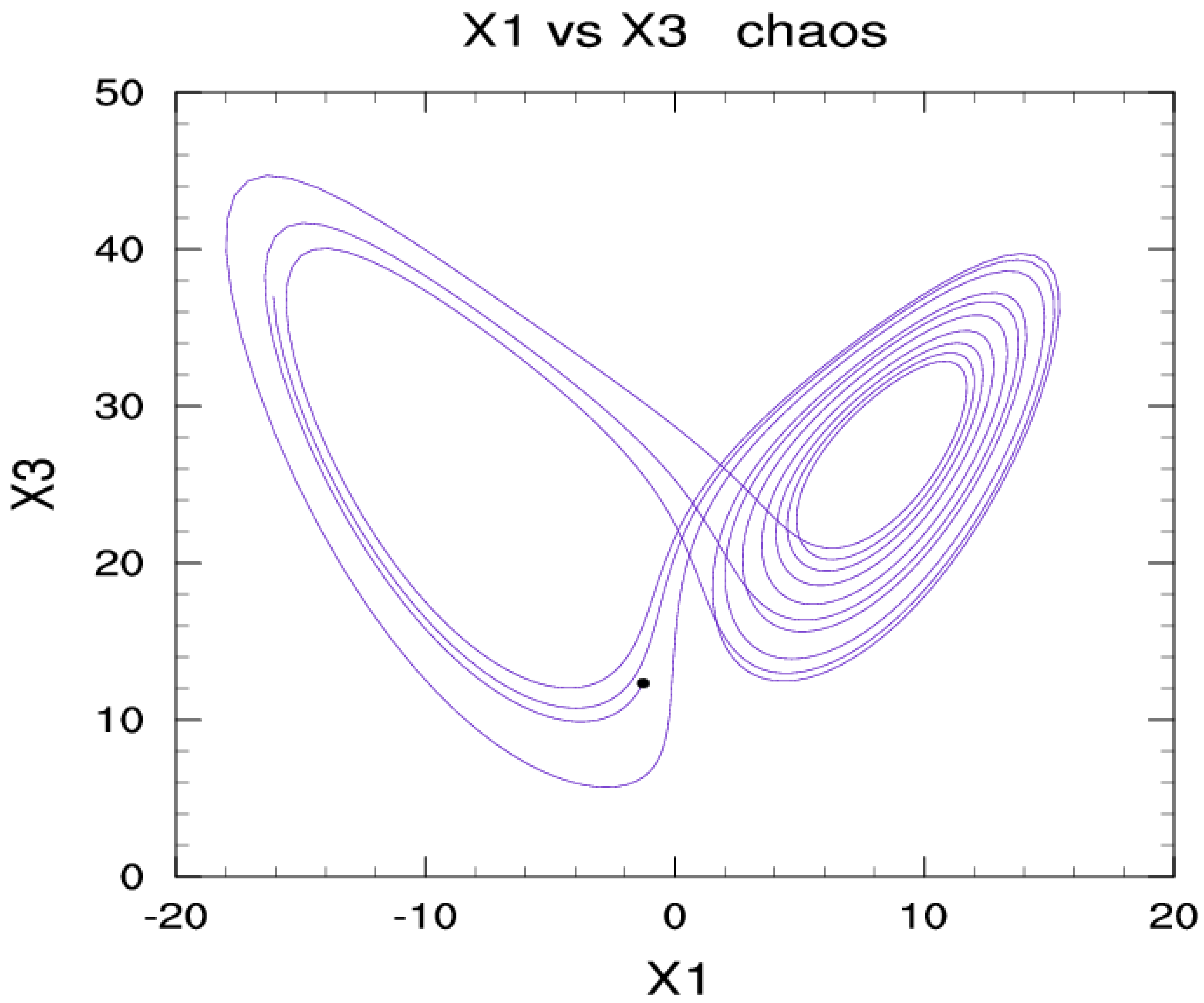
This is deterministic run from iteration 0 to 200



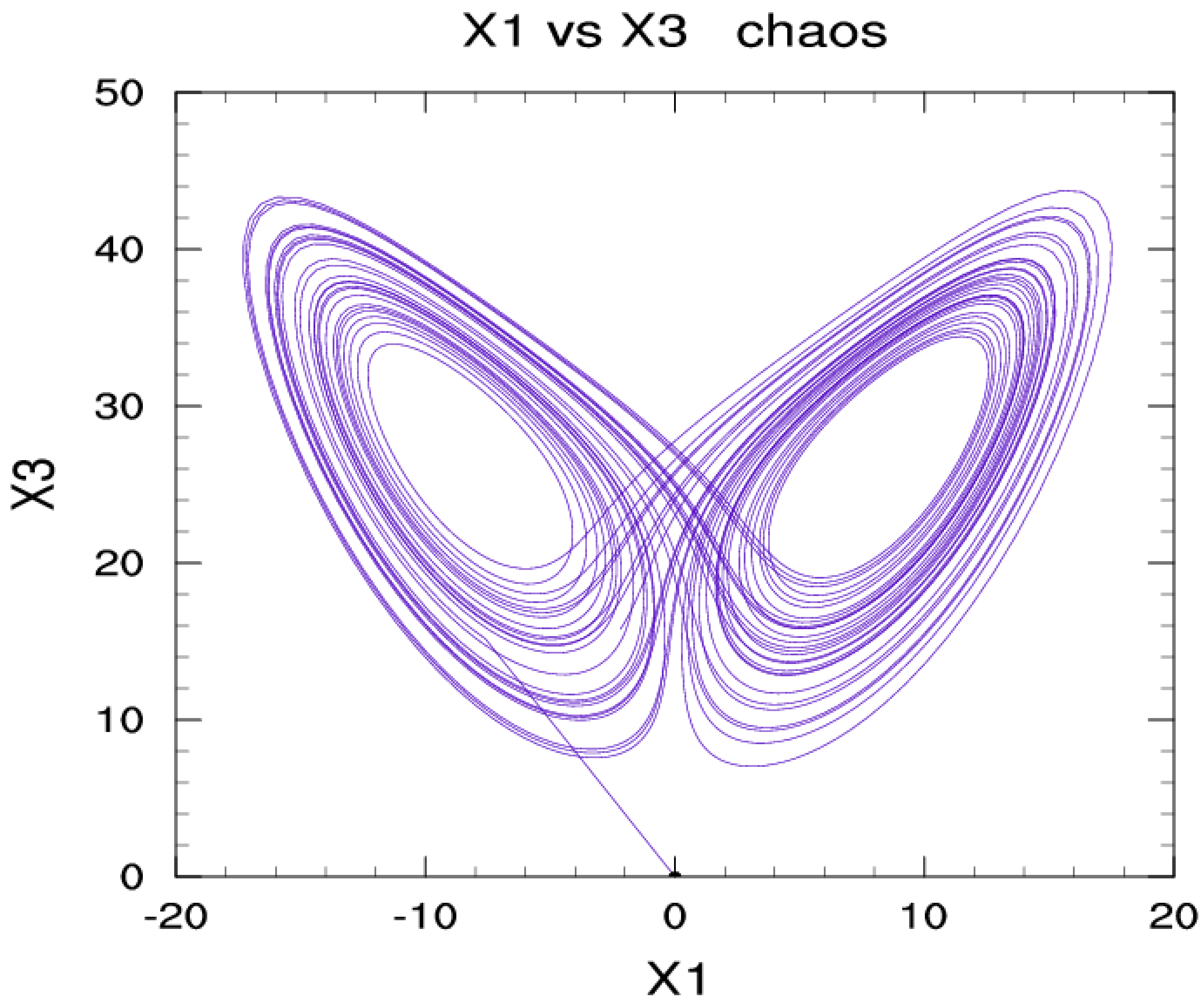
This is deterministic run from iteration 100 to 1450



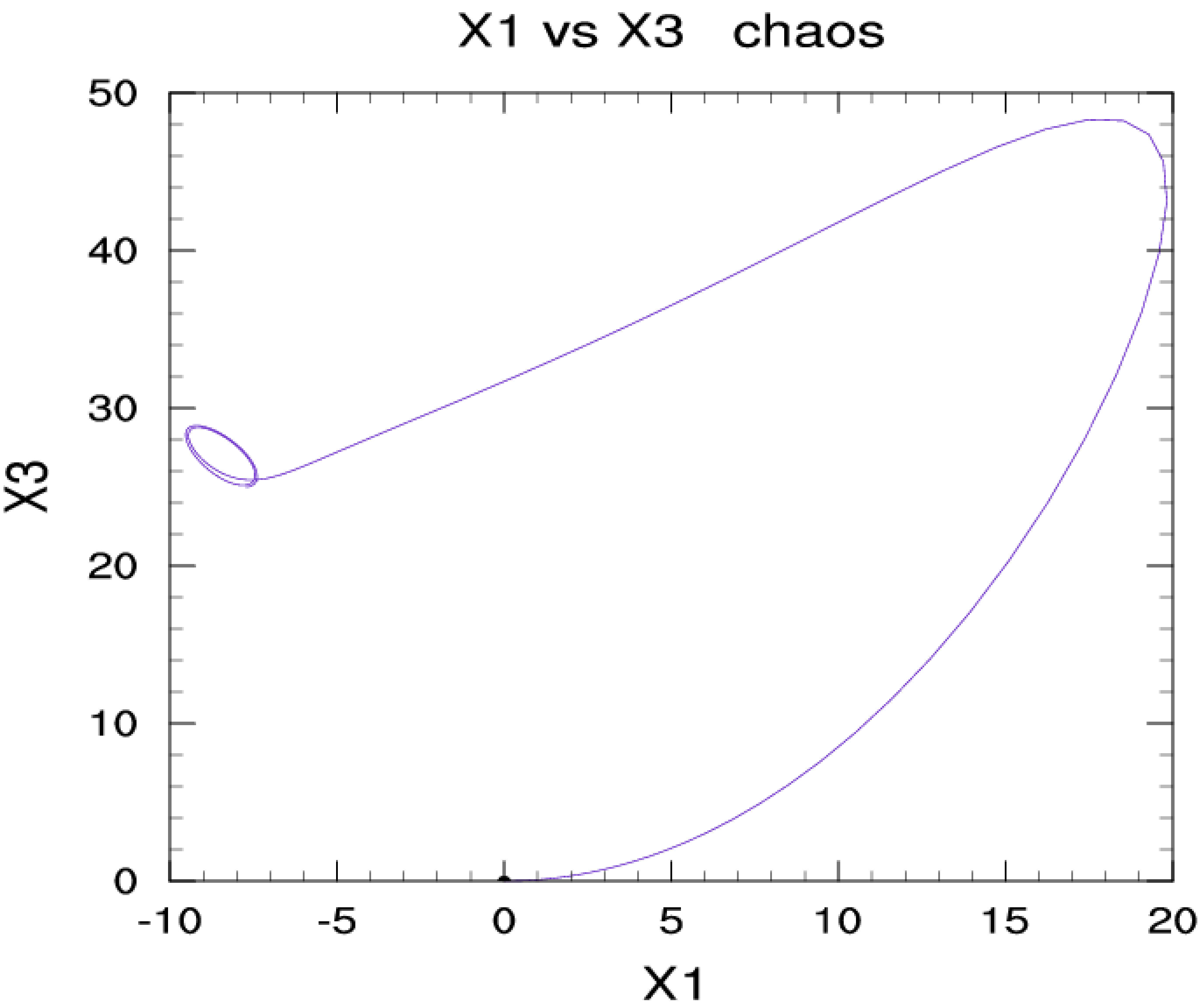
This is from iteration 1450 to 1900



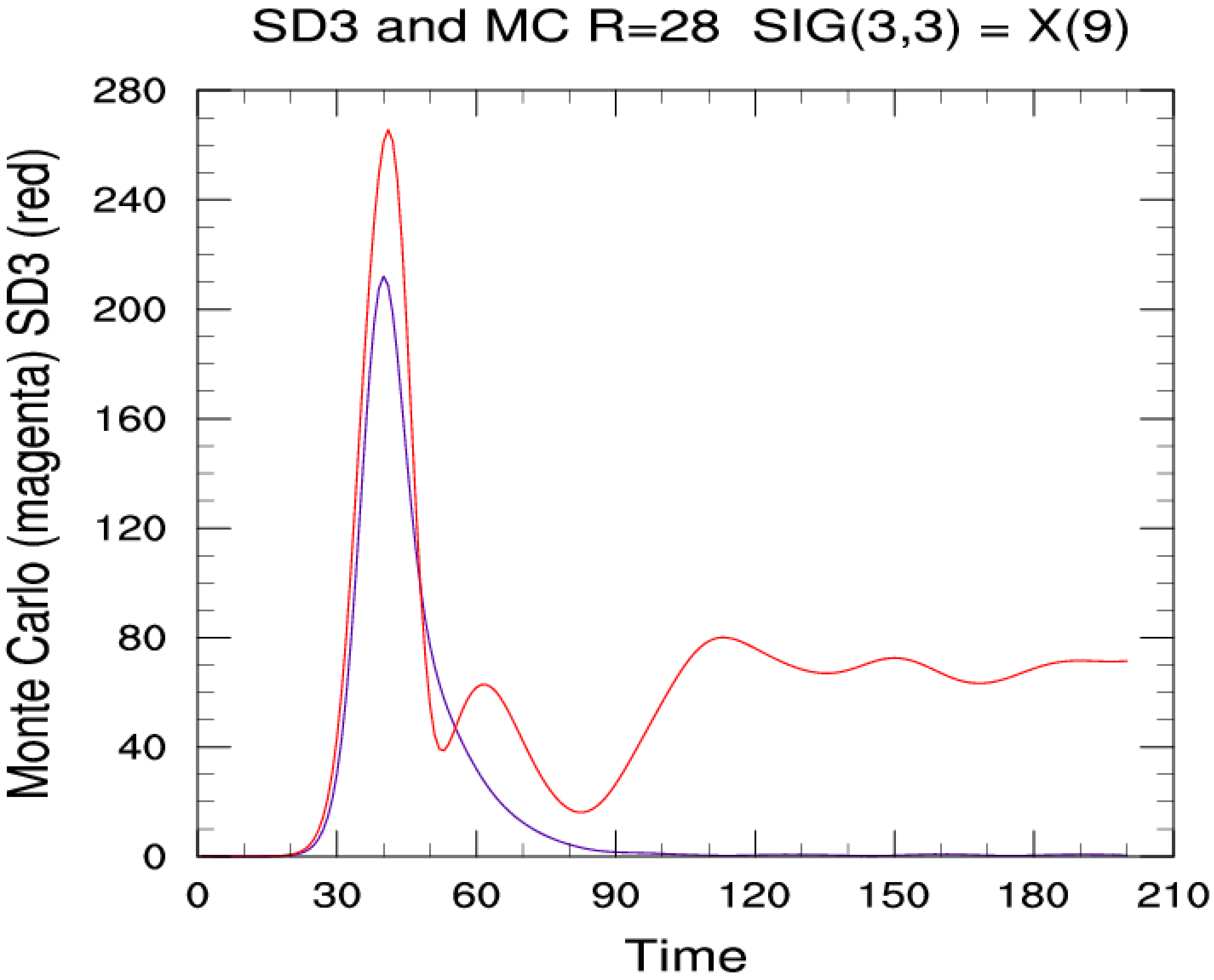
This from iteration 2000 to 3000



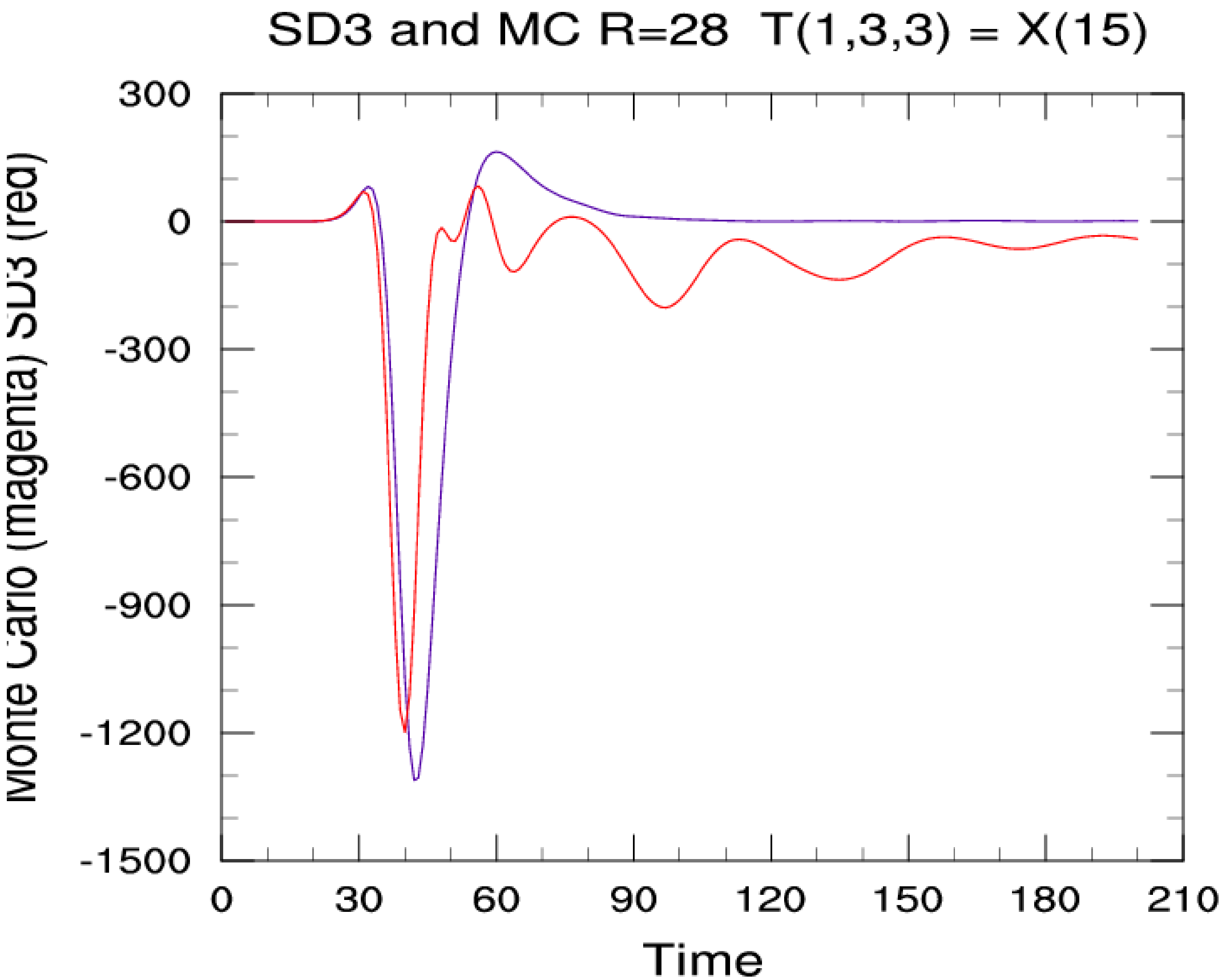
Lorenz set: R = 28; the final 8000 iterations of 16,000



This is deterministic run from iteration 0 to 200



The first 200 iterations – the initial random variance of X(3)



The first 200 iterations – the initial randomness of T(1,3,3); the final value is 0.0

Details of closure (an example)

$X(12)^{\cdot}$ = 0 with all the terms set to what the SD3 equations provided.
= … - 9060 … + λ(1,1,1,2) = 0 ; because λ(1,1,1,2) = 9060
Fix by adding a damping term to RHS and increase λ by 1.2λ; then

$X(12)^{\cdot}$ = … - 9060 + (1.2) (9060) – DK1 X(12) = 0

Therefore, DK1 = (0.2) (9060) / X(12) = 1812. / 400.64 = ~ 4.523

Therefore in the beginning the chaotic physics will drive X(12) perhaps crazily, but it will eventually settle down to its derived value

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$X(14)^{\cdot}$ = …- 4022 + … [λ(1,1,2,2) – λ(1,1,3,3)] = 0 because where the
λ difference = 10,735 – 6713 = 4022
Fix by adding a damping term and increase both λ's by factor of 1.2

Therefore, DK2 = (0.2) (4022) / X(14) = 804.4 / 198.24 = 4.0577

One wants λ(1,1,2,2) > λ(1,1,3,3). The kurtosis is of the two λ's is not important. [Defined as $B_2 = u_4 / (u_2)^2$ -- in a normal distribution $B_2 = 3$ and it is mesokurtic, platykurtic or leptokurtic – according as B_2 = 3, < 3, or > 3. The distribution is flatter about the mean if platykurtic – think of the flat bill of a platypus