# Closure of the Stochastic Dynamic Equations


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## 1. Introduction

The Stochastic Dynamic Equations (SDE) approach to dealing with uncertainty in initial conditions in numerical models, and the 2nd moments associated with them, was introduced into atmospheric science by Epstein (1969). The 3rd moment equations were outlined and discussed by Fleming (1971). The other major approach to dealing with such uncertainty is the Monte Carlo (MC) method. The MC approach draws upon a finite number of random deviates, usually from a normal distribution with a pre-defined variance. The SDE approach begins with an infinite ensemble of initial states, with that same predefined variance. The MC is thus an approximation of the SDE method. However, the SDE has a closure issue. Time derivatives of second moments also requires knowledge of third moments,  time derivatives of third moments involves fourth moments, and so on.

In a recent paper, Fleming (2014) reviewed the two approaches with uncertainty in the initial conditions of the Lorenz original strange attractor equations ( Lorenz, 1963). In that paper the use of both methods revealed the full statistical ensemble of relationships between the variables and moments in that equation set.

When a system is bounded and dissipative as the Lorenz system, all trajectories eventually tend toward some bounded set of zero volume in phase space. Using this fact, a long integration of the equations over time was used with a very large MC sample size. Flooding the attractor with many initial states, the time variance of a range of time averaging periods was used to judge the time independence of the moments. Combining the MC values with the ‘full’ SDE equation set produces the complete statistical relationships between the variables and moments. The ‘full’ SDE equation set includes 4th moments in 3rd moment equations, and with no assumptions on either of the moments.

This data set then affords the possibility to close the SDE equations set for those bound and dissipative systems. The Lorenz equations have both fixed point and chaotic solutions. The closure of the equations for both solution types is presented here. The closure methodology is slightly different for each solution type. One must address two issues for each type of solution: both the issue of initial explosive moment dispersion in phase space, and the final time independent value of the moments in phase space.

Examples are shown which compare the high resolution MC solutions with the SDE equation solutions  with the proper closure. Both issues are successfully addressed by the closure technique implemented as seen in the Figures presented.

## 2. Equations and nomenclature

The general form of a quadratic nonlinear deterministic data set and the SDE counterpart is provided below; the dummy indices p,q represent pairs of nonlinear terms):

$$\dot{X}_i = \sum_{p,q} a_{ipq} X_p X_q - \sum_{p} b_{ip} X_p + c_i$$

**The SDE equations follow:**

$$\dot{\mu}_i = \sum_{p,q} a_{ipq} ( \mu_p \mu_q + \sigma_{pq} ) - \sum_{p} b_{ip} \mu_p$$

$$\dot{\sigma}_{ij} = \sum_{p,q} a_{ipq} (\mu_p \sigma_{jq} + \mu_q \sigma_{jp} + T_{jpq}) + a_{jpq} (\mu_p \sigma_{iq} + \mu_q \sigma_{ip} + T_{ipq}) - \sum_{p} (b_{ip} \sigma_{jp} + b_{jp}\sigma_{ip})$$

$$\begin{aligned}\dot{T}_{ijk} = \sum_{p,q} \{ & a_{ipq} ( \mu_p T_{jkq} + \mu_q T_{jkp} - \sigma_{pq} \sigma_{jk} + \lambda_{jkpq} ) \\ & + a_{jpq} ( \mu_p T_{ikq} + \mu_q T_{ikp} - \sigma_{pq} \sigma_{ik} + \lambda_{ikpq} ) \\ & + a_{kpq} ( \mu_p T_{ijq} + \mu_q T_{ijp} - \sigma_{pq} \sigma_{ij} + \lambda_{ijpq} )\} - \sum_{p} ( b_{ip} T_{jkp} + b_{jp} T_{ikp} + b_{kp}T_{ijp})\end{aligned}$$

**where the $\mu(i)$ = the mean of X(i), $\sigma$ (i,j) = the covariance of X(i) X(j) , T(i,j,k) = the 3rd moment about the mean, and $\lambda$ (i, j, k, l ) is the 4$^{th}$ moment about the mean.**

**The Lorenz equations for his strange attractor offer a significant test for the closure of the SDE equations. These equations are:**

$$\dot{X} = P (Y - X) \qquad (1)$$

$$\dot{Y} = - X Z + R X - Y \qquad (2)$$

$$\dot{Z} = X Y - B Z \qquad (3)$$

where a ( • ) denotes a time derivative, and where **X, Y, Z** will be replaced with **X(1), X2), X(3)** to make subsequent notation somewhat easier to follow; and where **P = 10, B = 8/3,** and where stable fixed point (FP) solutions occur for **R < 24.74 and chaos occurs for R equal to or greater than 24.74.**

**The notation used here is:**

**X( i ) for i = 1 to 3 for the means of [ X, Y, Z ]**

**X( j ) for j = 4 to 9 for the covariance's: σ(1,1), σ(1,2), σ(1,3), σ(2,2), σ(2,3), σ(3,3)**

**X( k ) for k = 10 to 19 for the third moments: T( 1,1,1), T(1,1,2), T(1,1,3), T(1,2,2), T(1,2,3), T(1,3,3), T(2,2,2), T(2,2,3), T(2,3,3), T(3,3,3).**

**Solutions for equations (1) – (3) and the stochastic counterparts were performed in double precision with a fourth order Runge-Kutta scheme with a time step of 0.01.**

## 3. The stable fixed point solutions using R = 14

The deterministic solution of **(1) –(3)** with initial conditions for [X(1), X(2), X(3)] = [0,1,0] and the value of **R = 14 p**roduces the theoretical result of **X(3) = R - 1 = 13, and X(1) = X(2) = ± 5.888.** The MC result (with a very large sample size of 40,000) with the same initial conditions and a variance of **0.1** for each of the variables produces the result shown in **Figure 1**.

The value of **X(1) = X(2) = – 2.52,** and the value of **X(3), after some initial wandering,** is indeed **13, a constant value.**

Because **Z = X(3)** evolves to a constant value, all Z-deviates go to zero, so $2^{nd}$, $3^{rd}$, and $4^{th}$ moments with one or more 3's as an index will tend to zero. Thus, for example: **σ(1,3) = σ(2,3) = σ(3,3) = 0.0.** The MC results verify this and the further results **σ(1,1) = σ(1,2) = σ(2,2) = 28.316.** Fleming (2014) showed that the SDE equations, with the time tendency set to zero, reproduce the MC results. Just a few examples are shown below.

$$X(1)^{\bullet} = \mu(1)^{\bullet} = P\ (\mu(2) - \mu(1)) \tag{4}$$

$$X(2)^{\bullet} = \mu(2)^{\bullet} = -\mu(1)\,\mu(3) - \sigma(1,3) + R\,\mu(1) - \mu(2) \tag{5}$$

$$X(3)^{\bullet} = \mu(3)^{\bullet} = \mu(1)\,\mu(2) + \sigma(1,2) - B\,\mu(3) \tag{6}$$

$$X(4) = \sigma(1,1)^{\bullet} = 2P\,(\sigma(1,2) - \sigma(1,1)) \tag{7}$$

$$X(7)^{\bullet} = \sigma(2,2)^{\bullet} = -2\,[\,\mu(1)\,\sigma(2,3) + \mu(3)\,\sigma(1,2) + T(1,2,3) - R\,\sigma(1,2) + \sigma(2,2)\,] \tag{8}$$

$$X(9)^{\bullet} = \sigma(3,3)^{\bullet} = 2\,[\,\mu(1)\,\sigma(2,3) + \mu(2)\,\sigma(1,3) + T(1,2,3) - B\,\sigma(3,3)\,] \tag{9}$$

From the time tendency of **(4)** = 0, one obtains **X(1) = X(2)**. From **(5)** one finds:
**- X(1)X(3) + R X(1) –X(2) = 0 or [- X(3) + R - 1)] X(1) = 0 or X(3) = R - 1**

From **(6)** one finds: **$X(1)^2$ + X(5) = B X(3) = B** (**R - 1)** and from **(7), X(4) = X(5).**
**Combining results from (6) and (7) one has:**
**X(4) = B (R - 1) - $X(1)^2$ = ( 8/3) (13) - [ $(- 2.52)^2$ ] = 34.6667 - 6.3504 = 28.316**

The MC results give all $3^{rd}$ moments = 0 except:
**T(1,1,1) = T(1,1,2) = T(1,2,2) = T(2,2,2) = 142.723**

The MC results give all $4^{th}$ moments = 0 except:
**λ(1,1,1,1) = λ(1,1,1,2) = λ(1,1,2,2) = λ(1,2,2,2) = λ(2,2,2,2) = 1521.1**

A further progression through the full SDE equations and no assumptions on the $4^{th}$ moments reveals the same values for the $3^{rd}$ and $4^{th}$ moments as the MC results; e.g.,
**λ(1,1,1,2) = X(4) [ 4B (R - 1) - 3X(4) ] = (28.316) [ 53.719] = 1521.1**

These 4th moments are not “normal”. If they were, then for example:
**$\lambda(1,1,2,2) = \sigma(1,1)\ \sigma(2,2) + 2\sigma(1,2)\ \sigma(1,2) = 3\ (28.316)^2 = 2405.4$.** Thus the **4th moments are platykurtic (a broader distribution about the mean) with the ratio of $\lambda / \lambda_{normal} < 1$. Here, this ratio is referred to as FF = 1521.1 / 2405.4 = 0.63237.**

## 4. Closure of the SDE for fixed point solutions

The closure of the SDE equation set for these fixed point solutions should be fairly easy – and it is. However, the initial explosive randomness must be handled. The main idea in this closure is to let the physics drive the results.

The formerly used quasi-normal scheme (which had limited value) was borrowed from past turbulence theory. This assumes the normal form for the 4th moments in a 3rd moment prediction equation and adds a damping term **(DK)** to that equation; e.g., **$T(i, j, k)^{\bullet} = \ldots - (DK)\ T(i,j,k)$.** This would not work here as all the 4th moments are platykurtic, not “normal”.

The closure here has a few steps. First, the non-zero 4th moments all have the same **FF** value of **0.63237**, so the **normal form** of those 4th moments are multiplied by the forcing coefficient **FF**. Second, for consistency, this is done for all the 4th moments – even those that tend to zero, but may have appreciable values in the early transition phase. For example, **$\lambda(1,1,3,3)$** is replaced by **(FF) $[\sigma(1,1)\ \sigma(3,3) + 2\ \sigma(1,3)\ \sigma(1,3)]$ even though $\sigma(1,3)$** and **$\sigma(3,3)$** are tending toward zero. The third point is that **no** damping factor is applied to those 3rd moment equations with 4th moments that go to zero – the physics will drive the results.

The fourth and final point is that a damping term **is** applied to those 3rd moment equations that contain a non-zero 4th moment to cope with the initial randomness encountered. Iterative calculations found that values of **DK > 11.72** gave correct answers for all the variables. The value used for the fixed point solutions shown here was **DK = 12,** but values as large as five times this value gave the same results.

**Figure 2** shows the calculation of the MC solution over time versus that of the SD3 equations closed as discussed above. The MC sample size was as before equal to 40,000. One can be assured that the MC solution is statistically correct with that large a sample size. **Fig. 2** indicates that the two solutions agree in the final value of **$\sigma(2,2) = 28.316$.**

Equally important, the initial explosive randomness shown in **Fig. 2** is correctly handled. This initial randomness is seen in all the variables to some degree, and other examples are not shown to limit the length of this paper. All examples show the same results of the closure matching the MC results.

More examples will be shown in the next Section where, for chaos solutions, the explosive randomness is quite severe.

## 5. Closure of the SDE for chaotic solutions

The deterministic solution for the Lorenz equations for R = 28 is shown in **Figure 3.** This represents a significant challenge for the SDE. More than the obvious **visual** changes from the fixed point solution, $\mathbf{Z = X(3) \neq R\text{-}1}$ and the Z-deviates are important. Also the probability distributions of X(1) and X(2) are symmetric.

The time averaged MC calculations and the matching SDE results derived from the full equation set are shown in Tables 1 and 2. These include a 40,000 sample size and time averages over 4000 iterations for the MC as discussed in Fleming(2014). All of these time averaged moment values are now constant. They exactly balance the left hand (LHS) of the SDE equations, the time tendency, set to zero. No assumptions have been made within the SDE equations.

The MC and SDE calculations show that **X(1) = X(2) = 0.0.** The statistical symmetry of **X(1)** and **X(2)** imply that moments ($2^{nd}$, $3^{rd}$, and $4^{th}$) with an **odd** number of 1's and 2's in the indices will approach zero over time.

**The Tables show that only five $4^{th}$ moments are active (non-zero) and these occur in just four $3^{rd}$ moment prediction equations. Just these equations are shown below:**

$$\begin{aligned} X(12)^{\bullet} = T(1,1,3)^{\bullet} &= 2P\,[X(14) - X(12)] + X(1)\,X(11) + X(2)\,X(10) - B\,X(12) \\ &\quad - \sigma(1,1)\,\sigma(1,2) + \lambda(1,1,1,2) \end{aligned} \tag{10}$$

$$\begin{aligned} X(14)^{\bullet} = T(1,2,3)^{\bullet} &= P\,[X(17) - X(14)] - X(1)\,X(15) - X(3)\,X(12) + R\,X(12) \\ &\quad + X(1)\,X(13) + X(2)\,X(11) - (B + 1)\,X(14) \\ &\quad + \sigma(1,3)\,\sigma(1,3) - \sigma(1,2)\,\sigma(1,2) + \lambda(1,1,2,2) - \lambda(1,1,3,3) \end{aligned} \tag{11}$$

$$\begin{aligned} X(17)^{\bullet} = T(2,2,3)^{\bullet} &= -2\,[X(1)\,X(18) + X(3)\,X(14)] + 2R\,X(14) + (2 - B)\,X(17) \\ &\quad + X((1)\,X(16) + X(2)\,X(13) - 2\,[-\sigma(1,3)\,\sigma(2,3) + \lambda(1,2,3,3)] \\ &\quad - \sigma(1,2)\,\sigma(2,2) + \lambda(1,2,2,2) \end{aligned} \tag{12}$$

$$\begin{aligned} X(19)^{\bullet} = T(3,3,3)^{\bullet} &= 3\,[X(1)\,X(18) + X(2)\,X(15) - B\,X(19)] \\ &\quad + 3\,[-\sigma(1,2)\,\sigma(3,3) + \lambda(1,2,3,3)] \end{aligned} \tag{13}$$

Now one can examine the $4^{th}$ moment terms in the above equations from Table 2. These are quite different in the chaos case versus what was seen for the fixed point solutions. The non-zero $4^{th}$ moments are all different and their status relative to "normal" $4^{th}$ moments can be established.

From **(10)**, there is a single $4^{th}$ moment, **λ(1,1,1,2)** which has the value from Table 2 of **9060.1** The normal form of $\boldsymbol{\lambda(1,1,1,2) = 3\,\sigma(1,1)\,\sigma(1,2) = (3)(62.80)^2 = 11{,}831.5;}$ **therefore, FF1 = 9060.1 / 11,831.5 = 0.766.** This **λ** is platykurtic.

From **(11)**, there are two $4^{th}$ moments, **λ(1,1,2,2)** and **λ(1,1,3,3)** which have the values from Table 2 of **10,735** and **6,711.0** respectively. The normal form of **λ(1,1,2,2) =**

**$\sigma(1,1)\ \sigma(2,2)\ + 2\ \sigma(1,2)\ \sigma(1,2) = (62.8)\ (81.2) + (2)(62.8)^2 = 12{,}987$; therefore, FF2 = 10,735 / 12,987= 0.827.** The normal form of **$\lambda(1,1,3,3) = \sigma(1,1)\ \sigma(3,3) + 2\ \sigma(1,3)^2 = (62.8)\ (74.34) = 4{,}668.6$,** since **$\sigma(1,3) = 0$. FF3 = 6,711.0 / 4,668.6 = 1.44** and **$\lambda(1,1,3,3)$ has the ratio $\lambda / \lambda_{normal} > 1$ and** is **leptokurtic** (more peaked near the mean).

From **(12)**, there are two 4th moments, **$\lambda(1,2,2,2)$** and **$\lambda(1,2,3,3)$** which have the values from Table 2 of **13,774** and **5,021.6** respectively. The normal form of **$\lambda(1,2,2,2) = 3\ \sigma(1,2)\ \sigma(2,2) = (3)(62.8)\ (81.2) = 15{,}298$; therefore, FF4= 13,774 / 15,298 = 0.90.** The normal form of **$\lambda(1,2,3,3) = \sigma(1,2)\ \sigma(3,3) + 2\ \sigma(1,3)\ \sigma(2,3) = (62.8)\ (74.34) = 4668.6$ Therefore FF5 = 5,021.6 / 4,668.6 = 1.08.** This same 4th moment is also in **(13).**

Closing the SDE equation set for chaos requires two phases. In the 1st phase, a single damping term is used for those equations in which 3rd moments go to zero from the physics. This is required, though the physics drives these to zero, because of the initial explosive randomness which affects all the moments. A trial damping term **DK** is increased until the values, **X(1), X(2), X(6), X(8), X(10), X(11), etc. from Table 1, all approach 0.0 after a run of 4000 iterations.** The 4th moments approaching zero are put in **normal** form and multiplied by **FF = 1**. During the integration through the initial randomness and chaos, these 4th moments may look chaotic before progressing to zero.

Those non-zero 4th moments have been replaced with their **calculated** values by using the **FF1 – FF5** factors appropriately multiplying their **normal form** – again achieving their **calculated** values. It is important to **leave** the general **[- $\sigma(p,q)\ \sigma(k,l)$ ]** terms in the 3rd moment equations [see beginning of Section 2] **intact** – avoiding the error of incorporating these into the normal form of the 4th moments. These terms naturally arise from the SDE equation set and play an important feedback role in moment growth.

The 2nd phase methodology described here is to "over stimulate" those non-zero 3rd moment predictive equations so that a damping term can be calculated and applied to those predictive equations. The **FF1 – FF5** values are set to **(1.2) [FF1 – FF5] – the value 1.2 is arbitrary.** If this were not done, there could be no damping term applied -- the equations are already balanced with the **LHS = 0 by just** leaving the 4th moments as found from the MC results or the full SDE equations. Only by creating an **increased** imbalance, can a **positive** damping term be calculated. There is no balance until far into the integration as the time derivatives evolve through their plus and minus changes – even quite large changes initially. The damping term for each of the four active 3rd moment equations is computed to again balance the equations -- and applied in a sequence to observe the subsequent changes in key variables.

Tables 3 and 4 indicate changes in some of the key variables as the damping coefficients change. **These values exactly match those values as seen Table 1.** The results for Table 3 are for **FF = (1.2)[FF1 – FF5]** with the **DK** values optimized from the 1st phase with the initial values of FF1- FF5 = 6.0. The results for Table 4 are for **FF = (1.4)[ FF1-FF5]** with the **DK** values optimized with the initial values of FF1-FF5 = 7.0. This optimization does include these equated to some arbitrary value – the same for each.

## 6. Summary

The real success of the closure is seen in the Figures to follow. **Figure 4**. shows both **MC** and **SD3** calculations over 4000 iterations with both achieving the final value of **X(3) = 23.55.** The initial randomness has been captured

**Figure 5.** indicates both **MC** and **SD3** calculations over 4000 iterations with both arriving the final value of **σ(3,3) = 74.34**. The excessive initial randomness has been handled quite well.

**Figure 6.** reveals both **MC** and **SD3** calculations over 4000 iterations with both having the final value of **T(3,3,3) = 132.4**. The enormous explosive randomness has been treated perfectly.

The methodology employed here to close this bound and dissipative system can be applied to all such systems cast in the stochastic dynamic format. It should be highlighted that the closure methodology expressed here is for **nonlinear quadratic equations.** The general moment prediction equations shown at the beginning of Section 2 can be cast in an abbreviated form for a **single** nonlinear quadratic term (and temporarily ignoring the dummy indices which provide the constant coefficients for each. Thus:

$$\mu_i^{\bullet} = ( \mu_p \mu_q + \sigma_{pq} )$$

$$\sigma_{ij}^{\bullet} = (\mu_p \sigma_{jq} + \mu_q \sigma_{jp} + T_{jpq}) + (\mu_p \sigma_{iq} + \mu_q \sigma_{ip} + T_{ipq})$$

can be put in the following simplified form. Let any moment defined as **f , we have** $f_2$ **= 2nd moment about the mean (μ),** $f_3$ **=3rd moment, … $f_n$ be the n-th moment about the mean. Then one has the simple form:**

$$\mu^{\bullet} = \mu \mu + f_2$$

$$f_2^{\bullet} = 2 [ \mu f_2 + \mu f_2 + f_3 ]$$

$$f_n^{\bullet} = n [ 2 \mu f_n - f_2 f_{n-1} + f_{n+1} ] \text{ for } n = 3,4,\ldots$$

Thus prediction of a moment **n** involves a moment of **n+1.** Were the original deterministic equation nonlinear cubic $X^{\bullet} = X X X$, the general form for the predicted moments would be the more complex:

$$f_n^{\bullet} = n [ 3 \mu f_{n+1} + 3 \mu^2 f_n - 3 \mu f_2 f_{n-1} - f_3 f_{n-1} + f_{n+2} ]$$

Predicting a moment **n** involves a moment **n+2. A greater challenge, but in principle, the same logic would yield a solution.**

The **use of both MC and SDE approaches together** was shown to be quite advantages in Fleming (2014) and in this study of the closure of the **SDE** set.

## References


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**Tables**

| Moment | Var # | **MC Value** | **SD3 Value** | Moment | Var # | **MC Value** | **SD3 Value** |
|---|---|---|---|---|---|---|---|
| **μ(1)** | **X(1)** | **-.001** | **.000** | T(1,1,1) | **X(10)** | **.060** | **.000** |
| **μ(2)** | **X(2)** | **-.001** | **.000** | T(1,1,2) | **X(11)** | **0.060** | **.000** |
| **μ(3)** | **X(13** | **23.55** | **23.55** | T(1,1,3) | **X(12)** | **400.6** | **400.6** |
| **σ(1,1)** | **X(4)** | **62.8** | **62.8** | T(1,2,,2) | **X(13)** | **.040** | **.000** |
| **σ(1,2)** | **X(5)** | **62.8** | **62.8** | T(1,2,3) | **X(14)** | **198.2** | **198.2** |
| **σ(1,3)** | **X(6)** | **-.005** | **-.000** | T(1,3,3) | **X(15)** | **-.080** | **-.000** |
| **σ(2,2)** | **X(7)** | **81.20** | **81.20** | T(2,2,2) | **X(16)** | **.009** | **.000** |
| **σ(2,3)** | **X(8)** | **.001** | **-.000** | T(2,2,3) | **X(17)** | **84.83** | **84.83** |
| **σ(3,3)** | **X(9)** | **74.34** | **74.34** | T(2,3,3) | **X(18)** | **-.060** | **-.000** |
| | | | | T(3,3,3) | **X(19)** | **132.4** | **132.4** |

**Table 1. Calculated MC values and computed SD3 values from full equations for R = 28**

| Moment | Var # | **MC Value** | **SD3 Value** | Moment | Var # | **MC Value** | **SD3 Value** |
|---|---|---|---|---|---|---|---|
| **λ(1,1,1,1)** | **X(20)** | **9060.1** | **9060.1** | **λ(1,2,2,3)** | **X(27)** | **-.15** | **0.0** |
| **λ(1,1,1,2)** | **X(21)** | **9060.2** | **9060.1** | **λ(1,2,3,3)** | **X(28)** | **5021.5** | **5021.6** |
| **λ(1,1,1,3)** | **X(22)** | -.14 | **0.0** | **λ(1,3,3,3)** | **X(29)** | **-1.2** | **0.0** |
| **λ(1,1,2,2)** | **X(23)** | **10735** | **10735** | λ(2,2,2,2) | **X(30)** | | |
| **λ(1,1,2,3)** | **X(24)** | **-.003** | **0.0** | λ(2,2,2,3) | **X(31)** | | |
| **λ(1,1,3,3)** | **X(25)** | **6712.5** | **6713.0** | λ(2,2,3,3) | **X(32)** | | |
| **λ(1,2,2,2)** | **X(26)** | **13774** | **13774** | λ(2,3,3,3) | **X(33)** | | |
| | | | | λ(3,3,3,3) | **X(34)** | | |

**Table 2. Calculated MC values and computed SD3 values for R = 28.**
**X(20) and X(30) through X(34) are not in the SD3 equation set**

| Phase | DK | DK1 | DK2 | DK3 | DK4 | X(1) | X(12) | X(14) | X(17) | X(19) |
|---|---|---|---|---|---|---|---|---|---|---|
| | | For X12 | For X14 | For X17 | For X19 | | | | | |
| 1.1 | 5.5 | 6.0 | 6.0 | 6.0 | 6.0 | -3.1 | 250.6 | 138.4 | 144.0 | 294.3 |
| 1.2 | 16.8 | 6.0 | 6.0 | 6.0 | 6.0 | 0.0 | 379.0 | 193.1 | 111.7 | 284.9 |
| 2.1 | 16.8 | 4.52 | 6.0 | 6.0 | 6.0 | 0.0 | 400.1 | 196.9 | 114.8 | 289.2 |
| 2.2 | 16.8 | 4.52 | 4.06 | 6.0 | 6.0 | 0.0 | 401.3 | 200.2 | 96.37 | 293.3 |
| 2.3 | 16.8 | 4.52 | 4.06 | 8.80 | 6.0 | 0.0 | 400.6 | 198.2 | 84.83 | 290.9 |
| 2.4 | 16.8 | 4.52 | 4.06 | 8.08 | 22.8 | 0.0 | 400.6 | 198.2 | 84.83 | 132.4 |

**Table 3. Closure with FF = 1.2 FF. Values of damping coefficients at each stage of closure (using values of X(1), X(12), X(14), X(17) and X(19) as examples at 4000 iterations). Optimal values are shown in color red.**

| Phase | DK | DK1 | DK2 | DK3 | DK4 | X(1) | X(12) | X(14) | X(17) | X(19) |
|---|---|---|---|---|---|---|---|---|---|---|
| | | For X12 | For X14 | For X17 | For X19 | | | | | |
| 1.1 | 5.5 | 7.0 | 7.0 | 7.0 | 7.0 | -4.5 | 202.2 | 95.1 | 140.9 | 332.3 |
| 1.2 | 22.2 | 7.0 | 7.0 | 7.0 | 7.0 | 0.0 | 430.0 | 210.3 | 121.0 | 495.6 |
| 2.1 | 22.2 | 9.05 | 7.0 | 7.0 | 7.0 | 0.0 | 402.3 | 205.4 | 115.6 | 486.9 |
| 2.2 | 22.2 | 9.05 | 8.16 | 7.0 | 7.0 | 0.0 | 401.9 | 203.8 | 126.2 | 483.6 |
| 2.3 | 22.2 | 9.05 | 8.16 | 17.6 | 7.0 | 0.0 | 400.6 | 198.2 | 84.83 | 472.4 |
| 2.4 | 22.2 | 9.05 | 8.16 | 17.6 | 45.5 | 0.0 | 400.6 | 198.2 | 84.83 | 132.4 |

**Table 4. Closure with FF = 1.4*FF. Values of damping coefficients at each stage of closure (using values of X(1), X(12), X(14), X(17) and X(19) as examples at 4000 iterations). Optimal values are shown in color red.**

## Figures

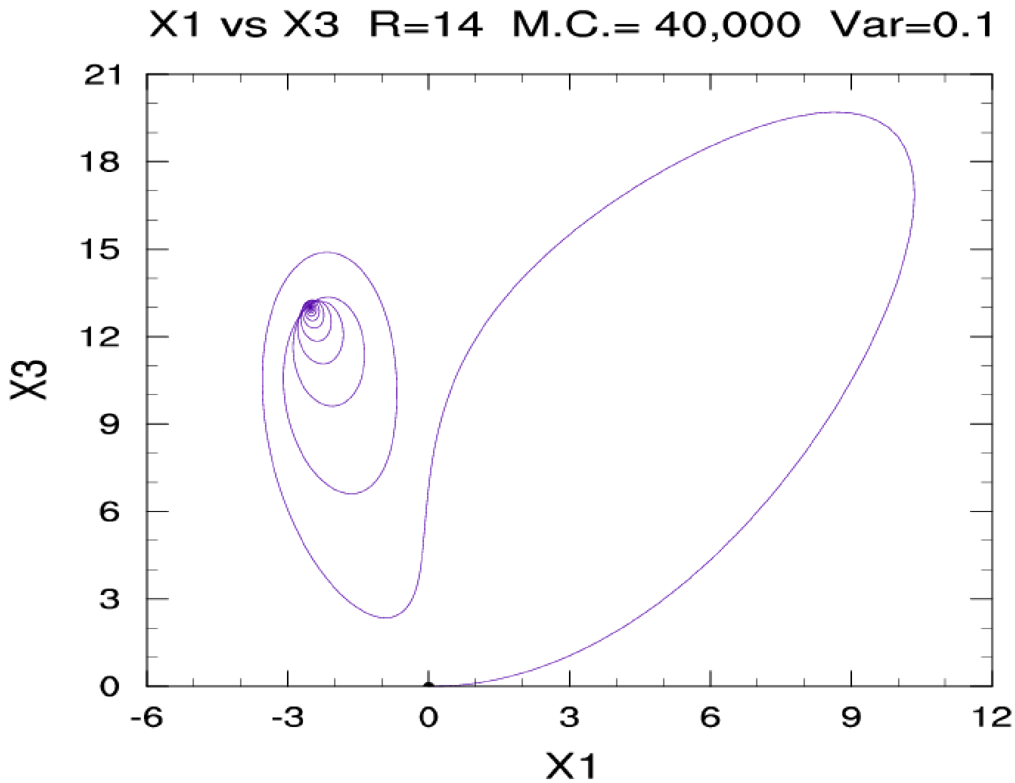


**Figure 1. R = 14 in Lorenz: [X1, X2, X3] = [0, 1, 0], initial variance X1 to X3 = 0.1, sample size = 40,000. After initial wandering X3 = R-1 = 13, and X1 = X2= -2.52**

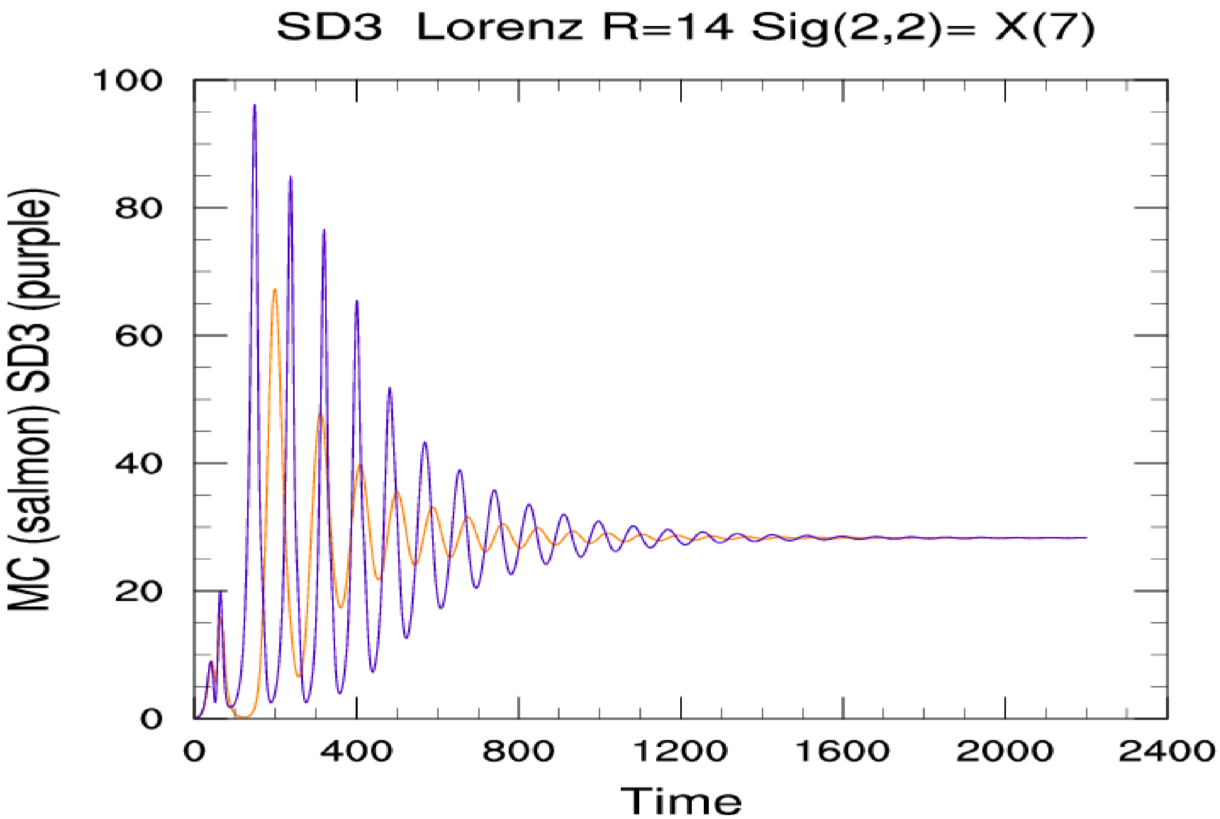


**Figure 2. $\sigma(2,2) = X(7)$ versus time. MC and SD3 agree on both initial "explosive randomness" and final value**

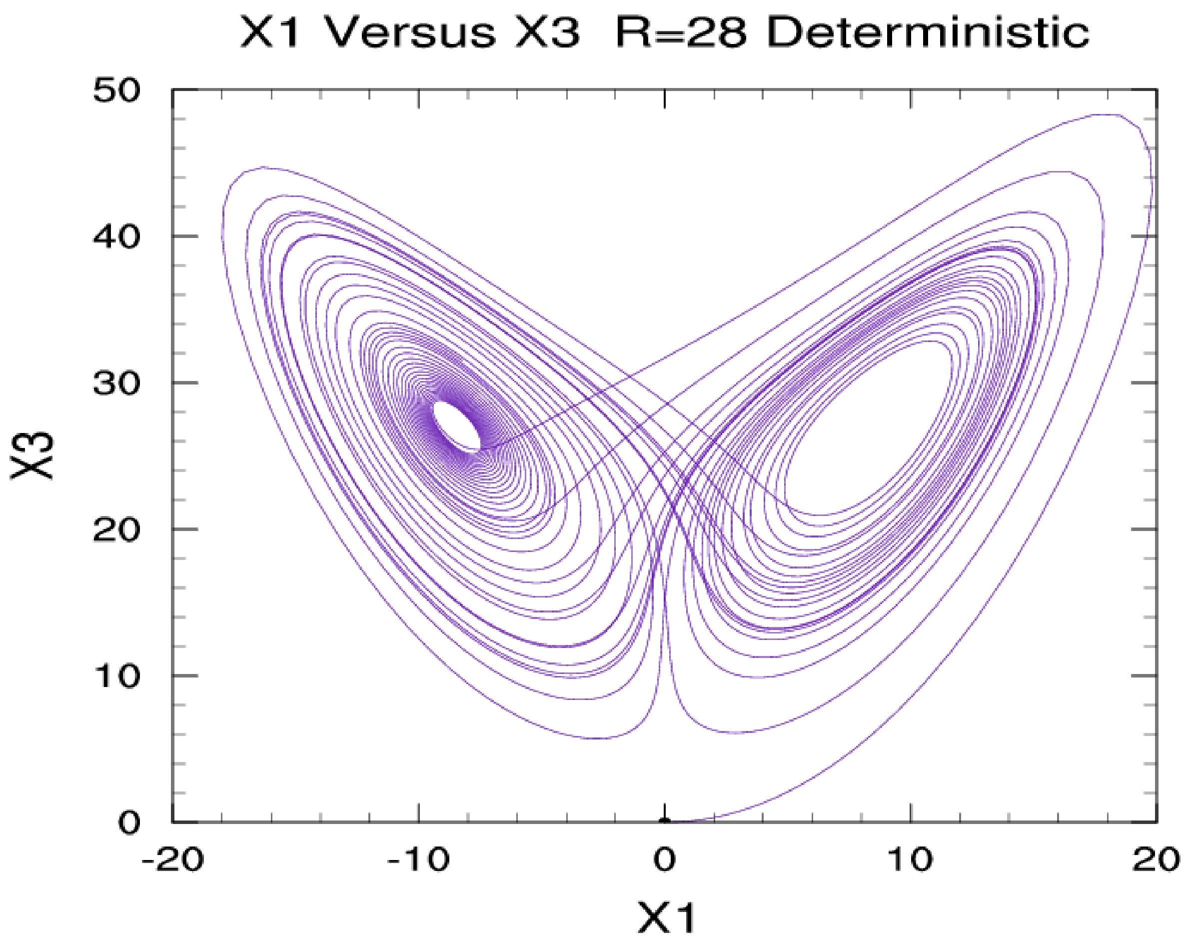


**Figure 3. Deterministic solution [0,1,0] R = 28. A formidable challenge!**

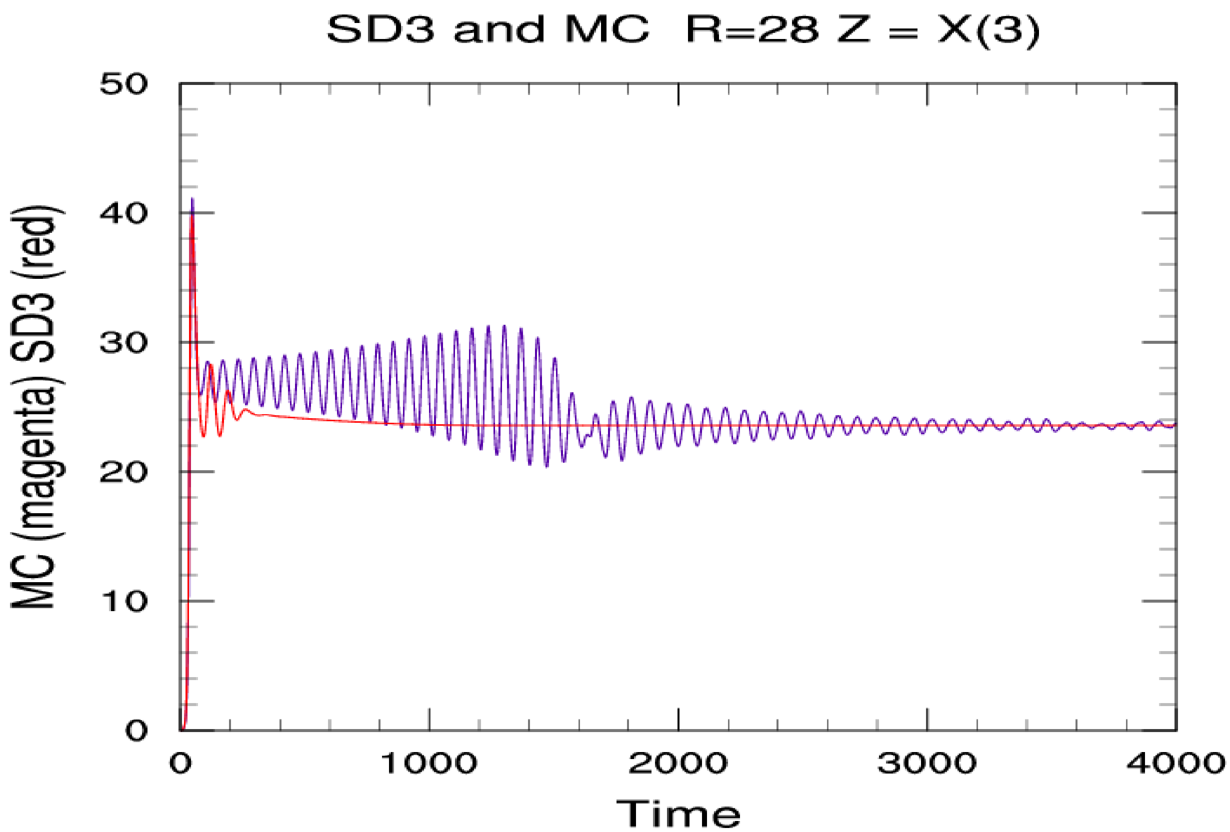


**Figure 4. X(3) as a function of time: MC and SD3 in excellent agreement**

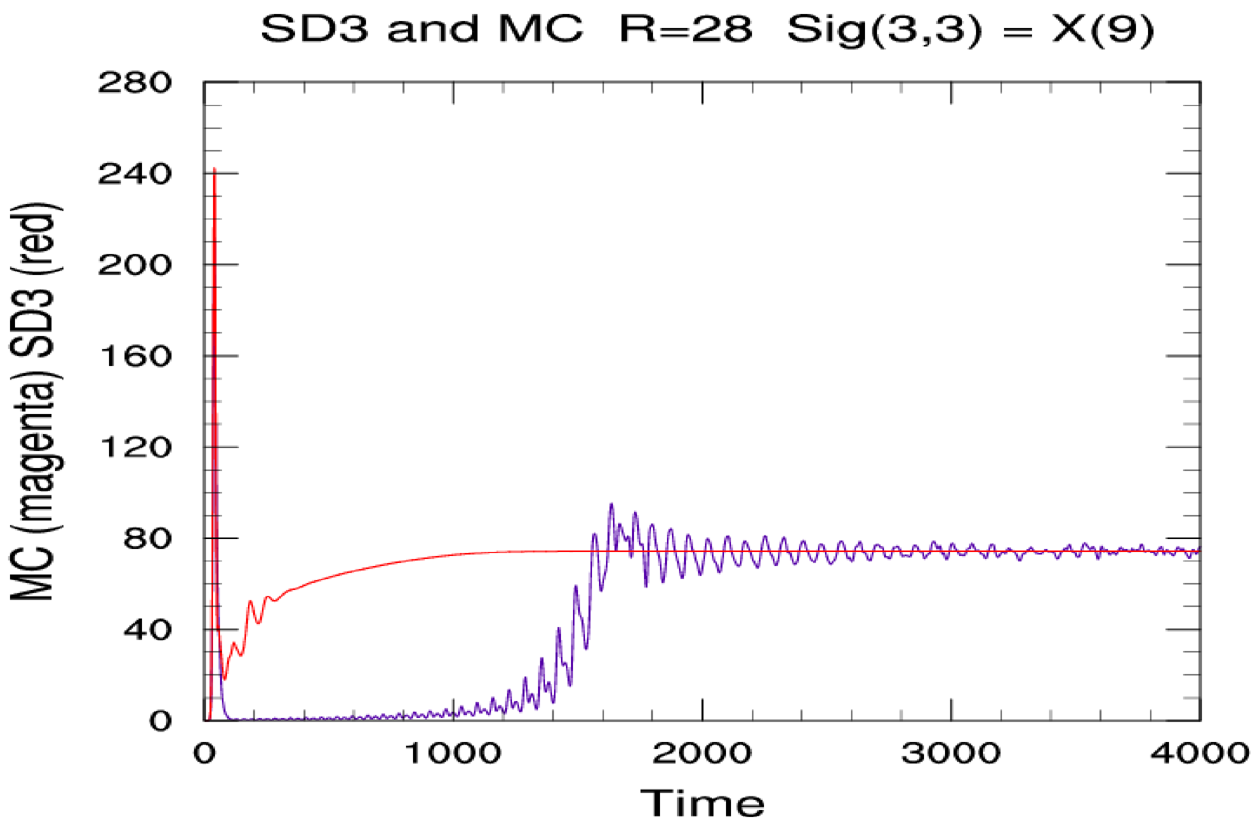


**Figure 5. MC and SD3 calculations achieve correct σ(3,3) = 74.3; the initial randomness is handled quite well**

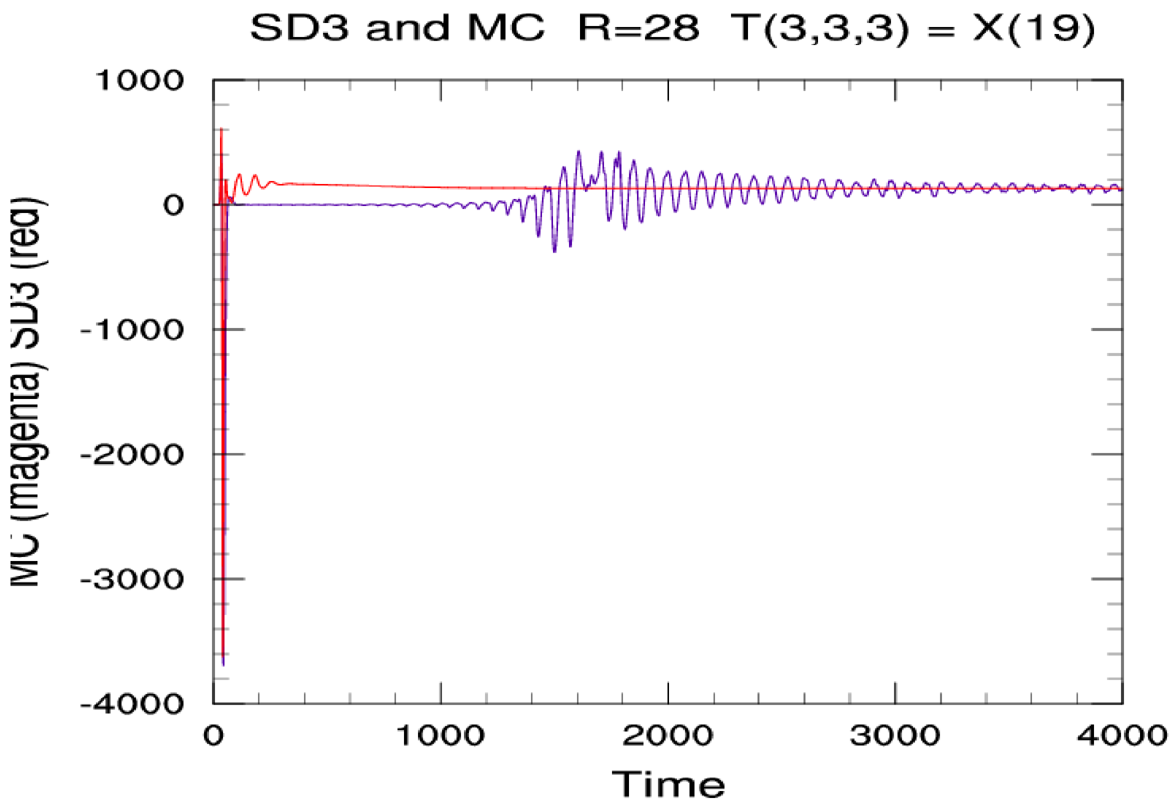


**Figure 6. MC and SD3 reach correct value of T(3,3,3) of 132.4; the initial randomness is handled extremely well**