Physical basis of cloud particle size distribution form

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Introduction

▪Several analytical forms of particle size distributions (PSDs) are used in model parameterizations and remote sensing retrievals, but which one is most appropriate to use?

Table. 1 Commonly used PSD forms

Exponential	$n(D) = N_0 e^{-\lambda D}$
Gamma	$n(D) = N_0 D^{\mu} e^{-\lambda D}$
Generalized gamma	$n(D) = N_0 D^{\mu} e^{-\lambda D^{\nu}}$
Lognormal	$n(D) = \frac{N_T}{\sqrt{2\pi}\ln\sigma D}\exp(-\frac{\ln^2(D/D_g)}{2\ln^2\sigma})$
Weibull	$n(D) = N_0 D^{\nu-1} e^{-\lambda D^{\nu}}$

▪Principle of maximum entropy (MaxEnt) states that for a group of probability density functions (PDFs) that satisfy given properties of a variable, the PDF with the largest information entropy for such a variable should be chosen. It predicts the most probable distribution for a system without regard to the detailed microphysical processes.

Conceptual model

For many numerical model parameterization and remote sensing retrievals, only bulk properties (macrostates) are known. How do we infer the microstate (particle size distributions, etc.) based on the macrostate information? What is the most probable PSD function if no other information is given?

Background:

1). For a typical cloud "unit" of 100 m x 100 m x 10 m, there are around N=10^{10} particles.

2). The macrostate of the cloud system is defined using bulk properties: total number concentration (N_T), total water content (*TWC*), extinction, radar reflectivity (*Z*), etc.

3). The microstate of the cloud system: the sizes, mass, area, terminal velocity of each individual particles

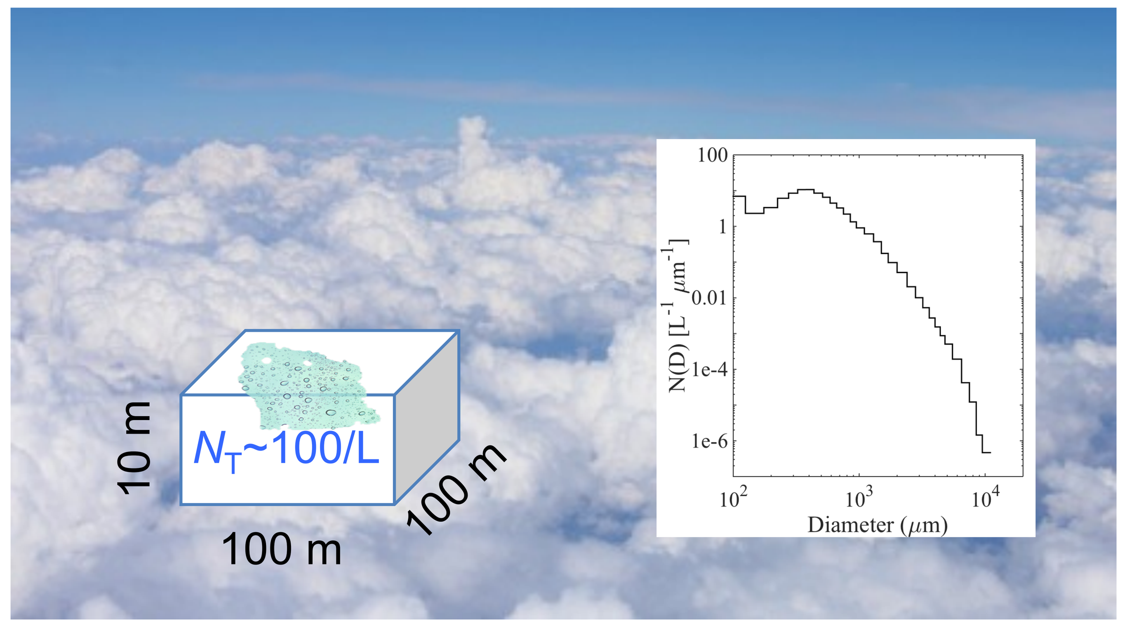

Fig. 1 Schematic of maximum entropy model

Questions:

Assuming N_T and TWC are known as in most models, with

$$N_T = \int_0^{\infty} N(D)dD \quad \text{and} \quad TWC = \int_0^{\infty} m(D)N(D)dD$$

where N(D) is the number distribution function, and m(D) is the mass-dimensional relations (usually power law), what is the most probable N(D), or probability density function (PDF, normalized form: p(D)=N(D)/N_T)?

Principle of maximum (relative) entropy (MaxEnt):

MaxEnt proposed by Jaynes (1957, 1968) is used widely in fields such as image processing, economics, ecology, mechanical engineering. MaxEnt states that entropy can be used as a criterion to choose a PDF.

Table. 2 MaxEnt examples

Constraints	Most probable distribution
Mean (E(x)=μ)	exponential distribution
Mean and variance (E(x)=μ and E[(x-μ)²]=σ²)	normal distribution

$$\underset{p(D)}{\arg\max} \quad \text{Entropy: } H = -\int_0^{\infty} p(D)\ln\frac{p(D)}{P_0(D)}dD$$

$$\text{subject to} \quad \int_0^{\infty} p(D)dD = 1$$

$$\int_0^{\infty} m(D)p(D)dD = \frac{TWC}{N_T}$$

For entropy of continuous distribution, prior information $P_0(x)$ is crucial. Otherwise, different results can be derived from the same assumptions.
Read a tale of two approaches

This can be solved by the method of Lagrangian multipliers:

$$L \equiv -\int_0^{\infty} p(D)\ln p(D)dD + \int_0^{\infty} p(D)\ln P_0(D)dD - \lambda_0[\int_0^{\infty} p(D)dD - 1] - \lambda_1[\int_0^{\infty} m(D)p(D)dD - IWC/N_T]$$

A Tale of Two Approaches

Common Assumptions:

1.M=a D^b

2.Number Concentration (N_T)

3.Total Water Content (TWC)

$$\int_0^{\infty} N(D)dD = N_T \qquad \int_0^{\infty} \alpha D^{\beta} N(D)dD = TWC$$

$$N(D) = C_1 exp(-\lambda_1 * D^{\beta})$$

$$\int_0^{\infty} N(M)dM = N_T \qquad \int_0^{\infty} MN(M)dM = TWC$$

$$N(M) = C_2 exp(-\lambda_2 M)$$

OR

$$N(D) = C_3 D^{\beta-1} exp(-\lambda_3 D^{\beta})$$

Different results from the same assumptions

Mathematical Solutions:

Using the method of Lagrangian multipliers:

$$L \equiv -\int_0^{\infty} p(D)\ln p(D)dD + \int_0^{\infty} p(D)\ln P_0(D)dD - \lambda_0[\int_0^{\infty} p(D)dD - 1] - \lambda_1[\int_0^{\infty} m(D)p(D)dD - IWC/N_T]$$

The general result can be solved to be in the form:

$$p(D) = \frac{1}{Z(\lambda_0, \lambda_1)} P_0(D) exp(-\lambda_0 - \lambda_1 m(D))$$

where $Z(\lambda_0, \lambda_1) = \int_0^{\infty} e^{-\lambda_0} P_0(D) e^{-\lambda_1 m(D)} dD$ is the normalization factor. By using m-D relation ($m = aD^b$), the above solution can be reorganized to the more conventional form:

$$n(D) = p(D)N_T = N_0 D^{\mu} exp(-\lambda D^{\nu}),$$

where $N_0 = N_T e^{-\lambda_0}/Z(\lambda_0, \lambda_1)$, $\lambda = \lambda_1 a$, and $\nu = b$.

Here P_0(D) is provided in the form ~D^{μ} because of prior assumption of total ignorance of number/area/mass/etc. over size, i.e. a uniform distribution.
The paradox of using old entropy definition is due to the underlying assumptions

Why maximize entropy?

Suppose that there are N cloud particles in a cloud unit. The mass and size of an individual particle is the result of many microphysical processes. For an individual particle, assume that its mass is a random variable, as long as the TWC of the system is limited. Then there are Ω ways that TWC can be distributed over N particles for any given p(D). Maximizing entropy equals maximizing Ω!

Conclusions

1). Principle of maximum (relative) entropy is proposed to derive the analytical form of cloud PSD;

2). Generalized gamma distributions are derived when TWC and N_T are known, and the prior information is given in the form of D^{μ};

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Acknowledgments

The work was supported by the office of Biological and Environmental Research (BER) of The U.S. Department of Energy DE-SC0001279 and DE-SC0008500) and by the National Science Foundation (AGS12-13311). Data were obtained from the ARM archive, sponsored by the US DOE Office of Science, BER, Climate and Environmental Sciences Division.

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