Equilibrium Phases in a Energy- Relative enstrophy Statistical Mechanics Model of Barotropic Flows on a Rotating Sphere — non- conservation of angular momentum

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1 Introduction

The statistical equilibrium of barotropic flows on a rotating sphere is simulated in a wide range of parameter space by Monte-Carlo methods. A spin-lattice Hamiltonian with a canonical constraint on the kinetic energy and a microcanonical constraint on the relative enstrophy is formulated as a convergent family of finite dimensional approximations of the barotropic vortex statistic on the rotating sphere. As this spin-lattice Hamiltonian model cannot be solved exactly since the nonzero rotation of the sphere will result in a difficult external field like term which is spatially inhomogeneous, Monte-Carlo simulations are used to calculate the spin-spin correlations and the mean nearest neighbor parity as order parameters or indicators of phase transitions in the system. Interestingly we find that at extremely high energy levels or negative temperatures, the preferred state is a superrotational equilibrium state aligned with the planetary spin. For large values of the planetary spin compared with the relative enstrophy there appears to be two phase transitions when the temperature varies from a numerically large negative value to a large positive value. The latter arises as constrained energy minimizers which can undergo a tilt instability of the axis of atmospheric rotation. This means the axis of atmospheric rotation does not always point in exactly the same direction of the axis of planetary spin but wobbles a little. A simple mean field theory which constrains the relative enstrophy only in an averaged sense and relaxes the angular momentum, gives critical temperatures that are consistent with the Monte-Carlo simulations.

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2 Barotropic Vorticity Equation on rotating sphere

The Barotropic Vorticity equation (BVE) on a rotating sphere is given by

$$\frac{Dq}{Dt} = q_t + (\vec{u} \cdot \vec{\nabla})q = 0,$$

where D/Dt is the material derivative, q is the total vorticity and $\vec{u}$ is the velocity. It states the conservation of total vorticity q by the relative flow $\vec{u}$ on the rotating sphere.

The BVE describes the evolution of a homogeneous, non-divergent, incompressible flow on the surface of the sphere. The BVE is the simplest possible model for ideal 2D flows on the rotating sphere. The barotropic vorticity model conserves the kinetic energy and any function of the vorticity as Casimirs of the governing equation.

2.1 Lattice approximation of the BVE

Given N fixed mesh points $\vec{x_1}, \vec{x_2}, \vec{x_3}, ..., \vec{x_N}$ on S^2 and the voronoi cells based on this mesh [3], by discretizing the vorticity field as a piecewise constant function ,we approximate the relative vorticity

$$w(x) = \sum_{j=1}^{N} s_j H_j(x),$$

where $s_j = w(j)$ and $H_j(x)$ is the characteristic function for the domain D_j, that is

$$H_j(x) = \begin{cases} 1 & x \in D_j \\ 0 & otherwise. \end{cases}$$

The domain D_j is the subset of the entire domain consisting of points which are closer to $\vec{x_j}$ than to any other mesh site $\vec{x_k}$ (also known as voronoi cell).

The choice of $\vec{x_j}$ is arbitrary with one constraint, that the areas A_j of each domain D_j are approximately equal.

Then we get spin-lattice Hamiltonian

$$H_N[q] = -\frac{1}{2}\sum_{i=1}^{N}\sum_{j\neq i}^{N} J_{ij}s_i s_j + \sum_{j=1}^{N} F_j s_j,$$

where $J_{ij} = \frac{16\pi^2}{N^2} log|1 - x_i \cdot x_j|$ and $F_j = \frac{2\pi}{N}\Omega\cos\theta_j$. The external fields F_j comes from the nonzero rotation $\Omega > 0$, and represent the coupling between the local relative vorticity and the planetary vorticity field. The presence of the second term is the source of the much richer mathematical and physical properties of BVE on the rotating sphere as compared to the Euler case on a non-rotating sphere.

2.2 Constrained variational theory for the BVE on the rotating sphere

Using the spherical harmonics eigenfunction expansion

$$\omega = \sum_{l\geq 1,m} \alpha_{lm}\psi_{lm} \qquad l = 1, .., +\infty, \qquad m = -l, ..0, ..l,$$

where ψ_{lm} is the orthonormal basis of spherical harmonics for the Hilbert space $L_2(S^2)$. We can expand the Hamiltonian function in terms of the orthonormal spherical harmonics,

$$\begin{aligned} H[q] &= -\frac{1}{2}\int_{S^2} dx\ \psi q \\ &= -\frac{1}{2}\int dx\ \psi(x)\,[w(x) + 2\Omega\cos\theta] \\ &= -\frac{1}{2}\langle w, G[w]\rangle - \Omega C\,\langle \psi_{10}, G[w]\rangle \\ &= \frac{1}{2}\sum_{l,m}\frac{\alpha_{lm}^2}{l(l+1)} + \frac{1}{2}\Omega C\alpha_{10}, \end{aligned}$$

where $G[\omega] = -\sum_{l,m}\frac{\alpha_{lm}}{l(l+1)}\psi_{lm}$ and $C = ||\cos\theta||_2$.

Since we choose $l \neq 0$, the total circulation of the vorticity field $\omega(x)$ is zero by the Stokes theorem, that is $TC = \int_{s^2}\omega dx = 0$. Then the only constraint is the relative enstrophy $\Gamma_r = \int_{s^2}\omega^2 dx = \sum_{l,m}\alpha_{lm}^2 = Q_{rel} > 0$.

Constrained variational problems for the BVE on the rotating sphere had been formulated and solved by Lim [4]. The main result is that at any rate of spin Ω and relative enstrophy Q_{rel}, the unique global energy maximizer for fixed relative enstrophy corresponds to solid-body rotation, $w^0_{Max}(Q_{rel}) = \sqrt{Q_{rel}}\psi_{10}$ in the direction of Ω. Another solution, the counter-rotating steady-state $w^0_{\min}(Q_{rel}) = -\sqrt{Q_{rel}}\psi_{10}$, is a constrained energy minimum provided the relative enstrophy is small enough, i.e., $Q_{rel} < \Omega^2 C^2$ where $C = ||\cos\theta\ ||_2$. If $\Omega^2 C^2 < Q_{rel}$, then $w^0_{\min}(Q_{rel})$ is a saddle point.

Given the conditions for $w^0_{\min}(Q_{rel})$ to be a local constrained minima, the solid-body rotation opposite to spin $w^0_{\min}(Q_{rel})$ is a nonlinearly stable equilibrium of the BVE. The global constrained maximizer $w^0_{Max}(Q_{rel})$ corresponding to solid-body rotation in the direction of spin, is always a nonlinearly stable equilibria of the BVE. For small relative enstrophy relative to the spin rate, the pro- and counter-rotating solid-body states are both nonlinearly stable. At higher relative enstrophy values, only the pro-rotating state is nonlinearly stable; the counter-rotating states are saddle points.

3 Energy-Relative enstrophy Model

The equilibrium statistical mechanics of the BVE on a rotating sphere is formulated on the basis of a canonical constraint on the kinetic energy , a microcanonical constraint on the relative enstrophy and total circulation is set to zero. That is

- Canonical constraint on the kinetic energy
- Microcanonical constraint on the relative enstrophy
- Total circulation is set to zero

Why did we choose this model? This set of constraints will yield mathematically elegant and physically significant statements and it can remove the low temperature defect in the classical energy-enstrophy theory (canonical in both kinetic energy and enstrophy). It also should be a better model to simulate the phase-transition since it is well defined at all temperatures.

We do not fix the angular momentum because that would make the rotating BVE look same as the non rotating case as Frederiksen did [5]. Frederiksen was not looking to model sup-rotation at all because the BVE with fixed angular momentum cannot gain or loose angular momentum from initial value of angular momentum in the atmosphere. Relative to the aim of getting a statistical mechanics model for super-rotation where we know the atmosphere gained its angular momentum from the solid planet, it is a better model to free the angular momentum from constraint except the implicit inequality constraint of being bounded above by square-root of relative enstrophy. So the dynamics and stability properties arising from this model are not exactly those of BVE itself, but rather the generalized barotropic flows on a rotating sphere, that exchange energy and angular momentum inviscidly with their respective reservoirs.

4 Partition function and Gibbs canonical-microcanonical probability

The spin-lattice partition function is ,

$$Z_N = \int (\Pi_{j=1}^N ds_j)\delta(NQ_N - 4\pi \sum_{j=1}^N s_j^2)\exp[-\beta H_N]$$

where H_N is spin-lattice Hamiltonian.

The Gibbs canonical-microcanonical probability is,

$$P_G = \frac{1}{Z_N}\exp[-\beta H_N]\delta(NQ_N - 4\pi \sum_{j=1}^N s_j^2).$$

5 Monte-Carlo simulations and some results

Monte-Carlo simulations in our energy-relative enstrophy model on the rotating sphere conserve both the discrete total circulation and the relative enstrophy. The main reason for not fixing angular momentum is to model super and sub-rotation in planetary atmosphere, and the tilt instability where the axis of angular momentum wobbles.

5.1 Super-Rotation

By recent variational theory for super-rotation in the BVE model on a rotating sphere[4], there are pro and counter-rotating solid body flows arise in the BVE as the nonlinearly stable states. The Monte-Carlo simulations agree with this theory. When β is negative, the first term of the spin-lattice Hamiltonian shows that any vortex surrounded by vortices of the same sign will get higher energy; the second term gives that the sites at north pole have the most positive values and the sites at the south pole have the most negative values to get higher energy, since the *cosθ* varies from 1 at north pole to -1 at the south pole. In this case the strong vortices of either sign are surrounded by other strong vortices of like sign and the greatest vorticity sites are at the north pole while the sites with the least vorticity are at the south pole.

The above state corresponds to the unique global energy maximizer in variational theory — at any spin rate Ω and fixed relative enstrophy Q_{rel} , the system will get the maximum energy with $w^0_{Max}(Q_{rel}) = \sqrt{Q_{rel}}\psi_{10}$ in the direction of Ω. It is a nonlinearly stable state (Figure 1). The color convention for positive vorticity is red and for negative is blue. With the predominance of red colors in the north hemisphere denotes the most probable state for negative β is a pro-rotating solid -body flow state.

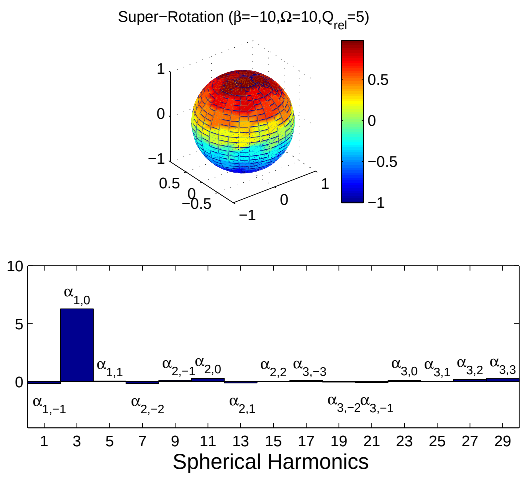

Figure 1: Negative β – Super-Rotation

5.2 Sub-Rotation

By variational theory [4], when β is positive and the relative enstrophy Q_{rel} is less than $\Omega^2 C^2$, there is a nonlinearly stable state getting local minimum energy with $w^0_{\min}(Q_{rel}) = -\sqrt{Q_{rel}}\psi_{10}$.

5.3 Phase Transition

According to the mean field theory given by Lim [1], there should exist two phase transitions at positive

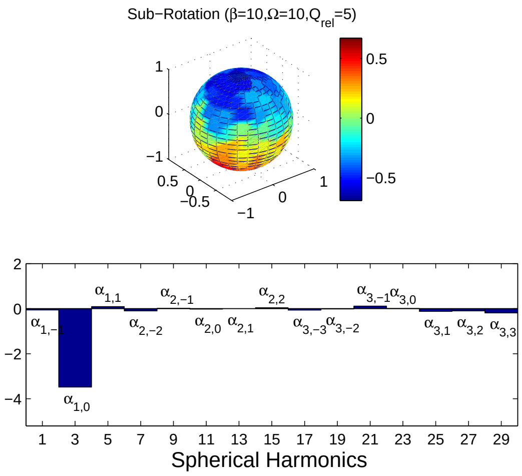

Figure 2: Positive β – Sub-Rotation

and negative temperatures when the planetary spin is large enough compared to relative enstrophy .

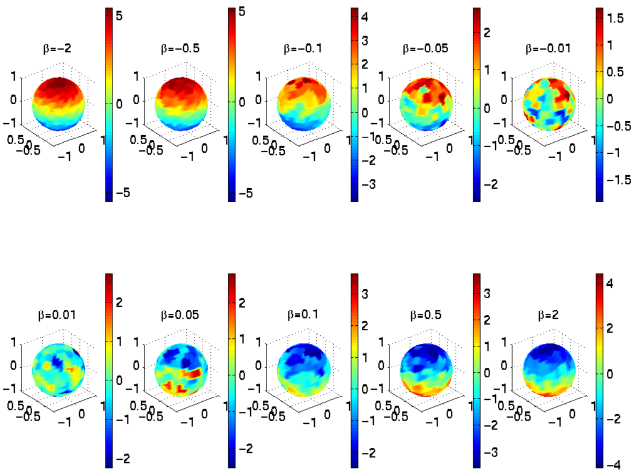

Figure 3: The most probable state vs. β

5.4 Wobble

Relaxation of angular momentum is a necessary step in the modeling of the important tilt instability where the rotational axis of the barotropic atmosphere tilts away from the fixed north-south axis of planetary spin.

The z component of the angular momentum is $\psi(1,0)$, the x and y components are $\psi(1,1)$ and $\psi(1,-1)$. Adding small amounts of x and y angular momenta to a large z angular momentum is equiva-

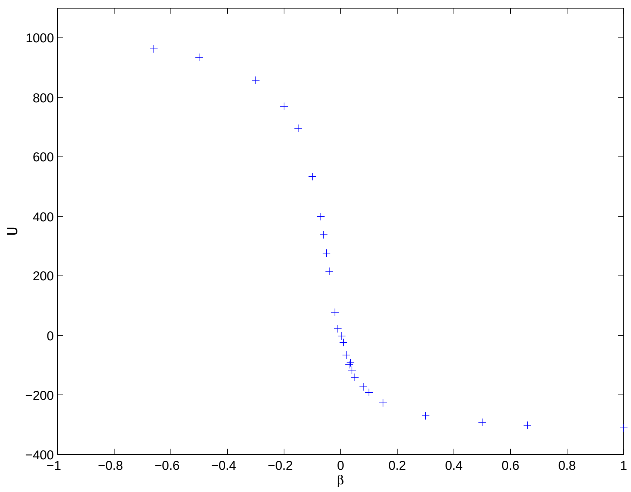

Figure 4: Internal energy vs. β

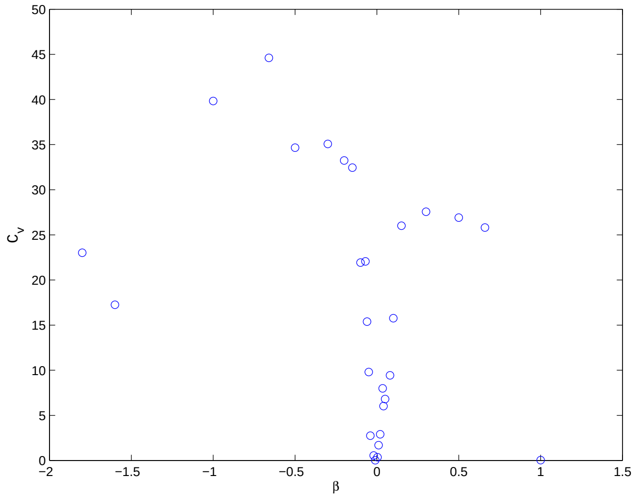

Figure 5: Specific heat vs. β

lent to tilting the axis of the atmospheric rotations away from the planetary spin axis (Figure 6).

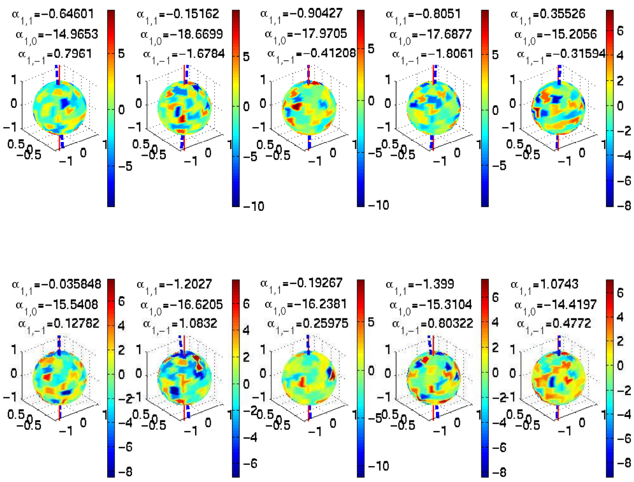

Figure 6: The dashed blue line denotes the rotational axis of the atmosphere while the red solid line is the planetary spin axis.

6 Conclusion

A detailed comparision with the Monte-Carlo simulation results reported here and the qualitative results from mean field theory [1] will be presented in Ding and Lim [2]. In addition, we also refer the reader to the report by Mavi and Lim [6] in the AMS proceedings, on the application of a one-step renormalization method known as the Bragg method to the barotropic flow problem on a rotating sphere.

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