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PHASE TRANSITIONS OF BAROTROPIC FLOW ON THE SPHERE BY THE BRAGG METHOD

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1 Introduction

We consider an ensemble of $2N$ vortices on a rotating sphere. This model is in the theory of a barotropic atmosphere with detectable phase transitions developed by Lim. The theory is relevant to weather systems where atmospheres have significant interactions with planetary surface. The Hamiltonian in these systems is given by coupling of vortices. The N point vortex problem for a finite number dynamic vortices is described [Newton(2001)]. In this paper we consider vortices on a fixed random lattice, similar to Lim[Lim(2005a)][Lim(2005b)], however, we will consider only vortices with spin of $+1$ or -1 as these spins are intended to model fluid flow, we enforce Stokes theorem by requiring N vortices of $+1$ spin and N vortices of -1 spin. We allow spins to reconfigure within the lattice with out further constraint and find the statistically preferred states.

A mean field method on a related model[Lim(2005c)], (where spins are allowed to take on a continuous range of values), has succesfully found phase transitions for the BVE on a Rotating Sphere. There as here, angular momentum of the fluids is conserved canonically, this leads to the exciting result that super-rotation and sub-rotational states are preferred in certain thermodynamic regimes. The Bragg mean field approach on the $2N$ vortex model, corroborates these previous results. These results are also in agreement with Monte Carlo analysis[Lim and Nebus(2004)], on the spherical logarithmic model. We shift from the previous spherical model used by Lim to a simplified discrete state model which enables us to use the Bragg method to approximate the free energy.

Our discrete vorticity model is a set $\mathbf{X} = \{\vec{x_i}\}$ of $2N$ random sites on the sphere with uniform distribution, each with a spin s_i in $\{+1, -1\}$, and interacts with every other site as a function of distance. Each spin also interacts with the planetary spin which is analogous an external magnetic field in the Ising model. Contribution to the kinetic energy from planetary spin varies zonally, however, this makes the external field inhomogeneous and difficult to deal with analytically. Bragg and Williams used a one step renormalization to investigate properties of order-disorder in the Ising Model of a ferromagnet. As the discretized BVE on a rotating sphere is similar to the Ising Model of a ferromagnet in an inhomogeneous field, we use the Bragg-Williams renormalization technique to infer the order-disorder properties of the fluid. In this setup we find a positive temperature phase transition at $0 < T < T_p$ to the sub-rotating state. In negative temperature there is a phase transition $T_n < T < 0$ to the super-rotating state.

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2 The Physical Picture

Time independent motion of incompressible inviscid single layer fluids can be completely described by the stream function. This is in the hopes that we can reproduce some of the occurring global phenomena. It is well known that fluid behavior is chaotic, and small scale fluctuations may give rise to large scale behavior, nevertheless some weather patterns can be reconstructed purely in equilibrium statistical mechanics. Thus we consider the atmosphere as a single layer of incompressible inviscid fluid. As our interest is in global weather patterns, we will consider the problem on the spherical geometry, with non-negative angular velocity $\Omega \geq 0$, and relative stream function ψ. The Barotropic Vorticity Equation is

$$\frac{D}{Dt}q = 0$$

where q discribes the total vorticity on the sphere,

$$q = \omega + 2\Omega\cos(\theta)$$

The first term ω is the vorticity relative to the rotating frame, and the second gives vorticity due to the angular velocity of the sphere. Recall that the vorticity is given by the Laplacian of the stream function,

$$\omega = \Delta\psi$$

In order to study the problem in a more tractable framework, we restate the problem in terms of local vortices. In this development we follow Lim[Lim(2005a)]. The relative zonal and sectoral flow is given by the gradient of the stream function and will be denoted as u_r and v_r respectively. In addition the flow due to rotation also contributes to the kinetic energy, it is entirely zonal and will be denoted as u_p. Thus we write

$$\psi = \Delta^{-1}q - \Delta^{-1}2\Omega\cos(\theta)$$

Note that Stokes theorem implies the total circulation on the sphere is zero,

$$\int_{S^2} q = 0$$

However there is zero total circulation of solid body rotation $2\Omega\cos(\theta)$ thus,

$$\int_{S^2} \omega = 0 \tag{1}$$

We now need the kinetic energy, which will be the Hamiltonian in our work,

$$\begin{aligned}
H'[q] &= \frac{1}{2}\int_{S^2} dx\left[(u_r+u_p)^2 + v_r^2\right] && (2)\\
&= \frac{1}{2}\int_{S^2} dx\left[u_r^2 + v_r^2 + 2u_r u_p + u_p^2\right] && (3)\\
&= -\frac{1}{2}\int_{S^2} dxq\psi + \frac{1}{2}\int_{S^2} dxu_p^2 && (4)\\
&= -\frac{1}{2}\int_{S^2} dx[\omega + 2\Omega\cos\theta]\Delta^{-1}\omega && (5)\\
&\quad +\frac{1}{2}\int_{S^2} dxu_p^2 && (6)\\
&= -\frac{1}{2}\left\langle\omega, \Delta^{-1}\omega\right\rangle && (7)\\
&\quad -\Omega C\left\langle\psi_{10}, \Delta^{-1}\omega\right\rangle + \frac{1}{2}\left\|u_p\right\|_2^2 && (8)
\end{aligned}$$

with $C = \|\cos\theta\|_2$. The zonal flow element $\frac{1}{2}\left\|u_p\right\|_2^2$ is invariant under variation of the steam function, and will be disregarded below.

3 The Discrete Model

In the above discussion the vorticity is a function on S^2 given by the laplacian of the stream function, we approximate the vorticity function by a step function constant over voronoi cells of the random lattice $\mathbf{X}$. In particular, the vorticity is approximated by,

$$\omega(x) = \sum_i H_i(x)s_i$$

where H_i is the indicator function for the voronoi cell at x_i. The vorticity s_i is assigned values ± 1 which models the direction of the spin over a cell. This has the advantage of preserving all orders of the vorticity, as is required by the BVE[Lim(2005c)]. Thus we have this simple form for the kinetic energy of the system,

$$H_N = -\sum_{(i,j)} J_{(i,j)}s_i s_j - \Omega\sum_i F_i s_i \tag{9}$$

The usefulness of this form is clear, it closely resembles the lattice models of solid state physics, and becomes accessible to the rich theory of methods of that field. The interaction coefficients are global and the external field term is position dependent,

$$\begin{aligned}
J_{(i,j)} &= \int_{S^2} dwH_i(w)G(H_j)(w) && (10)\\
F_i &= \int_{S^2} dw\cos\theta G(H_i)(w) && (11)
\end{aligned}$$

The $G(\cdot)$ is the Laplace-Beltrami operator on the sphere. The sum is over all lattice site pairs (x, x'). Note that as $\Omega \to 0$ the hamiltonian becomes the kinetic energy for the non-rotating regime. It can also be shown [Lim(2005c)] in the $N \to \infty$ limit,

$$\begin{aligned}
J(x_i, x_j) &\to \frac{16\pi^2}{N^2}\ln|1 - x_i\cdot x_j| && (12)\\
F_i &\to -\frac{2\pi}{N}\left\|cos\phi\right\|_2\psi_{10}(x_i) && (13)
\end{aligned}$$

Notice that the energy of two sites is of three flavors(++,--,+-) with only two different energies, this forms an important simplifying observation to the renormalization technique.

4 Bragg-Williams Approximation

The central feature of the Bragg method is the approximation of internal energy of a state by its long-range order [Huang(1988)]. We will have to alter the previous Bragg method, as the global order of the system is constrained microcanonically, to estimate internal energy from local order over domains on the sphere. The implicit assumption in this method, as in the original Bragg method is that the distribution of spins is homogeneous over a domain, thus, in a domain any spin selected has equal probability of being up. Specifically this is done by defining a partition on the sphere, into domains labeled $\{\xi\}$. For each domain ξ we define notation $N_\xi^+(N_\xi^-) \equiv$ number of sites in cell ξ which are positive(negative). Note $N_\xi = N_\xi^+ + N_\xi^-$. We define for every partition element ξ the local order parameter σ_ξ as:

$$\sigma_\xi \quad = \quad 2\frac{N_\xi^+}{N_\xi} - 1 \tag{14}$$

The method consists of approximation of important quantities by the probability of spin value by its renormalized domain. Then the probability of any spin in domain ξ to be up is $P_\xi^+ = \frac{1+\sigma_\xi}{2}$, and the probability of the spin being down is $P_\xi^- = \frac{1-\sigma_\xi}{2}$.

4.1 Statement of Equations

The notion developed above of local order leads to a simple method of quantifying interaction types. Specifically, pairwise interaction is dependent on several parameters, the above interpretation lead to derivations of equations relevant to the Hamiltonian. Pairwise interaction occurs between all sites in three types $(ss') = (++)$, $(--)$ and $(+-)$, depending on the spin of the interacting sites, we use probabilities of spin distribution to inform probabilities of spin interaction. We calculate the probability, labeled $P_{\xi\xi'}^{ss'}$, associated with these types. Clearly,

$$P_{\xi\xi'}^{++} + P_{\xi\xi'}^{--} + P_{\xi\xi'}^{+-} = 1 \tag{15}$$

Which is identical to the relation

$$P_{\xi\xi'}^{++} + P_{\xi\xi'}^{--} - P_{\xi\xi'}^{+-} = 1 - 2P_{\xi\xi'}^{+-} \tag{16}$$

Which expressions arise below in (19) and (23), and are the sole contribution of pairwise order to the free energy. Thus we are content to make explicit the order probability $P_{\xi\xi'}^{+-}$ only, as follows ($\xi' \neq \xi$):

$$Prob_{\xi\xi'}\{+-\} = \frac{N_{\xi\xi'}^{+-}}{N_\xi N_{\xi'}} = \frac{N_\xi^+}{N_\xi}\frac{N_{\xi'}^-}{N_{\xi'}} + \frac{N_\xi^-}{N_\xi}\frac{N_{\xi'}^+}{N_{\xi'}}$$

$$= \frac{1+\sigma_\xi}{2}\frac{1-\sigma_{\xi'}}{2} + \frac{1+\sigma_{\xi'}}{2}\frac{1-\sigma_\xi}{2} = \frac{1-\sigma_\xi\sigma_{\xi'}}{2}$$

Analogously we have,

$$Prob_{\xi\xi}\{+-\} = \frac{N_{\xi\xi}^{+-}}{\frac{1}{2}N_\xi(N_\xi - 1)} = \frac{N_{\xi\xi}^{+-}}{\frac{1}{2}N_\xi^2} = 2\frac{N_\xi^+}{N_\xi}\frac{N_\xi^-}{N_\xi}$$

$$= 2\frac{1+\sigma_\xi}{2}\frac{1-\sigma_\xi}{2} = \frac{1-\sigma_\xi^2}{2}$$

For N_ξ sufficiently large. Finally, we state the relation

$$1 - 2P_{\xi\xi'}^{+-} = \sigma_\xi\sigma_{\xi'}$$

We have made this estimation based on the assumption that the number of spins in any domain ξ is sufficiently large, in fact in the thermodynamic limit of lattice models it is standard to take the number of spins to infinity. This must be done, however, on the finite surface of the sphere. Therefore, we in the limit the values N_ξ approach infinity.

4.2 Estimation of Important Quantities

Internal Energy As per the above assumptions, namely, that the distribution of cells the ratio of distribution of positive and negative sites is homogeneous on a partition element, we can calculate the expected internal energy in terms of the vector $\{\sigma_\xi\}$.

$$\langle H_N \rangle$$

$$= \left\langle -\frac{1}{2}\sum_{\xi,\xi'}\sum_{(x,x')\in\xi\times\xi'} J(x,x')s_x s_{x'} - \Omega\sum_\xi\sum_{x\in\xi} F_x s_x \right\rangle$$

$$= -\frac{1}{2}\sum_{\xi,\xi'}\left\langle \sum_{(x,x')\in\xi\times\xi'} J(x,x')s_x s_{x'}\right\rangle - \Omega\sum_k\left\langle\sum_{x\in\xi_k} F_x s_x\right\rangle$$

The factor of $1/2$ arises due to double counting among the cells. J is the coupling constant between sites on the lattice, but, as we are course graining, we use only the expected coupling between sites. Thus we define the expected coupling constant

$$K_{\xi\xi'} = \langle \ln(1 - x\cdot x') | (x,x') \in \xi\times\xi' \rangle$$

Further, define

$$L_\xi = -\int_\xi \frac{dx}{V_\xi}\frac{\|cos\theta\|_2}{2}\psi_{10}(x)$$

the expected coupling of a site with the external field. Returning to the Hamiltonian, we focus on the individual terms,

$$\left\langle \sum_{(x,x')\in\xi\times\xi'} J(x,x')s_x s_{x'} \right\rangle \tag{17}$$

$$= N_\xi N_{\xi'} \langle s_x s_{x'} \rangle \langle J(x,x') | (x,x') \in \xi\times\xi' \rangle \tag{18}$$

$$= \left(P^{++}_{\xi\xi'} + P^{--}_{\xi\xi'} - P^{+-}_{\xi\xi'}\right) V_\xi V_{\xi'} K_{\xi\xi'} \tag{19}$$

$$= \sigma_\xi \sigma_{\xi'} V_\xi V_{\xi'} K_{\xi\xi'} \tag{20}$$

The expected interaction of a partition element with itself differs by a factor of $\frac{1}{2}$ to account for double counting.

$$\left\langle \sum_{(x,x')\in\xi\times\xi} J(x,x')s_x s_{x'} \right\rangle \tag{21}$$

$$= \frac{1}{2} N_\xi N_\xi \langle s_x s_{x'} \rangle \langle J(x,x') | (x,x') \in \xi\times\xi \rangle \tag{22}$$

$$= \frac{1}{2}\left(P^{++}_{\xi\xi} + P^{--}_{\xi\xi} - P^{+-}_{\xi\xi}\right) V_\xi V_\xi K_{\xi\xi} \tag{23}$$

$$= \frac{1}{2}\sigma_\xi \sigma_{\xi'} V_\xi V_{\xi'} K_{\xi\xi} \tag{24}$$

Finally, the external term becomes

$$\left\langle \sum_{x\in\xi} F_x s_x \right\rangle = N_\xi \langle s_x | x \in \xi \rangle \langle F_x | x \in \xi \rangle$$

$$= -\sigma_\xi \int_\xi dx \frac{\|cos\theta\|_2}{2} \psi_{10}(x) = \sigma_\xi V_\xi L_\xi$$

Thus we have the simple expression for the Bragg expected energy

$$U = -\frac{1}{2}\sum_{\xi,\xi'} \sigma_\xi \sigma_{\xi'} V_\xi V_{\xi'} K_{\xi\xi} - \Omega \sum_\xi \sigma_\xi V_\xi L_\xi$$

Entropy It should be noted that the nature of our model is constrained to a constant sized sphere, and therefore no extensive quantities exist, only intensive quantities. Thus the entropy is evaluated as in intensive quantity in the standard way as a function of $\{\sigma_\xi\}$.

$$S = -k_B \sum_\xi V_\xi \left[\frac{1+\sigma_\xi}{2}\ln\frac{1+\sigma_\xi}{2} + \frac{1-\sigma_\xi}{2}\ln\frac{1-\sigma_\xi}{2}\right]$$

Where the Shannon estimate of entropy is used, with entropy of variables weighted by area of domain.

The Free Energy The Helmholtz free energy is then,

$$\Psi = U - TS$$

$$= -\frac{1}{2}\sum_{\xi,\xi'} \sigma_\xi \sigma_{\xi'} V_\xi V_{\xi'} K_{\xi\xi} - \Omega \sum_\xi \sigma_\xi V_\xi L_\xi$$

$$+Tk_B \sum_\xi V_\xi \left[\frac{1+\sigma_\xi}{2}\ln\frac{1+\sigma_\xi}{2} + \frac{1-\sigma_\xi}{2}\ln\frac{1-\sigma_\xi}{2}\right]$$

Constrained to the set of σ a solution to

$$0 = \sum_\xi V_\xi \sigma_\xi \tag{25}$$

The goal is to find critical points of the free energy with respect to $\{\sigma_\xi\}$. Enforcing TC=0(25), we get a Lagrange multiplier problem. We must find the critical points, given by the simultaneous solution of the m equations,

$$\lambda V_\xi \left\{ \nabla_\sigma \sum_\xi \sigma_\xi \right\}_\xi = \{\nabla_\sigma F\}_\xi$$

$$= -V_\xi \Omega L_\xi - \sum_{\xi'} \sigma_{\xi'} V_\xi V_{\xi'} K_{\xi\xi'} + Tk_B V_\xi \left(\frac{1}{2}\ln\frac{1+\sigma_\xi}{1-\sigma_\xi}\right)$$

Along with the constraint equation (25). Which is equivalent to, $\forall\xi$

$$\sigma_\xi = \tanh\left[\beta(\Omega L_\xi + \lambda) + \beta\sum_{\xi'} \sigma_{\xi'} V_{\xi'} K_{\xi'\xi}\right]$$

In the continuum limit we take the number of spins to infinity. We can as well take the number of renormalized domains to infinity. In doing so we get a function $\sigma : S_2 \mapsto [-1,1]$. We get an analogous expression of the free energy,

$$\Psi[\sigma] = -\frac{1}{2}\int dx \int dy \sigma(x)\sigma(y)K(x,y) - \Omega\int dx\sigma(x)L(x)$$

$$+Tk_B \int dx \left[\frac{1+\sigma(x)}{2}\ln\frac{1+\sigma(x)}{2} + \frac{1-\sigma(x)}{2}\ln\frac{1-\sigma(x)}{2}\right]$$

From which we can easily recover the fixed point equation. The free energy must be extremized over the space of functions $\sigma : S^2 \mapsto [-1,1]$ so that $0 = \int_{S^2} dx\sigma(x)$ which was the constraint enforced by Stokes Theorem. Other physical constraints such as bounded enstrophy are enforced by virtue of the fixed point equation which can be derived in the continuous case similarly to the above.

$$\sigma(x) = \tanh\left[\beta(\Omega L(x) + \lambda) + \beta \int_{S^2} dy \sigma(y) K(x, y)\right] \tag{26}$$

(26) has the virtue of bounding the enstrophy pointwise. A simple newton scheme, at $\lambda = 0$ and no physical constraints enforced, successfully approximates the solution. The original discrete model was limited in its degrees of freedom, we now have continuum of freedom for the spin. Further the estimation of interaction became very accurate in the continuum model. The price paid is the artifact of the entropy estimation, however, this two degree of freedom estimation is exactly what enables the fixed point equation above. A graphic (figure 1) shows the spin states of some thermodynamic regimes.

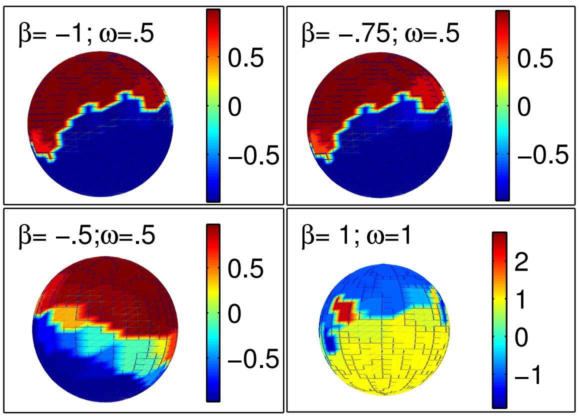

Figure 1: Some fixed point solutions

5 Polar State Criteria

To get a rough idea of the global behavior we use the Bragg method to approximate the long range order of two sets on the sphere. The order of the system varies continuously with the temperature and ambient spin. The system therefore exhibits both continuous and abrupt phase transitions. The system on a non-rotating sphere has an abrupt phase transition in negative temperature, that leads to a critical beta. Other thermodynamic regimes also exhibit continuous phase transitions. The existence of negative temperature is demonstrated below.

We partition the sphere into the northern and southern hemispheres. Under this configuration we maximize the detection of interaction of sites with the external field. In this case we have only parameters σ_n and σ_s representing local order of the northern and southern hemispheres respectively. Clearly then $\sigma_s = -\sigma_n$ from (1). This leads to equations,

$$\sigma_n = \tanh\left[\beta(\Omega L_1 + \lambda) + \beta\sigma_n V_n(K_{n,n} - K_{n,s})\right]$$

$$-\sigma_n = \tanh\left[-\beta(\Omega L_1 - \lambda) - \beta\sigma_n V_n(K_{n,n} - K_{n,s})\right]$$

This leads easily to the fixed point equation in one variable

$$\sigma_n = \tanh\left[\beta\Omega L_1 + \beta\sigma_n V_n(K_{n,n} - K_{n,s})\right] \tag{27}$$

Non-rotating The non-rotating case is given in equation 27 by setting $\Omega = 0$. For positive β the RHS of 27 is decreasing while the LHS increases, clearly, we have unique solution $\sigma_1 = 0$. In the negative temperature domain $\beta < 0$ we must maximize the free energy. The non-rotating system is given by $\Omega = 0$. For this case it is clear that a solution is found at $\sigma_1 = 0$, corresponding to a mixed state. Whether other fixed point exist depends on the slope of the RHS of (27) at $\sigma_1 = 0$. Graphically it is clear this situation arises when the slope of the RHS has a slope of one, thus there is a critical quantity β_c given by $\beta_c V_1(K_{11} - K_{12}) = 1$. Thus for

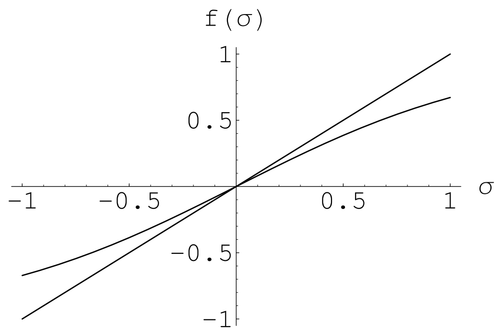

Figure 2: A graph of the fixed point when $\beta_c < \beta < 0, \Omega = 0$.

$0 > \beta > \beta_c = [V_1(K_{11} - K_{12})]^{-1}$ the point $\sigma_1 = 0$ is the only stationary point(see figure (2)). A check on the concavity shows this is indeed a maximum, as required. For $\beta < \beta_c < 0$ we have non-zero roots of the free energy $\sigma^- < 0$, $\sigma^+ > 0$, the tanh function is odd, thus $\sigma^- = -\sigma^+$(figure (3)).

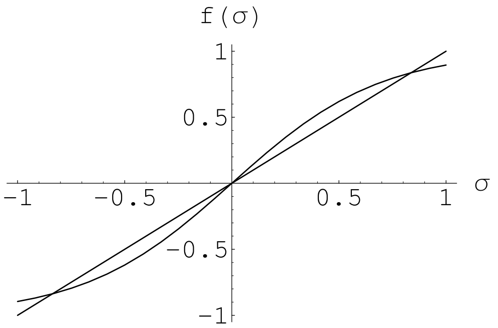

Figure 3: A graph of the fixed point when $\beta < \beta_c < 0, \Omega = 0$.

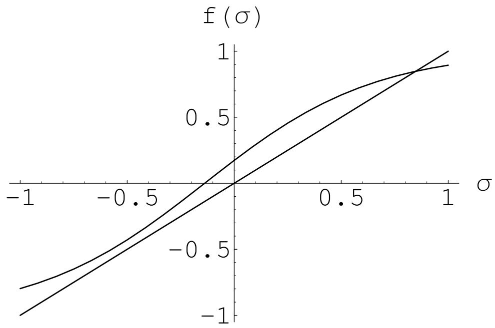

Figure 5: A graph of the fixed point when $\beta_c^{\Omega} < \beta < 0, \Omega_c > \Omega > 0$.

The free energy is symmetric here so we need only consider fixed points σ^+ and $\sigma = 0$. We compare the free energy for $0 < \sigma < 1$ to $\sigma = 0$.

$$
\begin{aligned}
0 &< \Psi(\sigma) - \Psi(0) && (28)\\
&= (K_{12} - K_{11})\sigma^2 + T\Delta S && (29)
\end{aligned}
$$

Where

$$\Delta S = k_B 2\pi \left[(1+\sigma)\ln(1+\sigma) + (1-\sigma)\ln(1-\sigma)\right]$$

We therefore have a polarized state for $\beta < \beta_c$.

Rotating In positive temperature for $\Omega > 0$, the RHS has zero at $\frac{\Omega L_1}{K_{12}-K_{11}}$, thus there is a solution $\bar{\sigma}_1 < 0$ (see figure 4).

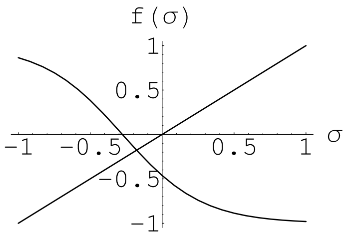

Figure 4: A graph of the fixed point when $\beta, \Omega > 0$.

In negative temperature, we note $\exists \beta_c^{\Omega}$ so that $\beta < \beta_c^{\Omega} < 0$ implies the existence of three fixed points (see figure 5) in the case $\Omega < \frac{V_1(K_{11}-K_{12})}{L_1} = \Omega_c$. Otherwise there is only one fixed point (see figure 6), indicating a super - rotational state.

In the low absolute valued negative temperature regime, we easily find that the negative fixed points $\sigma_1^-, \hat{\sigma}_1^- < 0$ are lower in absolute value than the positive fixed point $\sigma_1^+ > 0$, i.e. $\left|\sigma_1^-\right|, \left|\hat{\sigma}_1^-\right| < \left|\sigma_1^+\right|$. In physical terms this implies that the ordered state $\sigma_1^+ > 0$ (in which positive vorticity dominates the northern hemisphere), is 'more ordered' than the alternatives σ_1^- or $\hat{\sigma}_1^-$. The demonstration of this follows. In the thermal setting under consideration, i.e. $\Omega > 0, \beta < \beta_c^{\Omega} < 0$, the RHS of equation (27) has a zero at $\sigma^o < 0$. Negative solutions are at $\sigma^- = \sigma^o - \epsilon_i$ where $\epsilon_i > 0$, $i = 1, 2$. Denote T as the operator on the RHS of (27), acting on σ, then $\sigma^o - \epsilon_i = T(\sigma^o - \epsilon_i)$. But tanh is an odd function so

$$T(\sigma^o + \epsilon_i) = -\sigma^o + \epsilon_i > \sigma^o + \epsilon_i$$

Tanh is also an increasing function which implies,

$$T(-\sigma^o + \epsilon_i) > -\sigma^o + \epsilon_i$$

However, $T(1) < 1$, finally the continuity of tanh implies $\sigma^+ \in [-\sigma^o + \epsilon_i, 1]$. The preference of free energy

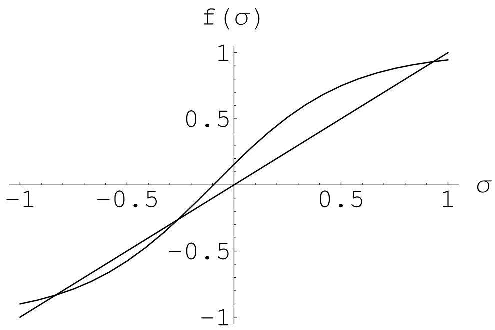

Figure 6: A graph of the fixed point when $\beta < \beta_c^{\Omega} < 0, \Omega_c > \Omega > 0$.

for the σ^+ state is then entirely due to the internal energy which prefers the super-rotational state. The sub-rotational state is preferred by entrophy; lesser, however, than the internal energy preference. We thereby gain the main results of the paper, namely, a simple analytic equation which gives the contribution of the first spherical harmonic to the stream function in any thermodynamic regime. The contribution of the first spherical harmonic being significant as it forms the only contribution to zonal angular momentum, which is the observed super-rotation phenomena we intended to describe.

6 Validity

Of the Continuous Limit As noted above, the model does not have an extensive limit from which we can derive intensive quantities. We have therefore bypassed the standard thermodynamic construction and directly built an intensive model. Convergence theorems support our continuum model. It is thus clear that the continuum model is consistent with the discrete model.

In order to create a continuous function of renormalized vorticity in our theoretical framework. We construct a sequence of lattice ensembles, the kth ensemble being of $N(k)$ lattice nodes. The renormalized domains at the kth step can also be defined by a random lattice, of $M(k)$ sites and containing all the spins sites in its voronoi cell. These quantities both approach infinity in the $k \to \infty$ limit, however, they will scale like, $N(k) = M(k)^2$. We are thus justified in considering σ to take on values continuously in [-1,1].

By the construction of the family of spin lattice and domains, we can consider sequences of decreasing sets in considering the interaction term in the Hamiltonian. Specifically, any point p on the sphere is contained in a domain P_k of the formulation at step k. Clearly, the sequence decreases $P_{k+1} \subset P_k \ \forall k$. It can be seen from (10) that the self interaction of the domains is negligible in the limit. For two points p and q on the sphere there is k so that, $p \in P_k$ and $q \in Q_k$ and $P_k \neq Q_k$.

We can thereby define the renormalized vorticity as a function on S^2 at the kth lattice as

$$\sigma^k(x) = \sum_{\xi} H_\xi(x)\sigma^k_\xi$$

then from the dominated convergence theorem we have that points converging in state space have converging Helmholtz free energy.

6.1 Temperature

It is well known that systems of vortices often have negative temperature. Indeed our model has negative temperature in both the rotating and non-rotating regimes. The existence of negative temperature can be seen by the following. Suppose we divide the sphere into northern and southern hemispheres, and further partition those into N_1, N_2 and S_1, S_2 respectively; in such a way that these sets are symmetric about the equator. This assumption simplifies the algebra, as we have some identities $V_{N_1} = V_{S_1}$ etc. Now let the order of these sets be given by $\sigma_{N_1} = 1$, $\sigma_{S_1} = -1$, $\sigma_{N_2} = \gamma$ and $\sigma_{N_2} = -\gamma$. We can write the energy and entropy in terms of these quantities,

$$U = -V_{N_1}^2 K_{N_1N_1} - \gamma^2 V_{N_2}^2 K_{N_2N_2}$$
$$-\gamma 2 V_{N_1} V_{N_2} K_{N_1N_2} V_{N_1} V_{S_1} K_{N_1S_1} + \gamma^2 V_{N_2} V_{S_2} K_{N_2S_2}$$
$$+\gamma V_{N_1} V_{S_2} K_{N_1S_2}$$
$$+\gamma V_{N_2} V_{S_1} K_{N_2S_1} - \Omega(2V_{N_1} L_{N_1} + \gamma V_{N_2} L_{N_2})$$
$$S = -k_B V_{N_2}\left[(1+\gamma)\ln\frac{1+\gamma}{2} + (1-\gamma)\ln\frac{1-\gamma}{2}\right]$$

Now from the Mean Value Theorem we have that there exists some temperature T_0 such that

$$\frac{1}{T_0} = \frac{\Delta S}{\Delta U} \tag{30}$$

Thus if we consider the secant from $\gamma \neq 0$ to $\gamma = 0$ we find ΔU and ΔS become

$$\Delta S = 2k_B V_{N_2}\ln 2$$

$$\Delta U = \gamma^2 V_{N_2}^2 (K_{N_1N_1} - K_{N_2N_2})$$
$$+\gamma V_{N_1} V_{S_2}(K_{N_1N_2} - K_{N_1S_2} - K_{N_2S_1}) + \gamma 2\Omega V_{N_2} L_{N_2}$$

Inspecting the denominator of the RHS of 30, it is clear at $\gamma = 1$ we get negative temperatures. On the other hand, if we set $\gamma = -1$, then the denominator becomes a sum of positive and negative terms. But, on taking V_{N_1} $\frac{1}{n}$ we find that the negative term approaches zero as

$$\gamma^2 V_{N_2}^2 (K_{N_1N_1} - K_{N_2N_2}) \approx -\frac{\ln n}{n^2}$$

The positive terms, however, approach zero as $\frac{1}{n}$ which gives us a positive temperature.

7 Conclusion

The results are in good agreement with those found through Monte Carlo Analysis by Ding and Lim[Ding and Lim(2006)] in a paper appearing in this conference. The complement to this technique, using probabilities of spin interactions to inform spin distribution is the implementation of the Bethe-Peierls approximation on this model, and remains to be attempted.

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