JP4.5 QUASI-WAVELET FORMULATIONS OF TURBULENCE WITH INTERMITTENCY AND CORRELATED FIELD PROPERTIES

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1. INTRODUCTION

Quasi-wavelets (QWs) are similar to customary wavelets in that they are based on translations and dilations of a parent function; however, their positions and orientations are random. An individual QW is roughly analogous to a turbulent eddy. A random ensemble of QWs, with size distribution and rotation rates chosen in a manner consistent with Kolmogorov's hypothesis, produces velocity fields with realistic spectral properties (Goedecke et al., 2004). Previous research has demonstrated the utility of QWs for modeling scattering of acoustic waves by turbulence (Goedecke et al., 2001; Wilson et al., 2004) and for synthesizing random, kinematic fields with statistical properties resembling actual turbulence (Goedecke et al., 2006). Other possible applications (pending further advances with QW models) include identification of coherent turbulent structures, formulation of subgrid-scale models in turbulence simulations, synthesis of random terrain elevations and geoligic structure, and electromagnetic and seismic wave scattering.

In this paper, earlier QW formulations are extended in two main ways. First, multiple QW fields with correlated properties are formulated. It is shown how a QW model can be constructed to produce a constant scalar flux layer. This technique could be used to model heat flux in the atmospheric surface layer. Second, a QW model with intermittency is developed based a multifractal formulation similar to Frisch et al. (1976). These extensions would be difficult (if not impossible) to systematically achieve with spatially infinite basis functions such as Fourier modes. They make possible the synthesis of wind and temperature fields in an atmospheric surface layer that have very realistic cross spectral and intermittent properties.

2. SYNOPSIS OF VELOCITY AND MONOPOLE SCALAR MODELS

Like customary wavelets, QWs are derived from translations and dilatations of a dimensionless, spatially localized parent function. We assume that the parent function $f(\xi)$ is spatially symmetric. Here ξ is the magnitude of the vector $\xi \equiv (\mathbf{r} - \mathbf{b}^{\alpha n})/a_\alpha$, $\mathbf{r}$ is the spatial coordinate, $\mathbf{b}^{\alpha n}$ is the center of the QW, and a_α is its size. The index α indicates the size class of the QW, with $\alpha = 1$ being the largest size and $\alpha = N$ the smallest. The index n indicates a particular QW within that size class. The size a_1 is associated with the outer scale of the turbulence and a_N with the inner scale.

To construct a scalar (e.g., temperature) QW model, let us write the field perturbation associated with an individual QW as

$$T^{\alpha n}(\mathbf{r}) = h^{\alpha n} \Delta T_\alpha f\left(\frac{|\mathbf{r} - \mathbf{b}^{\alpha n}|}{a_\alpha}\right), \tag{1}$$

where $h_{\alpha n}$ is a random sign factor and ΔT_α is an amplitude factor. The $h^{\alpha n}$ are assumed to be statistically independent and equal to $+1$ or -1 with equal probability. Therefore $\langle h^{\alpha n}\rangle = 0$ and $\langle h^{\alpha n} h^{\beta m}\rangle = \delta_{\alpha\beta}\delta_{nm}$, where $\delta_{\alpha\beta} = 1$ if and only if $\alpha = \beta$. The locations $\mathbf{b}^{\alpha n}$ are statistically independent and randomly distributed in the volume V.

The total field is found by summing the contributions of the individual QWs:

$$T(\mathbf{r}) = \sum_{\alpha=1}^{N} \sum_{n=1}^{N_\alpha} T^{\alpha n}(\mathbf{r}), \tag{2}$$

where N_α is the number of QWs for the size class α. We write the Fourier transform of the field as

$$\widetilde{T}(\mathbf{k}) = \frac{1}{(2\pi)^3} \int d^3 r\, T(\mathbf{r})\, e^{-i\mathbf{k}\cdot\mathbf{r}}. \tag{3}$$

Applying to Eqs. (2) and (1), we have

$$\widetilde{T}(\mathbf{k}) = \sum_{\alpha=1}^{N} \sum_{n=1}^{N_\alpha} \widetilde{T}^{\alpha n}(\mathbf{k}), \tag{4}$$

where

$$\widetilde{T}^{\alpha n}(\mathbf{k}) = h^{\alpha n} a_\alpha^3 \Delta T_\alpha \exp\left(-i\mathbf{k}\cdot\mathbf{b}^{\alpha n}\right) F(k a_\alpha), \tag{5}$$

and

$$F(y) = \frac{1}{(2\pi)^3} \int d^3\xi\, f(\xi)\, e^{-i\mathbf{y}\cdot\xi} \tag{6}$$

is the spectral parent function.

The scalar spectral density may be defined by the following equation

$$\Phi_T(\mathbf{k}) = \frac{(2\pi)^3}{V} \left\langle \left|\widetilde{T}(\mathbf{k})\right|^2 \right\rangle. \tag{7}$$

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Substituting with (5) and applying the independence of the sign factors yields

$$\Phi_T(\mathbf{k}) = \sum_{\alpha=1}^{N} \Phi_T^\alpha(\mathbf{k}), \tag{8}$$

where

$$\begin{aligned}\Phi_T^\alpha(\mathbf{k}) &= (2\pi)^3 n_\alpha \left\langle \left|\widetilde{T}^{\alpha n}(\mathbf{k})\right|^2 \right\rangle \\ &= (2\pi)^3 n_\alpha a_\alpha^6 \Delta T_\alpha^2 F^2(ka_\alpha),\end{aligned} \tag{9}$$

and $n_\alpha = N_\alpha / V$ is the number density of the class α.

To formulate a solenoidal representation of a turbulent velocity field, we write the velocity field of each QW, $\mathbf{v}^{\alpha n}(\mathbf{r})$, as $\nabla \times \mathbf{A}^{\alpha n}(\mathbf{r})$. Here $\mathbf{A}^{\alpha n}(\mathbf{r})$ is a vector potential given by

$$\mathbf{A}^{\alpha n}(\mathbf{r}) = \mathbf{\Omega}^{\alpha n} a_\alpha^2\, g(\xi), \tag{10}$$

where $\mathbf{\Omega}^{\alpha n}$ is the angular velocity vector of the QW. For a homogeneous, isotropic turbulence model, the $\hat{\mathbf{\Omega}}^{\alpha n}$ are statistically independent and distributed with random uniformity over all directions. The symmetric function $g(\xi)$ is analogous to $f(\xi)$ for the scalar case. The presence of a_α^2 in the definition provides dimensional consistency. Writing out the curl of the potential leads to the following result for the rotational velocity field associated with the QW:

$$\mathbf{v}^{\alpha n}(\mathbf{r}) = \mathbf{\Omega}^{\alpha n} \times (\mathbf{r} - \mathbf{b}^{\alpha n}) \left(-\xi^{-1} \partial g / \partial \xi\right). \tag{11}$$

Transformation of Eq. (11) leads to

$$\widetilde{\mathbf{v}}^{\alpha n}(\mathbf{k}) = i\,(\mathbf{k} \times \mathbf{\Omega}^{\alpha n}) \exp(-i\mathbf{k} \cdot \mathbf{b}^{\alpha n})\, a_\alpha^5 G(ka_a). \tag{12}$$

We may define the spectral density tensor $\Phi_{ij}(\kappa)$ of the velocity fluctuations as

$$\Phi_{ij}(\mathbf{k}) = \frac{(2\pi)^3}{V} \left\langle \tilde{v}_i(\mathbf{k}) \tilde{v}_j^*(\mathbf{k}) \right\rangle, \tag{13}$$

where $\langle\rangle$ is the ensemble average (average of a large number of random realizations) over the $\hat{\mathbf{\Omega}}$ and $\mathbf{b}$ variables. Substituting with Eq. (12) and recalling that the QWs have statistically independent orientations, we find

$$\Phi_{ij}(\mathbf{k}) = \sum_{\alpha=1}^{N} \Phi_{ij}^\alpha(\mathbf{k}), \tag{14}$$

where

$$\Phi_{ij}^\alpha(\mathbf{k}) = (2\pi)^3 n_\alpha \left\langle \tilde{v}_i^{\alpha n}(\mathbf{k}) \left[\tilde{v}_j^{\alpha n}(\mathbf{k})\right]^* \right\rangle. \tag{15}$$

Substitution with (12) leads to

$$\begin{aligned}\Phi_{ij}^\alpha(\mathbf{k}) &= (2\pi)^3 n_\alpha a_\alpha^{10} G^2(ka_\alpha) \\ &\quad \langle [(\mathbf{k} \times \mathbf{\Omega}^{\alpha n}) \cdot \mathbf{e}_i] [(\mathbf{k} \times \mathbf{\Omega}^{\alpha n}) \cdot \mathbf{e}_j] \rangle,\end{aligned} \tag{16}$$

in which $\mathbf{e}_i$ is the unit vector along the ith coordinate axis. Since the angular velocity magnitude is the same for all members of a size class, we may define $\Omega_\alpha = |\mathbf{\Omega}^{\alpha n}|$. Writing out the components of this equation and noting that the cross correlations of the angular velocity components (e.g., $\left\langle \Omega_x^{\alpha n} \Omega_y^{\alpha n} \right\rangle$) are zero, whereas the autocorrelations (e.g., $\left\langle (\Omega_x^{\alpha n})^2 \right\rangle$) must equal $\Omega_\alpha^2/3$, we have

$$\Phi_{ij}^\alpha(\mathbf{k}) = \frac{(2\pi)^3}{3} \left(k^2 \delta_{ij} - k_i k_j\right) n_\alpha a_\alpha^{10} \Omega_a^2 G^2(ka_\alpha). \tag{17}$$

As is well known in the theory of isotropic turbulence, the spectral tensor can be written as $\Phi_{ij}(\mathbf{k}) = E(k) \left(k^2 \delta_{ij} - k_i k_j\right) / \left(4\pi k^4\right)$, where $E(k)$ is the turbulent kinetic energy (per unit mass) spectrum (Batchelor, 1953). Thus

$$E(k) = \frac{32\pi^4}{3} k^4 \sum_{\alpha=1}^{N} n_\alpha a_\alpha^{10} \Omega_a^2 G^2(ka_\alpha). \tag{18}$$

To mimic the properties of turbulence, we choose n_α, ΔT_α, and Ω_α in a manner consistent with Kolmogorov's hypothesis. First, we assume that the packing fraction, defined as

$$\varphi = n_\alpha a_\alpha^3, \tag{19}$$

is the same for all QW sizes. In the inertial subrange, the rotation rate Ω_α should depend only on the eddy size (a_α in our notation) and the dissipation rate of specific turbulent kinetic energy, ϵ (dimensions length^2 divided by time^3). By dimensional analysis, we must have

$$\Omega_\alpha = c_\Omega a_\alpha^{-2/3} \epsilon^{1/3}, \tag{20}$$

where c_Ω is a constant. Similarly, ΔT_α should depend only on the eddy size, dissipation rate, and the scalar destruction rate χ_T (dimensions T^2 divided by time). Hence

$$\Delta T_\alpha = c_T a_\alpha^{1/3} \chi_T^{1/2} \epsilon^{-1/6}, \tag{21}$$

where c_T is a constant (not to be confused with the structure-function parameter). Applying these relationships to (8) and (17), we have

$$\Phi_T(\mathbf{k}) = (2\pi)^3 \varphi c_T^2 \chi_T \epsilon^{-1/3} \sum_{\alpha=1}^{N} a_\alpha^{11/3} F^2(ka_\alpha), \tag{22}$$

and

$$E(k) = \frac{32\pi^4}{3} \varphi c_\Omega^2 \epsilon^{2/3} k^4 \sum_{\alpha=1}^{N} a_\alpha^{17/3} G^2(ka_\alpha). \tag{23}$$

The final step involves setting the dependence of the eddy sizes a_α on α and converting the two previous expressions to integrals. As with normal wavelets, we assume that the ratio of one size class to the next is constant. The following equation has this property

$$a_\alpha = a_1 e^{-\mu(\alpha-1)}. \tag{24}$$

Here, μ controls the spacing between the size classes. In the limit of small μ, the summations can be replaced by integrals according to the rule

$$\sum_{\alpha=1}^{N} \to \mu^{-1} \int_{a_N}^{a_1} \frac{da}{a}. \tag{25}$$

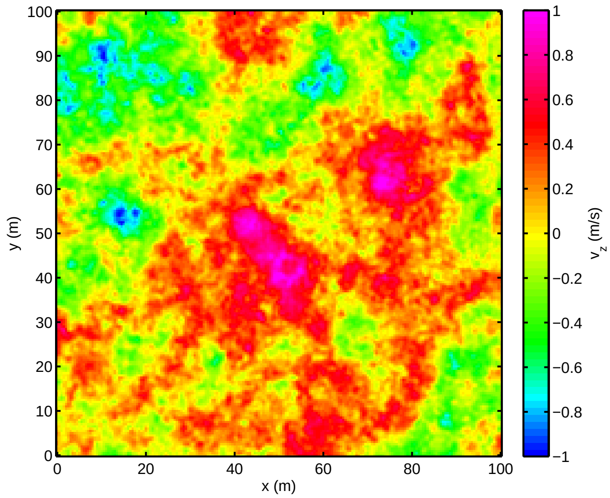

FIG. 1: Visualization of a velocity field synthesized with quasi-wavelets. Shown is the vertical velocity component in a horizontal plane.

We thus have for (22) and (23)

$$\Phi_T(\mathbf{k}) = \frac{(2\pi)^3 \varphi}{\mu} c_T^2 \chi_T \epsilon^{-1/3} k^{-11/3} \int_{ka_N}^{ka_1} dy\, y^{8/3} F^2(y), \tag{26}$$

and

$$E(k) = \frac{32\pi^4 \varphi}{3\mu} c_\Omega^2 \epsilon^{2/3} k^{-5/3} \int_{ka_N}^{ka_1} dy\, y^{14/3} G^2(y). \tag{27}$$

For wavenumbers within a well developed inertial subrange, $a_1 \gg k^{-1} \gg a_N$, the integrals are nearly constant. Hence $\Phi_T(\mathbf{k}) \sim \chi_T \epsilon^{-1/3} k^{-11/3}$ and $E(k) \sim \epsilon^{2/3} k^{-5/3}$, as would be expected.

Figure 1 is an example realization of a QW wavelet velocity field. Shown is the vertical velocity field in a horizontal plane. To generate this realization, the QWs were randomly positioned inside a volume with dimensions 150m by 150 m by 50 m. A Gaussian QW parent function was used as described in Goedecke et al. (2001). The packing fraction φ was set to 1 and μ to 0.693, which corresponds to $a_\alpha / a_{\alpha+1} = 2$. The largest QW size is $a_1 = 50$ m and the smallest $a_N = 0.5$ m. (Although this value for a_N is larger than the Kolmogorov microscale, it is not necessary to generate smaller eddies in this case since they would not be visible at the resolution of the visualizations.) The visualizations show a 100 m by 100 m cross section in the $x-y$ plane. A 25-m buffer on each side mitigates edge effects that otherwise would result from missing large QWs that are partly inside the volume but whose centers lie outside of it.

3. DIPOLE SCALAR MODEL

The scalar model described in the previous section can be described as a *monopole* scalar model, since each QW is isotropic and has only sign associated with it. We could also construct a *dipole* scalar model by making each QW proportional to a function that has a single axis of symmetry, and is positive in one region and negative in another. The utility of the dipole scalar model will become apparent later when we consider fluxes. The dipole can be created by differentiating the spherically symmetric function $f(\xi)$ along an axis passing through its origin. Designating this axis as $\mathbf{d}^{\alpha n}$, we set

$$T^{\alpha n}(\mathbf{r}) = \Delta T_\alpha a_\alpha (\mathbf{d}^{\alpha n} \cdot \nabla) f(\xi). \tag{28}$$

The axis $\mathbf{d}^{\alpha n}$ represents the random orientation of the dipole. Taking the Fourier transform of (28) yields

$$\widetilde{T}^{\alpha n}(\mathbf{k}) = i a_\alpha^4 \Delta T_\alpha (\mathbf{d}^{\alpha n} \cdot \mathbf{k}) \exp(-i\mathbf{k} \cdot \mathbf{b}^{\alpha n}) F(ka_\alpha), \tag{29}$$

from which

$$\Phi_T^\alpha(\mathbf{k}) = (2\pi)^3 n_\alpha \left\langle \left| \widetilde{T}^{\alpha n}(\mathbf{k}) \right|^2 \right\rangle \tag{30}$$

$$= (2\pi)^3 n_\alpha a_\alpha^8 \Delta T_\alpha^2 \left\langle (\mathbf{k} \cdot \mathbf{d}^{\alpha n})^2 \right\rangle F^2(ka_\alpha).$$

For an isotropic model, $\langle d_i^{\alpha n} d_j^{\alpha n} \rangle = \delta_{ij}/3$. Hence

$$\left\langle (\mathbf{k} \cdot \mathbf{d}^{\alpha n})^2 \right\rangle = \langle k_1^2 d_1^2 + k_2^2 d_2^2 + k_3^2 d_3^2 \rangle = k^2/3, \tag{31}$$

and

$$\Phi_T(\mathbf{k}) = \frac{(2\pi)^3}{3} n_\alpha a_\alpha^8 \Delta T_\alpha^2 k^2 F^2(ka_\alpha). \tag{32}$$

Substituting with (19) and (21), we have

$$\Phi_T(\mathbf{k}) = \frac{(2\pi)^3}{3} \varphi c_T^2 \chi_T \epsilon^{-1/3} k^2 \sum_{\alpha=1}^{N} a_\alpha^{17/3} F^2(ka_\alpha). \tag{33}$$

In the limit of small μ, this becomes

$$\Phi_T(\mathbf{k}) = \frac{(2\pi)^3 \varphi}{3\mu} c_T^2 \chi_T \epsilon^{-1/3} k^{-11/3} \int_{ka_N}^{ka_1} dy\, y^{14/3} F^2(y). \tag{34}$$

This result differs from (26) only by a factor of $1/3$ and by the $y^{14/3}$ (instead of $y^{8/3}$) under the integral. Note in particular that, as before, $\Phi_T(\mathbf{k}) \sim \chi_T \epsilon^{-1/3} k^{-11/3}$ in the inertial subrange.

4. VELOCITY-SCALAR COVARIANCE

An important feature of the atmospheric surface layer is that it has nearly constant vertical fluxes of momentum and heat across its depth. In this section, we consider formulation of a constant scalar flux QW model, such as would apply to heat flux. Our modeling approach assumes each QW has both a scalar field and velocity field associated with it. The center locations for the two fields, $\mathbf{b}^{\alpha n}$, coincide.

As is well known in the literature on boundary-layer meteorology, the turbulent flux of a scalars in direction i is proportional to the covariance between the scalar and the velocity component in that direction. The covariance can be determined by integrating the cross spectrum over the wavenumber space. For a particular size class α, the cross spectrum between the scalar and velocity component i is

$$\Phi_{Ti}^{\alpha}(\mathbf{k}) = (2\pi)^3 n_\alpha \left\langle \widetilde{T}^{\alpha n}(\mathbf{k}) \left[\widetilde{v}_i^{\alpha n}(\mathbf{k})\right]^* \right\rangle . \quad (35)$$

Suppose first we construct the scalar perturbations $\widetilde{T}^{\alpha n}(\mathbf{k})$ with the monopole equation (5), and the velocity field with (12). Then

$$\Phi_{Ti}^{\alpha}(\mathbf{k}) = i(2\pi)^3 \varphi a_\alpha^5 \Delta T_\alpha \Omega_\alpha \quad (36)$$
$$\mathbf{e}_i \cdot \left(\mathbf{k} \times \left\langle h^{\alpha n} \widehat{\mathbf{\Omega}}^{\alpha n} \right\rangle\right) F(ka_\alpha) G(ka_a),$$

where $\widehat{\mathbf{\Omega}}^{\alpha n} \equiv \mathbf{\Omega}^{\alpha n}/\Omega_\alpha$ is the unit vector along the axis of rotation. Regardless of the value of $\left\langle h^{\alpha n} \widehat{\mathbf{\Omega}}^{\alpha n} \right\rangle$, the preceding equation is an odd function of each of the wavenumbers k_1, k_2, and k_3. Hence when it is integrated over the wavenumber space, it is zero. Therefore the monopole scalar model is incapable of producing a flux. This outcome could have been anticipated from a simple physical argument. Each QW has an "updraft" and "downdraft" of equal strength distributed about its axis of rotation. Since the scalar field has the same sign and is symmetrically distributed in the updraft and downdraft regions, the net flux must be zero.

Let us next consider the dipole scalar model. This model allows the scalar field to have different signs in the updraft and downdraft. In this case, the cross spectrum for the size class α is

$$\Phi_{Ti}^{\alpha}(\mathbf{k}) = (2\pi)^3 \varphi a_\alpha^6 \Delta T_\alpha \Omega_\alpha \quad (37)$$
$$\mathbf{e}_i \cdot \left\langle (\mathbf{k} \cdot \mathbf{d}^{\alpha n}) \left(\mathbf{k} \times \widehat{\mathbf{\Omega}}^{\alpha n}\right) \right\rangle F(ka_\alpha) G(ka_a).$$

If the $\mathbf{d}^{\alpha n}$ and $\widehat{\mathbf{\Omega}}^{\alpha n}$ are independent random variables, $\left\langle (\mathbf{k} \cdot \mathbf{d}^{\alpha n}) \left(\mathbf{k} \times \widehat{\mathbf{\Omega}}^{\alpha n}\right) \right\rangle = (\mathbf{k} \cdot \langle \mathbf{d}^{\alpha n} \rangle) \left(\mathbf{k} \times \left\langle \widehat{\mathbf{\Omega}}^{\alpha n} \right\rangle\right) = 0$, and the cross spectrum vanishes. Some statistical dependence between $\mathbf{d}^{\alpha n}$ and $\widehat{\mathbf{\Omega}}^{\alpha n}$ are required to produce a flux. Some recasting of (37) will help us better understand the flux-producing relationship. By a well known identity involving the vector and dot product, $\mathbf{d}^{\alpha n} \times \left(\mathbf{k} \times \widehat{\mathbf{\Omega}}^{\alpha n}\right) = \mathbf{k}\left(\mathbf{d}^{\alpha n} \cdot \widehat{\mathbf{\Omega}}^{\alpha n}\right) - \widehat{\mathbf{\Omega}}^{\alpha n} (\mathbf{d}^{\alpha n} \cdot \mathbf{k})$. Applying $\mathbf{k}\times$ to both sides, and recognizing that $\mathbf{k} \times \mathbf{k} = 0$, yields $(\mathbf{k} \times \mathbf{d}^{\alpha n}) \times \left(\mathbf{k} \times \widehat{\mathbf{\Omega}}^{\alpha n}\right) = -\left(\mathbf{k} \times \widehat{\mathbf{\Omega}}^{\alpha n}\right)(\mathbf{d}^{\alpha n} \cdot \mathbf{k})$. Hence

$$(\mathbf{k} \cdot \mathbf{d}^{\alpha n}) \left(\mathbf{k} \times \widehat{\mathbf{\Omega}}^{\alpha n}\right) = -(\mathbf{k} \times \mathbf{d}^{\alpha n}) \times \left(\mathbf{k} \times \widehat{\mathbf{\Omega}}^{\alpha n}\right). \quad (38)$$

Furthermore, $-(\mathbf{k} \times \mathbf{d}^{\alpha n}) \times \left(\mathbf{k} \times \widehat{\mathbf{\Omega}}^{\alpha n}\right) = \widehat{\mathbf{\Omega}}^{\alpha n}[\mathbf{k} \cdot (\mathbf{d}^{\alpha n} \times \mathbf{k})] - \mathbf{k}\left[\mathbf{k} \cdot \left(\mathbf{d}^{\alpha n} \times \widehat{\mathbf{\Omega}}^{\alpha n}\right)\right]$. The first term on the right side of this equation can be shown to vanish. Finally, we have

$$\Phi_{Ti}^{\alpha}(\mathbf{k}) = (2\pi)^3 \varphi a_\alpha^6 \Delta T_\alpha \Omega_\alpha k_i \quad (39)$$
$$\left(\mathbf{k} \cdot \left\langle \widehat{\mathbf{\Omega}}^{\alpha n} \times \mathbf{d}^{\alpha n} \right\rangle\right) F(ka_\alpha) G(ka_a).$$

Hence, when $\left\langle \widehat{\mathbf{\Omega}}^{\alpha n} \times \mathbf{d}^{\alpha n} \right\rangle$ is non-zero, a flux may be generated. This can be understood from a mental picture in which the dipole is perpendicular to the rotational axis of the QW. In that case, the signs of the scalar field in the updraft and downdraft are opposite. The flux is thus finite and symmetric in planes passing through the rotational axis.

The correlation $\mathbf{u}_\alpha = \left\langle \widehat{\mathbf{\Omega}}^{\alpha n} \times \mathbf{d}^{\alpha n} \right\rangle$ can be expected to depend on the size class α. For larger eddies, which carry most of the flux, we would expect $u_\alpha = |\mathbf{u}_\alpha|$ to be comparatively large. At smaller scales, the turbulence tends toward isotropy, and therefore u_α becomes small. Let us suppose $\mathbf{u}_\alpha$ is described by a power-law relationship:

$$\mathbf{u}_\alpha = \mathbf{u}_1 \left(\frac{a_\alpha}{a_1}\right)^\nu. \quad (40)$$

Substituting this relationship into (39), applying (20) and (21), and summing over QW size classes, we have

$$\Phi_{Ti}(\mathbf{k}) = (2\pi)^3 \varphi c_\Omega c_T \chi_T^{1/2} \epsilon^{1/6} a_1^{-\nu} (\mathbf{k} \cdot \mathbf{u}_1) \quad (41)$$
$$k_i \sum_{\alpha=1}^{N} a_\alpha^{17/3+\nu} F(ka_\alpha) G(ka_a).$$

In the limit of small μ,

$$\Phi_{Ti}(\mathbf{k}) = \frac{(2\pi)^3 \varphi}{\mu} c_\Omega c_T \chi_T^{1/2} \epsilon^{1/6} (\mathbf{k} \cdot \mathbf{u}_1)(ka_1)^{-\nu} \quad (42)$$
$$k_i k^{-17/3} \int_{ka_N}^{ka_1} dy\, y^{14/3+\nu} F(y) G(y).$$

As before, in the inertial subrange, the integral is approximately constant. When (42) is integrated over the wavenumber space to determine the covariance, only terms consisting of even functions of the wavenumber components survive. Therefore, in the dot product $\mathbf{k} \cdot \mathbf{u}_1$, only $k_i \mathbf{e}_i \cdot \mathbf{u}_1$ contributes to the covariance.

5. INTERMITTENT TURBULENCE

Intermittency refers to the tendency of turbulence to occur in bursts of activity. More precisely, it can be defined as concentration of the TKE dissipation rate in certain regions of space. Up to this point, we have assumed that the eddies positions (the $\mathbf{b}^{\alpha n}$) were smoothly distributed in space. In intermittent turbulence, however, the eddies must occupy a progressively smaller space as α increases. To accomodate this behavior, we suppose that the space available to the eddies, the so-called *active region*, decreases according to a power law

$$V_\alpha = V \left(\frac{a_\alpha}{a_1}\right)^\lambda. \quad (43)$$

The packing fraction φ for the eddies within the active region V_α is assumed to be constant (independent of class); that is, the *eddies fill the same amount of space within the active region, regardless of their size*. The average of the packing fraction for size class α over all of space is

$$\varphi_\alpha = \frac{N_\alpha a_\alpha^3}{V} = \frac{N_\alpha a_\alpha^3}{V_\alpha} \left(\frac{a_\alpha}{a_1}\right)^\lambda = \varphi \left(\frac{a_\alpha}{a_1}\right)^\lambda. \quad (44)$$

Hence the space-averaged packing fraction decreases with increasing α.

Eq. (44) is a generalization of (3.2) in Frisch et al. (1976), where the authors consider the case $a_\alpha = 2^{-(\alpha-1)}a_1$ and $\varphi = 1$. They hypothesize that the packing fraction φ_α is given by an equation

$$\varphi_\alpha = \beta^{\alpha-1}, \tag{45}$$

where β is a model parameter. With $a_\alpha = 2^{-(\alpha-1)}a_1$, (44) becomes $\varphi_\alpha = 2^{-\lambda(\alpha-1)}$. Hence the parameter $\beta = 2^{-\lambda}$. From Eqs. (3.2) and (3.9) in Frisch et al, $\beta = 2^{D_f-3}$, where D_f is the fractal dimension. Therefore $\lambda = 3 - D_f$. For turbulence, the authors suggest $D_f \approx 2.5$.

The discussion so far in this section addresses only the size of the space the eddies occupy, but not where this space may be. Frisch et al. (1976) suggest that the space occupied by the $\alpha+1$ class is a subset entirely contained by the space occupied by the α class ($V_{\alpha+1} \subset V_\alpha$). Although Frisch et al. explicitly discuss the case where the characteristic size of one generation is half is predecessor, we may presume that the argument applies to smoother distributions of eddy sizes, as well. Such a construction implies that the positions of the eddies in the $\alpha+1$ class are not independent of the positions of the α class.

Fortunately, the assumption that the eddy positions $\mathbf{b}^{\alpha n}$ are statistically independent and distributed uniformly in a volume was never actually used in the previous derivation for non-intermittent turbulence. The derivation leading up to (18) is unchanged. What must be modified is the subsequent discussion of the packing fraction and rotation rate. Using the α-dependent packing fraction, (18) becomes

$$\begin{aligned} E(k) &= \frac{32\pi^4 k^4}{3} \sum_{\alpha=1}^{N} \varphi_\alpha \Omega_\alpha^2 a_\alpha^7 G^2(ka_\alpha) \\ &= \frac{32\pi^4 \varphi k^4 a_1^{-\lambda}}{3} \sum_{\alpha=1}^{N} \Omega_\alpha^2 a_\alpha^{7+\lambda} G^2(ka_\alpha). \end{aligned} \tag{46}$$

The specific kinetic energy for the size class α is

$$E_\alpha \sim \varphi_\alpha v_\alpha^2 \sim \varphi_\alpha \Omega_\alpha^2 a_\alpha^2. \tag{47}$$

The energy transfer within the active volume still takes place over a time $t_\alpha \sim \Omega_\alpha^{-1}$.Hence the rate of energy transfer for the size class α is

$$\epsilon_\alpha \sim E_\alpha / t_\alpha \sim \varphi_\alpha \Omega_\alpha^3 a_\alpha^2. \tag{48}$$

For conservation of energy, ϵ_α must again equal $\epsilon\varphi$. (The correct relationship is $\epsilon_\alpha = \epsilon\varphi$ rather than $\epsilon_\alpha = \epsilon\varphi_\alpha$, since the dissipation occurs by hypothesis within the active region. If the eddies are space-filling within the active region, then we would have $\epsilon_\alpha = \epsilon$.) Therefore, solving (48) for Ω_α, we have

$$\Omega_\alpha = c_\Omega^2 \left(\varphi_\alpha/\varphi\right)^{-1/3} a_\alpha^{-2/3} \epsilon^{1/3} = c_\Omega^2 a_1^{\lambda/3} a_\alpha^{-(2+\lambda)/3} \epsilon^{1/3}. \tag{49}$$

Eq. (49) implies that the eddies, in comparison to the non-intermittent formulation, the eddies must spin faster as the size decreases. Substituting (49) into (46) yields

$$E(k) = \frac{32\pi^4 \varphi c_\Omega^2 \epsilon^{2/3} k^4 a_1^{-\lambda/3}}{3} \sum_{\alpha=1}^{N} a_\alpha^{(17+\lambda)/3} G^2(ka_\alpha). \tag{50}$$

Substituting with (24) and taking the limit of a continuous distribution of size classes, we have

$$E(k) \simeq \frac{32\pi^4 \varphi c_\Omega^2 \epsilon^{2/3} k^{-5/3} \left(ka_1\right)^{-\lambda/3}}{3\mu} \tag{51}$$

$$\times \int_{ka_N}^{ka_1} dy\, y^{(14+\lambda)/3} G^2\left(y\right). \tag{52}$$

This is a generalization of (27) to intermittent turbulence. When $\lambda = 0$, (51) reduces to (27). In the inertial subrange, where the integral is approximately constant, it agrees with (3.8) in Frisch et al. (1976).

The method for constructing the fractal active volume described in preceding analysis leaves out a lot of practical details that are needed to synthesize a random field with QWs. For example, it assumed that the turbulent regions are either active or completely inactive. Since the QWs taper off gradually, completely inactive regions cannot be created. Also, Frisch et al. (1976) assume the size classes exactly fill the active region, which makes it unclear how to handle a region composed of a fractional number of eddies. Most importantly, the formulation in this paper has random QW positions; therefore, it is unclear how to position the QWs so they are contained within the active region.

To resolve some of the problems, we adopt here the following simplified procedure. To create fractal active regions, we generate the active volume by removing cubes of size a_α^3 at each stage until the desired V_α is obtained. The eddy centers are randomly positioned anywhere within this active region. As a result, much of the eddy may actually be outside of the active volume. Since this process does not correspond directly to the one described above, the relationship between λ and the fractal dimension D_f cannot be expected to be the same.

Figure 2 is an example realization of a QW wavelet velocity field including intermittency ($\lambda = 0.5$). Except for the intermittency, it was created in the same manner as Figure 1. The intermittent version has a considerably smoother appearance outside of the regions of intense turbulent activity.

6. CONCLUSION

Quasi-wavelets provide a method for constructing random fields from spatially compact functions that have a resemblence to actual turbulent eddies. Their spatial compactness is very well suited to the problem of constructing an intermittent turbulence field. In particular, we have shown in this paper that intermittent turbulence described by the well known beta model can be readily constructed. We have also shown that constant-flux layers,

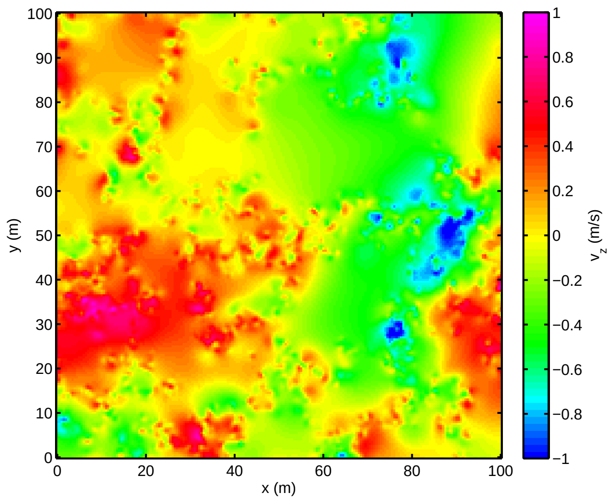

FIG. 2: Same as Figure 1, except that intermittency ($\lambda = 0.2$) has been included in the QW realization.

such as occur in the atmospheric surface layer, can be constructed from a scalar/velocity QW model in which the orientations are correlated.

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