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# MULTIPLE SCALING METHOD AND ITS APPLICATION TO THE CALCULATION OF RADIATIVE FLUXES

Tsuneaki Suzuki[1*], Teruyuki Nakajima[2], and Masayuki Tanaka[3]

[1]Frontier Research Center for Global Change, JAMSTEC, Japan
[2]Center for Climate System Research, The University of Tokyo, Japan
[3]Department of Environmental information Engineering, Tohoku Institute of Technology, Japan

## 1. INTRODUCTION

For the accurate evaluation of earth radiation budget and the precise simulation of climate by a general circulation model, rapid yet accurate radiation code is indispensable. The scaling of radiative transfer equation is one of methods realizing such an accurate radiation calculation. Van de Hulst and Grossman (1968) introduced the early scaling method called the similarity relation. This scaling method, however, was not mature and was applied only to the calculation of reflection for semi-infinite layer. Joseph et al. (1976) showed afterward that the scaling method using delta-function approximation improved the accuracy of radiative fluxes computed by the Eddington approximation. Wiscombe (1977) extended this scaling method to arbitrary-streams approximation, and this is called the delta-M method.

For the calculation of the radiative intensity, an excellent method has been developed, using both the scaling and the correction of single scattering radiation (Nakajima and Tanaka, 1988). This method, however, could not guarantee good accuracy in the calculation of radiative fluxes, because the energy conservation of radiative transfer equation was broken down in this approximation. It seems that more efficient approximations for the flux computation have not been proposed after the delta-M method. In this study we try to develop the new scaling method for calculating radiative fluxes.

* *Corresponding author address:* Tsuneaki Suzuki, Frontier Research Center for Global Change, JAMSTEC, 3173-25, Showa-machi, Kanazawa-ku, Yokohama, Kanagawa 236-0001, Japan; e-mail: tsuneaki@jamstec.go.jp

## 2. SCALING ALGORITHMS OF RADIATIVE TRANSFER EQUATION

### 2.1 *Multiple scaling method*

In this section, we formulate the multiple scaling method on the basis of the successive orders method (Tanaka, 1966; Lenoble, 1985). First, we consider the radiative transfer equation,

$$\mu \frac{dI(\tau,\mu)}{d\tau} = -I(\tau,\mu) + J(\tau,\mu)\,, \quad (1)$$

$$J(\tau,\mu) = \frac{\omega_0}{2}\int_{-1}^{1} P(\mu,\mu')I(\tau,\mu')\,d\mu'\,, \quad (2)$$

for each scattering order which is given by

$$\mu \frac{dI^{(0)}(\tau,\mu_0)}{d\tau} = -I^{(0)}(\tau,\mu_0)\,, \quad (3)$$

$$\mu \frac{dI^{(1)}(\tau,\mu)}{d\tau} = -I^{(1)}(\tau,\mu) + J^{(1)}(\tau,\mu)\,, \quad (4)$$

$$\vdots$$

$$\mu \frac{dI^{(n)}(\tau,\mu)}{d\tau} = -I^{(n)}(\tau,\mu) + J^{(n)}(\tau,\mu)\,, \quad (5)$$

where $I^{(n)}(\tau,\mu)$ is the azimuthally averaged intensity of the n-th order scattering at the optical depth $\tau$ and for the direction of cosine of the zenith angle $\mu$, and $J^{(n)}(\tau,\mu)$ is the source term of the n-th order scattering with the phase function $P(\mu, \mu')$:

$$J^{(1)}(\tau,\mu) = \frac{F_0 \omega_0}{4\pi} e^{-\tau/\mu_0} P(\mu,\mu_0), \quad (6)$$

$$J^{(n)}(\tau,\mu) = \frac{\omega_0}{2} \int_{-1}^{1} P(\mu,\mu') I^{(n-1)}(\tau,\mu') d\mu'. \quad (7)$$

Here, $\mu_0$ is the cosine of the solar zenith angle, $\omega_0$ is the single scattering albedo and $F_0$ is the solar irradiance at the top of atmosphere.

Now, we derive the formulae of the multiple scaling method. Assuming that the truncation fraction of the n-th order scattering is given by $f^{(n)}$ independently, we introduce the approximate phase function of $P^{(n)}(\mu,\mu')$ instead of $P(\mu,\mu')$ by using the delta-function:

$$P^{(n)}(\mu,\mu') = 2 f^{(n)} \delta(\mu - \mu') + (1 - f^{(n)}) \wp^{(n)}(\mu,\mu'), \quad (8)$$

$$\wp^{(n)}(\mu,\mu') = \sum_{m=0}^{2M-1} (2m+1) \chi_m^{(n)} P_m(\mu) P_m(\mu'), \quad (9)$$

$$\chi_m^{(n)} = \frac{\chi_m - f^{(n)}}{1 - f^{(n)}}, \quad (10)$$

where the coefficient of $\chi_m$ is the moment of phase function $P(\mu,\mu')$ with respect to the Legendre polynomial, and $P_m$ is the m-th order Legendre polynomial. The number of $M$ is the order of approximation.

After some transformations of equations, we obtain a set of radiative transfer equations scaled by the multiple scaling method,

$$\mu \frac{dI^{(0)}(\tau^{(0)},\mu_0)}{d\tau^{(0)}} = -I^{(0)}(\tau^{(0)},\mu_0), \quad (11)$$

$$\mu \frac{dI^{(1)}(\tau^{(1)},\mu)}{d\tau^{(1)}} = -I^{(1)}(\tau^{(1)},\mu) + \hat{J}^{(1)}(\tau^{(0)},\mu), \quad (12)$$

$$\vdots$$

$$\mu \frac{dI^{(n)}(\tau^{(n)},\mu)}{d\tau^{(n)}} = -I^{(n)}(\tau^{(n)},\mu) + \hat{J}^{(n)}(\tau^{(n-1)},\mu), \quad (13)$$

where we define the scaled quantities for each scattering order:

$$d\tau^{(n)} = (1 - \omega_0 f^{(n+1)}) d\tau, \quad (14)$$

$$\omega_0^{(n)} = \frac{(1 - f^{(n)})\omega_0}{1 - \omega_0 f^{(n+1)}}, \quad (15)$$

$$\hat{J}^{(n)}(\tau^{(n-1)},\mu) = \frac{\omega_0^{(n)}}{2} \int_{-1}^{1} \wp^{(n)}(\mu,\mu') I^{(n-1)}(\tau^{(n-1)},\mu') d\mu'. \quad (16)$$

## 2.2 *Double scaling method*

It is difficult to solve fast the radiative transfer equation in terms of the successive orders method except for the case of optically thin atmospheres. Then we transform Eqs. (12) and (13) to the ordinary radiative transfer equation with two truncation fractions. In Eqs. (12) and (13) we give truncation fractions as follows:

$$f^{(n)} = \begin{cases} f^{(1)} & (n = 1) \\ f^{(2)} & (n \geq 2) \end{cases}. \quad (17)$$

Next we sum up those equations for all scattering orders, and have the radiative transfer equation for the diffuse radiation as follows:

$$\mu \frac{dI(\tau^{(1)},\mu)}{d\tau^{(1)}} = -I(\tau^{(1)},\mu) + \frac{\omega_0^{(2)}}{2} \int_{-1}^{1} \wp^{(2)}(\mu,\mu') I(\tau^{(1)},\mu') d\mu' + \frac{F_0 \omega_0^{(1)}}{4\pi} \wp^{(1)}(\mu,\mu_0) e^{-\tau^{(0)}/\mu_0}, \quad (18)$$

where the scaled optical thickness and the single scattering albedo are given by

$$d\tau^{(0)} = (1 - \omega_0 f^{(1)}) d\tau, \quad (19)$$

$$d\tau^{(1)} = (1 - \omega_0 f^{(2)}) d\tau, \quad (20)$$

$$\omega_0^{(1)} = \frac{(1 - f^{(1)})\omega_0}{1 - \omega_0 f^{(2)}}, \quad (21)$$

$$\omega_0^{(2)} = \frac{(1 - f^{(2)})\omega_0}{1 - \omega_0 f^{(2)}}, \quad (22)$$

and $I$ is the intensity of the diffuse radiation summed up for all scattering orders.

This is the formula of the double scaling method which has the two truncation fractions; one scaling factor is a truncation fraction for the phase function of the integral term and the other is that for the phase function of the internal source term.

# 3. APPLICATION OF DOUBLE SCALING METHOD TO DISCRETE ORDINATES METHOD

## 3.1 *Determination of scaling factors*

In section 2.2, we derived the double-scaled radiative transfer equation. To solve this equation, however, a problem remains; we should determine the appropriate values of two truncation fractions. The first truncation fraction $f^{(1)}$ is associated with the internal source term which consists of the single scattering radiation of incident solar irradiance. The second $f^{(2)}$ is associated with the integral source term which consists of the radiation with higher order scatterings.

First, we consider $f^{(1)}$. When single scattering radiation prevails, it is enough to determine the ratio of the number of the forward scattered photons to that of the backward scattered photons, in order to calculate radiative fluxes accurately in atmospheric layers.

Here, we introduce the backscattered fraction (Wiscombe and Grams 1976)

$$\beta(\mu_0) = \frac{1}{2}\int_0^1 P(-\mu, \mu_0)\, d\mu. \qquad (23)$$

We define $\beta(\mu_0)$ as the backscattered fraction of the true phase function and define $\beta^*(\mu_0, f)$ as that of the phase function approximated by the delta-function

$$P(\mu, \mu') = 2f\delta(\mu - \mu') + (1 - f)P^*(\mu, \mu'), \qquad (24)$$

$$P^*(\mu, \mu') = \sum_{n=0}^{2M-1} (2n+1)\omega_n^* P_n(\mu) P_n(\mu'), \qquad (25)$$

$$\omega_n^* = \frac{\omega_n - f}{1 - f}, \qquad (26)$$

where $M$ is the number of streams in the hemisphere in terms of the discrete ordinates method. The backscattered fraction is the ratio of the backward scattering photons to all photons. Then the condition

$$min\left[\Delta\beta(f)\right], \qquad (27)$$

where

$$\Delta\beta(f) = \left|\beta(\mu_0) - \beta^*(\mu_0, f)\right|, \qquad (28)$$

is plausible for determining $f^{(1)}$. The condition of Eq. (27) has no meaning practically, when the $\Delta\beta$ is not sensitive to truncation fraction $f$, i.e., $\Delta\beta_{\max}$ is almost the same as $\Delta\beta_{min}$; $\Delta\beta_{\max}$ and $\Delta\beta_{min}$ are the maximum and minimum values of $\Delta\beta(f)$, respectively. In this case, according to the delta-M method, we give the truncation fraction $f^{(1)}$ by the moment of phase function with respect to Legendre polynomials, $\omega_{2M}$. From above consideration, the truncation fraction $f^{(1)}$ is given by

$$\begin{cases} f^{(1)}\ by\ Eq.(27) & \Delta\beta_{\max} - \Delta\beta_{min} \geq 0.05 \\ f^{(1)} = \omega_{2M} & \Delta\beta_{\max} - \Delta\beta_{min} < 0.05 \end{cases}. \qquad (29)$$

Secondly, we consider $f^{(2)}$. The scaling factor of $f^{(2)}$ is the truncation fraction for the phase function included in the integral source term, and the term describes the radiation fields of higher order scattering. When the radiation field consists of photons experiencing many scatterings, it is considered that the value of the truncation fraction at a certain scattering angle does not so much affect the accuracy of calculated radiative fluxes. Then we give the truncation fraction $f^{(2)}$ by $\omega_{2M}$ according to the delta-M method.

### 3.2 *Application to discrete ordinates method*

We solved the radiative transfer equation using the discrete ordinates method (Nakajima and Tanaka 1986), which were scaled by the double scaling method or the delta-M method. To calculate radiative fluxes, we adopted a shifted double Gaussian quadrature according to Nakajima and Tanaka (1988).

Figure 1a shows the transmittance of homogeneous layer with $\tau_0 = 1$ and the Henyey-Greenstein phase function of the asymmetry factor $g = 0.8$, and we present the absolute accuracy of radiative fluxes of the two-stream approximation using the delta-M method and the double scaling method in Figs. 1b and 1c, respectively. When we computed radiative fluxes by the delta-M method, large errors occurred at large solar zenith angles, especially, in optically thin regions where single scattering

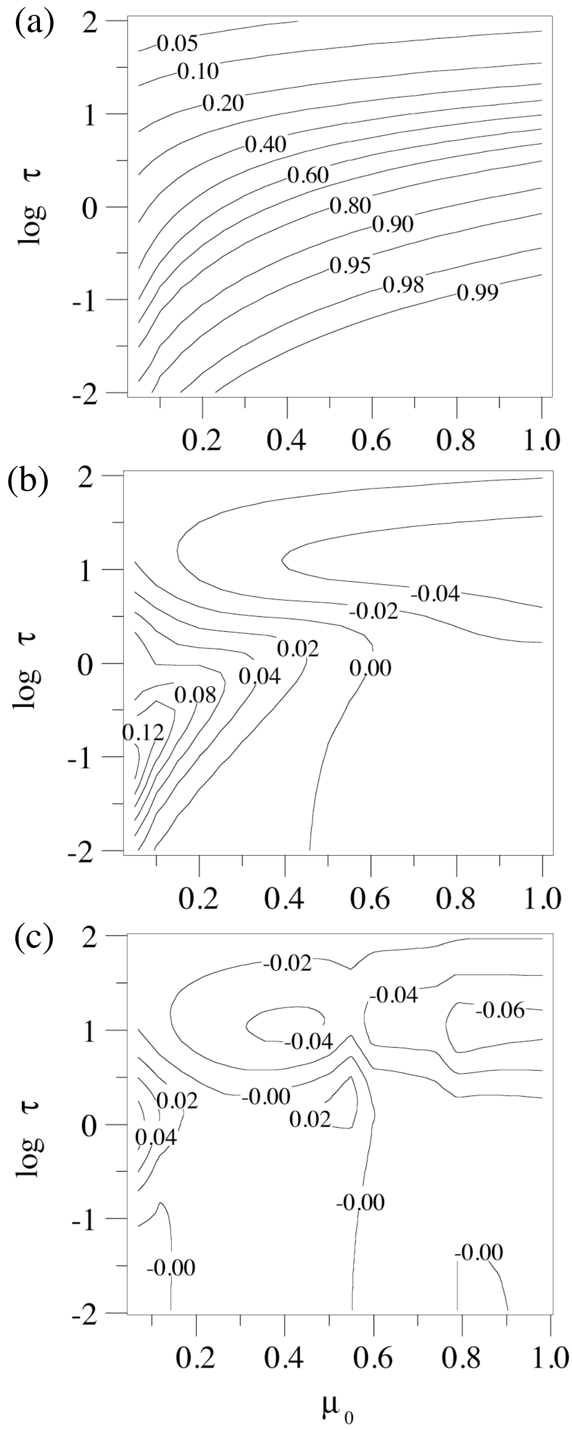


Figure 1. (a) Exact value of transmittance. Absolute accuracy of transmittance for (b) the delta-M method and (c) the double scaling method in two-stream approximation. These are shown as a function of logarithmic optical thickness and cosine of the solar zenith angle. We used the Henyey-Greenstein phase function of $g = 0.8$ and the single scattering albedo $\omega_0 = 1$.

radiation prevails. These errors decreased by using the double scaling method. The improvement by the double scaling method results mainly from the appropriateness of the truncation fraction $f^{(1)}$. The double scaling method improves the accuracy in the region where single scattering radiation prevails, almost keeping the accuracy in the region where multiple scattering radiation dominates. We also examined the effect of the double scaling method in the four-stream approximation. The improvement is more apparent than that in the two-stream approximation (not shown).

**4. SUMMARY AND CONCLUSIONS**

In this study, we presented new scaling formulae to calculate radiative fluxes. These new scaling methods were developed on the basis of the successive orders method. First, we showed the multiple scaling method which introduced the scaling factors for each scattering order independently. This scaling method, however, can be applied only to the radiative transfer equation divided for each scattering order, and this formula cannot be solved fast except for the condition of optically thin atmospheres. Then we derived the double scaling method which scales the ordinary radiative transfer equation using two scaling factors.

We proposed the way to determine the scaling factors $f^{(1)}$ and $f^{(2)}$, and applied the double scaling method to two-stream and four-stream approximations. In order to examine the effect of the double scaling method, we solved the radiative transfer equation by the discrete ordinates method using this scaling method and the delta-M method.

Comparing the result of the double scaling method with that of the delta-M method, we found that the double scaling method improved the accuracy of radiative fluxes at large solar zenith angles, especially, in optically thin regions, and that this scaling method kept the accuracy in the region where multiple scattering dominates.

In addition, the double scaling method can calculate radiative fluxes as fast as traditional two-steam approximations, once we prepare a table of truncation fraction $f^{(1)}$ for the phase function. As a conclusion, the double scaling method is useful to compute radiative fluxes in a general circulation model.

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**References**

Joseph, J.H., W.J. Wiscombe, and J.A. Weinman, 1976: The delta-Eddington approximation for radiative flux transfer. *J. Atmos. Sci.*, **33**, 2452-2459.

Lenoble, J., 1985: *Radiative transfer in scattering and absorbing atmospheres: Standard computational procedures*, A. Deepak Publishing, 300 pp.

Nakajima, T., and M. Tanaka, 1986: Matrix formulations for the transfer of solar radiation in a plane-parallel scattering atmosphere. *J. Quant. Spectrosc. Radiat. Transfer*, **35**, 13-21.

Nakajima, T., and M. Tanaka, 1988: Algorithms for radiative intensity calculations in moderately thick atmospheres using a truncation approximation. *J. Quant. Spectrosc. Radiat. Transfer*, **40**, 51-69.

Tanaka, M., 1966: On a method of solving an equation of radiative transfer. *Sci. Rep. Tohoku Univ., Ser. 5, Geophys.*, **17**, 125-145.

van de Hulst, H.C., and K. Grossman, 1968: Multiple light scattering in planetary atmospheres. *The Atmospheres of Venus and Mars*, J.C. Brant and M.B. McElroy, Eds., Gordon and Breach, 35-55.

Wiscombe, W.J., 1977: The delta-M method: Rapid yet accurate radiative flux calculations for strongly asymmetric phase functions. *J. Atmos. Sci.*, **34**, 1408-1422.

Wiscombe, W.J., and G.W. Grams, 1976: The backscattered fraction in two-stream approximations. *J. Atmos. Sci.*, **33**, 2440-2451.