P1.56 DIFFUSION GROWTH OF SOLID AND HOLLOW HEXAGONAL ICE COLUMNS

Chiou-Jiu Chen and Pao K. Wang*
University of Wisconsin-Madison, Madison, Wisconsin

1. INTRODUCTION

Hexagonal ice columns are one of the main ice crystal types in atmospheric clouds and they can be solid or hollow. It has been found that a large percentage of ice columns found in cirrus clouds is the hollow type. The diffusion growth rate of hollow ice columns has not been studied rigorously and compared with that of the solid columns. This study examines the diffusion growth of stationary hexagonal ice columns under a specified environmental condition. Two sets of solutions are shown. One is the simplified analytical solutions, and the other is numerical, which allows us to do more flexible tests. In both experiment sets, different degrees of hollowness are considered so as to understand its impacts on the growth. Some relevant variables, such as cross sectional area and equivalent depth, will also be discussed.

2. METHODS

2.1 Analytical Method

The mass growth rate of a stationary ice crystal by diffusion process is described by (Pruppacher and Klett, 1997):

$$\frac{dm}{dt} = 4\pi\, D_v\, C\,(\rho_\infty - \rho_a), \tag{1}$$

where m is the ice mass, D_v the diffusivity of water vapor in air, and ρ_∞ and ρ_a the water vapor densities in the environment and over the ice crystal surface respectively. C is the ice capacitance which is a function of ice size and habit. In this study, the ice capacitances for solid and hollow hexagonal columns developed by Chiruta and Wang (2005) were adopted. Based on the electrostatic analogy, they proposed that the ice capacitance of solid and hollow hexagonal columns could be represented by

$$C = a_1 a + a_2 c\,. \tag{2}$$

a_1 and a_2 are two constants, 0.751 and 0.491 respectively. The capacitance depends on a and c and is independent of the degree of hollowness. The geometrical parameters describing a hexagonal-columnar crystal are illustrated in figure 1. Recent laboratory experiments of Bailey and Hallett (2004) for ice crystal growth in -20 to -70C range suggested that the electrostatic analogy may overestimate the growth rates due to the none-evenly distributed water vapor pressure over the ice crystal surface. The details are not yet known. Since the present study focuses on the relative growth rates, the conclusions are most likely little affected by the absolute growth rates.

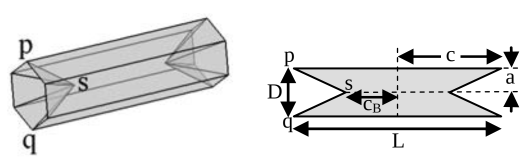

Figure 1. The scheme for parameters. L is the length, D the diameter, and c and a are half length and radius respectively. c_B is the length from the center to the tip of the cavity.

The volumes of solid and hollow hexagonal columns can be expressed as

$$V_s = 3\sqrt{3}a^2 c \tag{3a}$$

$$V_h = 2\sqrt{3}a^2 c + \sqrt{3}a^2 c_B\,. \tag{3b}$$

V is for volume and the subscripts s and h denote solid and hollow.

If the ice density is assumed to be constant, then (1) can be rewritten as

$$\frac{dV}{dt} = \frac{4\pi\, D_v\, C\,(\rho_\infty - \rho_a)}{\rho_i}, \tag{4}$$

where ρ_i is the density of ice. Taking the time derivatives of Eq. 3(a) and 3(b) and assuming that 1) growths are along c-axis only for both solid and hollow ice columns, and 2) c_B is a constant, we also get the time derivative of volume as

$$\frac{dV}{dt} = \alpha\, a_o^2 \frac{dc}{dt}\,. \tag{5}$$

For solid case, $\alpha = \alpha_s = 3\sqrt{3}$; for hollow case, $\alpha = \alpha_h = 2\sqrt{3}$.

By equating the Eq. (4) and (5), we obtain

$$\frac{dc}{dt} = \frac{1}{\alpha\, a_o^2}\frac{4\pi\, D_v\,(\rho_\infty - \rho_a)}{\rho_i}C = \Lambda' C\,. \tag{6}$$

Note that ρ_a can be obtained from the ideal gas law for water vapor:

$$\rho_{a(T_a)} = \frac{e_{sat(a,T_a)}}{R_v T_a}\,. \tag{7}$$

where T_a is the surface temperature, R_v the gas constant for water vapor and $e_{sat\,(a,Ta)}$ the saturation water vapor pressure over the surface. Because steady state is considered here, the surface temperature is a constant. If we further neglect the curvature and solute effects on the saturated vapor pressure term, then ρ_a is a constant too. Thus, Eq.

* *Corresponding author address:* Pao K. Wang, Univ. of Wisconsin-Madison, Dept. of Atmospheric and Oceanic Sciences, Madison, WI 53706; e-mail: pao@windy.aos.wisc.edu .

(6) shows that for a fixed environmental condition, the length growth rate is proportional to the ice capacitance only with Λ' the proportionality constant.

Eq. (2) says that the ice capacitance depends on the length. To solve Eq. (6), the changing variable was used. Let the new variable $c^* = a_o a_1 + a_2 c$, which is also equal to the ice capacitance. Take the time derivative of c^* and plug in Eq. (6), we get

$$\frac{d\ln(c^*)}{dt} = a_2\Lambda' = \Lambda \cdot \tag{7}$$

The solution of Eq. (7) is

$$c^*_{(t)} = c^*_o \exp(\Lambda t). \tag{8}$$

Change variable c^* back to c, the final solution is

$$c_{(t)} = \frac{(a_1 a_o + a_2 c_o)\exp(\Lambda t) - a_o a_1}{a_2}, \tag{9}$$

where the constant Λ is

$$\Lambda = \frac{a_2}{\alpha\ a_o^2}\frac{4\pi\ D_v\ (\rho_\infty - \rho_a)}{\rho_i}.$$

Note that the constant Λ includes the parameter α, which could be $3\sqrt{3}$ or $2\sqrt{3}$, depending on whether the column is solid or hollow.

Although the solution is compact, the water vapor density is not directly measured usually. In the following, we replace the water vapor density by temperature and water vapor pressure using the ideal gas law. Under the steady state condition, the surface temperature is determined by the energy balance between latent heat released and heat conducted out. Follow the same procedure as in Pruppacher and Klett (1997), we could find the relationship between the surface temperature T_a and the environmental temperature T_∞.

$$T_a = T_\infty(1+\delta), \tag{10}$$

where the δ is

$$\delta = \frac{L\rho_i\alpha\ a_o^2}{4\pi\ ka_2 T_\infty}\frac{d\ln c^*}{dt}.$$

Again ignoring the curvature and solute effects and the small high-order terms, we get the final solution.

$$c_{(t)} = \frac{(a_1 a_o + a_2 c_o)\exp(\gamma t) - a_o a_1}{a_2}, \tag{11}$$

where

$$\gamma = \frac{S_\infty - 1}{\frac{\alpha\ a_o^2 \rho_i}{4\pi\ D_v a_2}\frac{R_v T_\infty}{e_{sat(\infty,T_\infty)}} + \frac{\alpha\ L\rho_i a^2}{4\pi\ k\ a_2 T_\infty}\left(\frac{L_s}{R_v T_\infty} - 1\right)}.$$

The solution is exactly like Eq. (9) except the argument in the exponential function. S_∞ is the saturation ratio, L_s the latent heat of sublimation, and k the thermal conductivity for air. All other variables have been defined previously. Note again that the α could be for solid and hollow ice column.

2.2 Numerical Method

In the above analytical derivations, we have assumed growth is along c-axis only and c_B is fixed. To get the growth rate with relaxed restrictions, we use numerical method by allowing growth along both c- and a-axes and a flexible c_B.

The steps are shown in the flowchart in figure 2. We decided to use empirical relations of length and diameter based on observations. For a given initial length, the corresponding diameter is determined by the specified length-diameter relationship (L-D). With the length, diameter and a desired degree of hollowness, the mass can be calculated by Eq. (3) multiplied by ice density. The ice capacitance is determined by Eq. (2). In order to compute the mass growth rate, we also need to determine the surface temperature. A simple iteration method is used to solve the implicit equation of the mass growth rate and surface temperature at the same time by keeping the energy balance between the released latent heat and heat conduction. We add the mass increase to the original mass and distribute the new mass into new length and diameter by the same L-D relationship, then proceed to the next time loop.

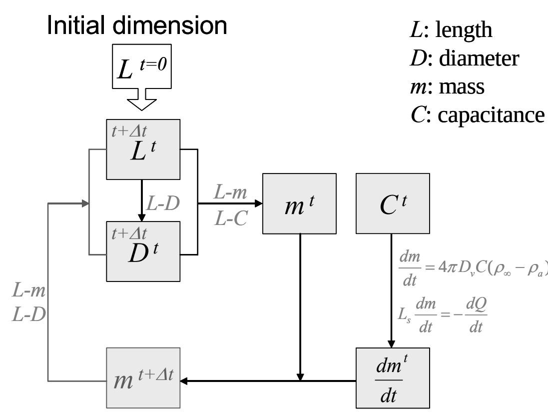

Figure 2: The flowchart of numerical computation of length growth.

3. RESULTS

In this section, the length growth and the changes of the corresponding variables will be shown. The environmental condition is kept fixed; that is, p = 200 mb, T = -50°C and RHi = 110% (RHi is the relative humidity with respect to ice). The initial length of the ice crystal is 10 μm, and the corresponding diameter which follows the power law relationship (see Pruppacher and Klett, 1997),

$$D = 0.26\ L^{0.927} \tag{12}$$

is 4.305 μm.

3.1 The Analytical Solutions

The analytical solutions are shown in figure 3. Figure 3(a) is the length growth versus time for both solid and hollow ice columns. The length grows exponentially with time for both cases; however, the hollow one grows faster than the solid one. This is because that the argument γ in the exponential function is larger for hollow ice column due to the smaller α than the solid ice column, The initial condition of the two cases are both a $10\ \mu m$ long solid ice column, so the initial mass increases are the same due to the same ice capacitance from the same length and diameter. After the increased masses are redistributed with diameters fixed, however, the length of the hollow ice column becomes longer than the solid one. Upon further growth, the capacitance of the hollow column becomes larger than the solid column due to the longer length, and hence its mass increases even faster. Redistributing the larger mass increase makes the hollow column even longer. Such a positive feedback makes the hollow ice column growing faster than the solid column. Figure 1(b) shows the ratio of length of hollow ice column to solid ice column. The ratio increases exponentially with time. In the first minute, the length of hollow ice column grows to 8 times longer than the solid one. For a larger initial length of 30μm (not shown), the ratio exceeds 8 in just 8 minutes.

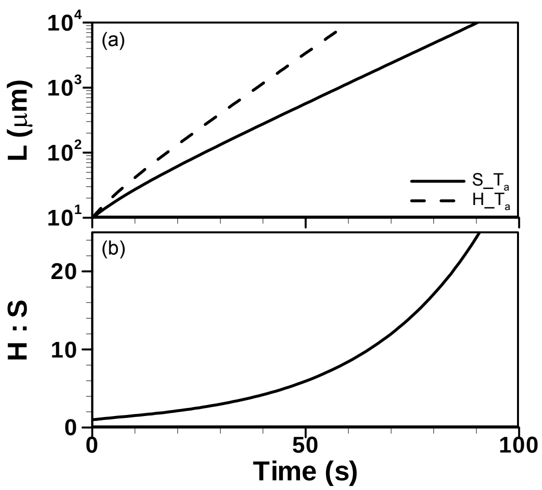

Figure 3. (a) The length of solid (solid) and hollow (dashed) ice columns as function of time. (b) The ratio of length of hollow ice to that of solid ice.

Figure 4 shows the variations of other geometrical variables corresponding to the length growth in figure 3. The mass of both solid and hollow ice columns increase exponentially (figure 4(a)). The cross sectional area, calculated by length multiplying diameter, increases exponentially too (figure 4(b)). The equivalent depth in figure 4(c) is defined as the volume divided by the cross sectional area. For solid and hollow ice columns, the equivalent depths are

$$D_{eq_s} = \frac{3\sqrt{3}}{4}a \tag{13a}$$

$$D_{eq_h} = \frac{2\sqrt{3}}{4}a + \frac{\sqrt{3}}{4}a\frac{c_B}{c}. \tag{13b}$$

Because we assume that the diameter is fixed, from

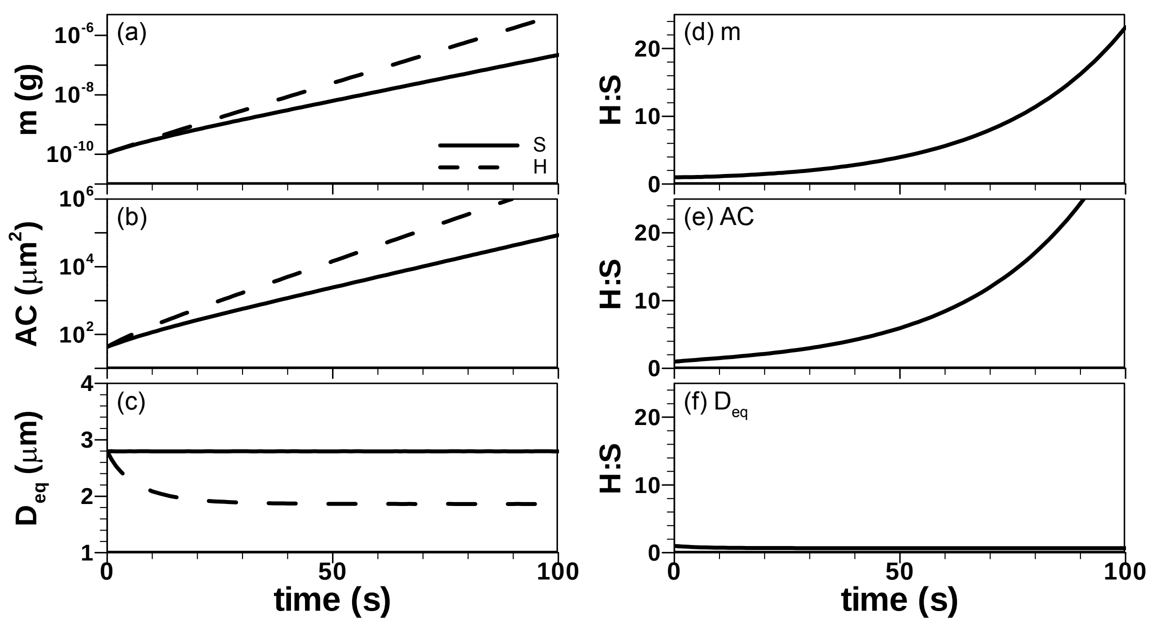

Figure 4. The diagnosed (a) mass, (b) cross section area, and (c) equivalent depth corresponding to figure 3. The solid lines are for solid ice column, and the dashed lines are for hollow ice column. (d) The solid line is the ratio of mass of hollow ice column to that of solid ice column. (e) and (f) are similar as (d), but for cross section area and equivalent depth respectively.

Eq. (13a) the equivalent depth for solid ice column should be a constant. For the hollow ice column, while the length grows with c_B fixed, the ratio of c_B/c in the second term of Eq. (13b) is becoming smaller and smaller as c increases. The equivalent depth for hollow ice column then will be a constant too, while the radius a is fixed. The ratio of the two constants, $D_{eq\,h}$ to $D_{eq\,s}$, is 2/3 as shown in figure 4(f). Contrast to the constant ratio of equivalent depth, the ratios of mass (figure 4(d)) and cross sectional area (figure 4(e)) increase exponentially with time.

3.2 The Solutions for Power Law Type L-D Relation

In this section, we allow the ice crystal to grow in both c- and a-axes, and the dimensions of length and diameter follow the L-D relationship given by Eq. (12). Three degrees of hollowness are considered. The first is solid, c_B $/c$=1. The second is half-hollowed, c_B $/c$=0.5. The last one is sheath (Nakaya, 1954), total-hollowed so that c_B/c=0. The results are shown in figure 5.

The lengths grow rapidly in the beginning and then slow down (figure 5(a)). After about 1 hour, the lengths grow to about 200-300 μm and increase approximately linearly with time. Contrast to the former exponential growth behavior where the diameter is held constant, the length growth in this case is more complicated. The mass increase is redistributed in both length and diameter. It needs more mass to increase 1 μm in diameter than 1 μm in length. So, the increase in dimensions is smaller in this situation. But, the ice capacitance, a linear combination of length and diameter, weights the diameter more, and it could cause more mass increase under certain aspect ratios. Combining these effects, we obtain the linear growth behavior

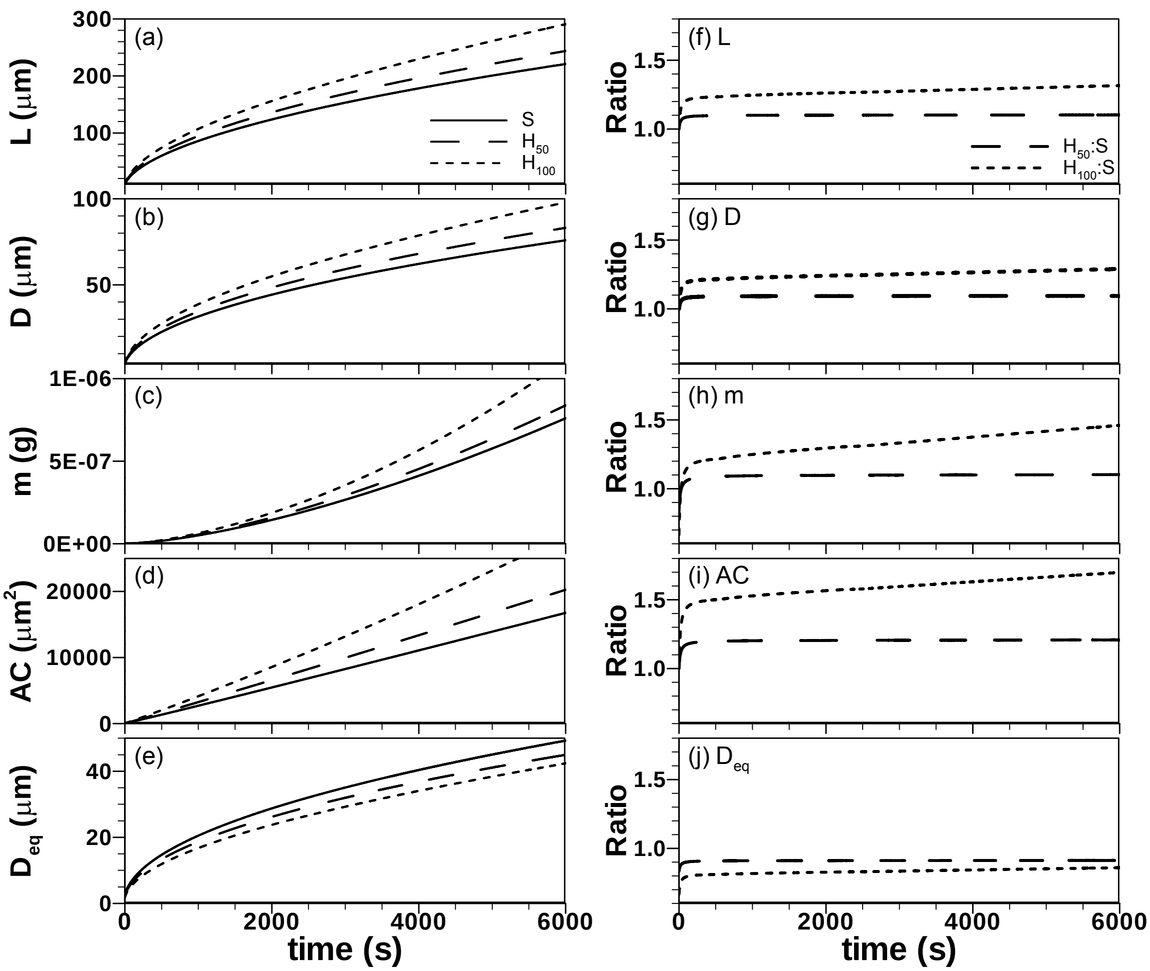

Figure 5: Temporal variation of (a) length , (b) The diameter, (c) mass, (d) cross sectional area, and (e) equivalent depth corresponding to (a). Solid curves are for solid ice column; dashed for half-hollowed ice column and dotted for total-hollowed ice column (sheath). Right panels are the ratio of half-hollowed to solid ice column (dashed) and the ratio of total-hollowed to solid ice column (dotted). (f)-(j) are corresponding to variables in left panels (a)-(e).

as shown. The slopes, which are the length growth rates, are different for the three kinds of ice columns. The slope of sheath is the steepest and the slopes of the half-hollowed and solid ice columns are similar and flatter. The ratios of the lengths of sheath and half-hollowed ice columns to the solid ice column are shown in figure 5(f). The length of half-hollowed ice column remains at about 1.1 times longer than the solid one. On the other hand, the ratio of sheath to solid ice column increases rapidly to 1.2 in first 100 seconds, then gradually increases to 1.3 in the next 1.5 hours. The diameters diagnosed from Eq. (12) are plotted in figure 5(b). The aspect ratio (defined as length divided by diameter) changes from 2.3 to 3 while length grows from 10 to 300 μm. Unlike the length growth, the mass growth rates increase with time (figure 5(c)), however, the ratios have the similar pattern as the ratios of lengths, that is, about constant ratio for half-hollowed to solid ice column and gradually increasing ratio for sheath to solid ice column (figure 5(h)). As to cross sectional area, the ratio increases with time for sheath, but remains approximately constant for half-hollowed and solid ice columns. The ratios for cross sectional area have similar pattern as the length and mass.

In this experiment set, the ice crystals start with the same dimension but different degrees of hollowness, so they start with the same cross sectional areas but different masses. It is interesting that the initially lightest sheath grows to become heavier than the solid in merely 30 seconds. The variations of the equivalent depth are shown in figure 5(e) and the ratios are shown in (j). The ratio of half-hollowed to solid is still about constant at ~0.91, but that for sheath to solid column is gradually increasing from 2/3 to 0.85.

4. CONCLUSIONS

In this study, we compared the length growth of solid and hollow columnar ice crystals in two experiment sets. In the simplified analytical solutions, the diameter and c_B are held fixed. The growth rate of length, mass and cross sectional area are exponential with time for both hollow and solid ice column. The hollow ice column grows faster than the solid one. The differences of length, mass, and cross sectional area between solid and hollow ice columns increase exponentially with time, too. As to the equivalent depth, after growing tens of seconds, the ratio of the equivalent depth of hollow to solid ice column remains constant such that the equivalent depth of hollow ice column is 2/3 of that of solid ice column. In the numerical experiments, the ice columns with three degrees of hollowness are allowed to grow in both a- and c-axes following a power-law type dimensional relationship. The results show that the length growth tends to be linear with time. The length growth rate of sheath is the largest, and the length growth rates of half-hallowed and solid ice columns are similar and smaller than the sheath. The ratios of the length, mass, and cross sectional area of half-hollowed to solid are becoming constant after growing tens of seconds, but that for sheath to solid increase gradually with time. The ratios of the equivalent depths have the similar tendency, but the sheath has the smallest equivalent depth and the solid ice column the largest. It is interesting to note that the ratios change extremely rapidly in just the first minute and then suddenly slow down. In the future, it would be interesting to investigate how sensitive the short wave and long wave radiations will respond to such growth behavior.

5. REFERENCES

Bailey, M. and J. Hallett, 2004: Growth Rates and Habits of Ice Crystals between −20° and −70°C. *J. Atmos. Sci.*, *61*, No. 5, pp. 514–544.

Chiruta, M. and P. K. Wang, 2005: The capacitance of solid and hollow hexagonal ice columns. *Geophys. Res. Lett.*, **32**, L05803.

Nakaya U., 1954: *Snow crystals, natural and artificial*. Cambridge, Harvard University Press, 510 pp.

Pruppacher, H. R. and J. D. Klett, 1996: *Microphysics of Clouds and Precipitation*. Springer, 976 pp.