12.3 A THEORETICAL STUDY OF WAVES FORCED BY ISOLATED TOPOGRAPHY

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Abstract

In this study, analytic and numerical methods are used to examine the nonlinear dynamics of gravity waves forced by an isolated mountain. The topographic gravity waves take the form of a packet, localized in the horizontal direction and comprising a continuous spectrum of horizontal wavenumbers centered at zero. The wave packet propagates upwards in a density-stratified shear flow and reaches a critical level, where the horizontal mean wind is the same as the wave phase speed. The governing nonlinear time-dependent equations are solved numerically to study the nonlinear interactions that take place between the wave packet and the mean wind. These interactions lead to the transfer of momentum to the mean flow by the packet and the development of static instabilities in the vicinity of the critical level. To obtain further insight into the results of the numerical simulations, analytic solutions of the governing linearized equations are derived.

1. INTRODUCTION

Density-stratified air flow over mountains may give rise to internal gravity waves which affect the general circulation of the atmosphere though momentum and energy transport and deposition. Such waves are referred to as topographic or mountain waves or as lee waves, since they generally occur downstream of the source.

Theoretical studies of gravity wave propagation in stratified shear flows have helped advance our understanding of the mechanisms for topographic wave generation and propagation and gravity-wave–mean-flow interactions. By means of simplifications such as the hydrostatic approximation and the Boussinesq approximation the governing equations can be simplified to an extent that they become tractable to analytic or relatively inexpensive numerical solutions and, thus, give some insight into the results obtained using more sophisticated models. A well-known example is the model of Long (1953) which assumes the fluid flow to be two-dimensional, steady and hydrostatic and, under further assumptions, leads to a single linear Helmholtz equation for the flow field, which can be readily solved. Analytic solutions of nonlinear time-dependent non-hydrostatic models of topographic waves are harder to derive, but there have been numerous numerical simulations at varying levels of approximation [see, for example, Baines (1989) or Wurtele, Sharman and Datta (1996) for reviews].

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An important feature of gravity wave propagation in the atmosphere is the *critical-level interaction*. Such interactions take place when gravity waves reach a level in the atmosphere where their intrinsic phase speed is zero, i.e., where the background wind speed equals the wave phase speed, and they act as a mechanism for momentum and energy deposition, wave breaking and, in some cases, the onset of turbulence. The literature on forced gravity wave critical levels begins with Booker and Bretherton's (1967) pioneering study of the linear problem. Later, a series of analytic studies was undertaken by Brown and Stewartson (1982) extending Booker and Bretherton's solution to include nonlinearities.

There have been studies of topographic waves forced by flow over an isolated mountain that have included investigations of critical-level phenomena. An importance issue in that context is the concept of *resonance* as defined in the papers of Clark and Peltier (1984) and Bacmeister and Pierrehumbert (1988). In these studies, the flow was said to be resonant for a certain discrete set of distances between the source and the critical level. In this resonant state, Clark and Peltier (1984) predicted that there would be a large drag force on the mountain. Bacmeister and Pierrehumbert (1988) noted that the exact locations of the resonant levels actually depend on the height of the mountain.

More recently, Campbell and Maslowe (2003) carried out a nonlinear numerical simulation of the evolution of a forced gravity wave packet and found similarities between their results and those of Bacmeister and Pierrehumbert (1988), namely, a prolonged state of absorption of the wave packet by the mean flow and an observed outward outward flux of wave activity in the horizontal direction. However, they did not investigate the possibility of resonance. The goal of this project is to extend that study to the specific case of topographic waves forced by an isolated mountain using analytic as well as numerical methods, and exploring the possibility of resonance, among other issues. Some preliminary results are presented here. Linear analytic solutions are described in Section 3 and linear and nonlinear numerical solutions are presented in Section 4.

2. MODEL FORMULATION

The governing equations for this study are the equations of motion for a two-dimensional fluid defined in terms of Cartesian coordinates x and z in the horizontal and vertical directions, respectively. The Boussinesq approximation is made, i.e., density variations with altitude are neglected in the acceleration terms, but retained only in the buoyancy force term in the vertical momentum equation. With this approximation the continuity equation is simply

$$u_x + w_z = 0, \tag{1}$$

where u and w are the horizontal and vertical components of the velocity and the subscripts denote partial differentiation. This allows us to define a streamfunction by

$$u = -\Psi_z, \;\; w = \Psi_x, \tag{2}$$

so that the horizontal and vertical momentum equations can be combined into a single equation for the streamfunction. Throughout this study we shall work with nondimensional variables and parameters. The various quantities are made nondimensional on the basis of typical length scales L and H in the horizontal and vertical directions respectively, U a typical velocity scale, and φ the dimensional amplitude of the waves at the source level. The scale L is related to the width of the mountain, while H corresponds to the dimensional vertical wavelength of the waves.

The function $\Psi(x, z, t)$ is the total streamfunction; it is decomposed into a contribution from the steady basic flow and a time-dependent perturbation in the form of a gravity wave packet:

$$\Psi(x, z, t) = \bar{\psi}(z) + \varepsilon\psi(x, z, t). \tag{3}$$

The basic flow is a shear flow depending on z and independent of x; it has velocity $(\bar{u}(z), 0)$, where $\bar{u}(z) = -\bar{\psi}_z$, and density $\bar{\rho}(z)$. It is assumed that $\psi \sim O(1)$, so the parameter ε gives a measure of the magnitude of the waves relative to that of the basic flow and, hence, defines the height of the mountain. The total density is also decomposed into a steady basic part $\bar{\rho}$ and a time-dependent perturbation $\varepsilon\rho(x, z, t)$.

The governing equations for the evolution of the perturbation are

$$\begin{aligned}\zeta_t + \bar{u}\zeta_x - \bar{u}^{''}\psi_x + g(\bar{\rho})^{-1}\rho_x + \varepsilon(\psi_x\zeta_z - \psi_z\zeta_x)&\\ -Re^{-1}\nabla^2\zeta + Re^{-1}\varepsilon^{-1}\bar{u}^{'''} = 0,&\end{aligned} \tag{4}$$

where

$$\zeta = \nabla^2\psi \tag{5}$$

is the perturbation vorticity, and

$$\begin{aligned}\rho_t + \bar{u}\rho_x + \bar{\rho}^{'}\psi_x + \varepsilon(\psi_x\rho_z - \psi_z\rho_x)&\\ -Re^{-1}Pr^{-1}\nabla^2\rho - Re^{-1}Pr^{-1}\varepsilon^{-1}\bar{\rho}^{''} = 0.&\end{aligned} \tag{6}$$

As before, the subscripts x and z denote partial differentiation, while the primes denote differentiation with respect to z. The Laplacian operator is nondimensional and takes the form

$$\nabla^2 = \delta\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2},$$

where $\delta = H^2/L^2$ is the square of the aspect ratio and gives a measure of the magnitude of the vertical scales to the horizontal. The limit $\delta \rightarrow 0$ is the "long-wave limit", or more precisely it corresponds to a wave packet of long horizontal extent. The limit of small δ is also a condition for the flow to be hydrostatic (see, for example, Baines, 1995). The constant g is the acceleration due to gravity and Re and Pr are the Reynolds number and Prandtl number respectively. The viscous and heat conduction terms are included in the nonlinear numerical simulations for the stabilizing effect that they have on the solutions; however, it is assumed that $Re \gg 1$, which is an appropriate assumption for geophysical flows. The Prandtl number Pr is set to 0.72, the value of the Prandtl number for air. The last term in each of equations (4) and (6) must be included because the mean density and velocity profiles used in our numerical simulations do not satisfy the equations with the viscous and heat conduction terms included; however, the inclusion of these terms is found to have a negligible effect on the qualitative behaviour of the numerical solutions.

3. LINEAR ANALYTIC SOLUTIONS

In this section, analytic solutions of the linearized equations are discussed. The goal of this project is to extend these solutions to take into account the nonlinear terms, but in this paper only linear solutions are presented. The linear analytic solutions help us to interpret and understand the numerical solutions which are described in Section 4. The linearized equations are obtained by setting ε to zero in equations (4)–(6) to zero and further omitting the viscous and heat conduction terms. The resulting equations can then be combined to give a single equation for the streamfunction of the wave packet:

$$\left[\frac{\partial}{\partial t}+\bar{u}\frac{\partial}{\partial x}\right]^2\nabla^2\psi-\bar{u}''(z)\left[\frac{\partial}{\partial t}+\bar{u}\frac{\partial}{\partial x}\right]\psi_x+N^2\psi_{xx}=0, \tag{7}$$

where N is the Brunt-Väisälä frequency or buoyancy frequency and is defined as:

$$N^2=-\frac{g}{\bar{\rho}}\frac{d\bar{\rho}}{dz}. \tag{8}$$

Equation (7) shall be used as a simple model for the time evolution of small-amplitude gravity waves forced by flow over an isolated mountain. The domain of our solution is taken to be the semi-infinite region $-\infty < x < \infty$, $0 < z < \infty$. The mountain is represented by a Gaussian function which is used as a lower boundary condition for the amplitude of the perturbation streamfunction, i.e.,

$$\psi(x,0,t)=e^{-\mu^2x^2}. \tag{9}$$

The parameter μ is assumed to be small (relative to unity).

The configuration studied here is a special case of the more general configuration involving a mountain range with multiple peaks, a simple representation of which is given by the lower boundary condition

$$\psi(x,0,t)=\text{Re}\{e^{-\mu^2x^2}e^{ikx}\}. \tag{10}$$

The parameter μ is assumed to be sufficiently small that the horizontal extent of the mountain range is much greater than the distance between individual peaks and there are several peaks within the mountain range. The general problem described by equation (7) with the boundary condition (10) can be solved using the *method of multiple scaling*. In this method we define two horizontal scales: the “fast” scale defined by the variable x and a “slow” scale defined by the variable $X=\mu x$. The boundary condition can then be written as $e^{-X^2}e^{ikx}$ and we seek solutions of the form $\psi(x,X,z,t)=\phi(X,z,t)e^{ikx}$. When this expression is substituted into equation (7), an equation for ϕ is obtained where each x derivative in (7) is replaced by the linear operator $ik+\mu\frac{\partial}{\partial X}$.

In our problem where $k=0$, there is only a slow scale and no fast scale. The boundary condition is e^{-X^2} and we seek solutions of the form $\psi(X,z,t)=\phi(X,z,t)$, so each x derivative in (7) is replaced by $\mu\frac{\partial}{\partial X}$.

3.1 *Steady solution*

The simplest configuration to study analytically is that in which the solution is independent of time, i.e., the wave packet takes the form $\phi(X,z)$, and $\bar{u}$ is a linear function of z. In that case (7) becomes

$$\bar{u}^2\left(\delta\mu^2\frac{\partial^2}{\partial X^2}+\frac{\partial^2}{\partial z^2}\right)\phi_{XX}+N^2\phi_{XX}=0. \tag{11}$$

We can find a solution in even powers of the parameter μ:

$$\phi\sim\phi^{(0)}(X,y)+\delta\mu^2\phi^{(1)}(X,y)+\delta^2\mu^4\phi^{(1)}(X,y)+\dots \tag{12}$$

In the long wave limit ($\delta=0$), the solution is given just by the first term in the series. If δ is $O(1)$, the leading-order term in the series solution is a reasonable approximation for the solution, since μ is small. This term satisfies

$$\bar{u}^2\phi^{(0)}_{zzXX}+N^2\phi^{(0)}_{XX}=0. \tag{13}$$

In a situation where there is no vertical shear, i.e., where $\bar{u}$ is constant, we can solve for ϕ_{XX} and then integrate twice with respect to X to obtain a solution of the form

$$\phi^{(0)}\sim A(X)e^{-\frac{N}{\bar{u}}z}+B(X)e^{\frac{N}{\bar{u}}z}+\alpha(z)X+\beta(z). \tag{14}$$

The functions $A(X)$, $B(X)$, $\alpha(z)$ and $\beta(z)$ are easily found from the boundary conditions. We require that $\phi\to0$ as $X\to\pm\infty$ and, thus, $\alpha(z)$ and $\beta(z)$ must both be zero. The boundary condition at $z=0$ is (9) and the upper boundary condition is that ϕ be finite as $z\to\infty$, which means the function $B(X)$ must be zero. Thus, the solution is

$$\phi^{(0)}\sim e^{-X^2}e^{-\frac{N}{\bar{u}}z}. \tag{15}$$

So with increasing altitude the wave packet amplitude decays and, to leading order in the parameter μ, the basic shape of the packet is unchanged from its shape at the source level. Having found $\phi^{(0)}$, the subsequent terms in the series (12) can be obtained successively. These terms describe the modifications to the shape of the packet as it propagates upwards.

In a situation where there is wind shear, i.e., where $\bar{u}$ depends on altitude z, the behaviour of the solution depends on whether there are any levels where the wind changes direction from easterly to westerly or vice versa, i.e., levels where $\bar{u}=0$. Such levels are critical levels, since the wave packet phase speed is zero, and they correspond to values of z for which the equation (13) is singular.

To understand what happens at a critical level, let us consider the simplest horizontal wind profile for which one can occur: the linear profile $\bar{u}(z)=a(z-z_c)$, where z_c is the altitude at which the critical level occurs and $a=\bar{u}'(z)$ is a constant. In that case, the solution of equation (13) can be shown to be

$$\begin{aligned}\phi^{(0)}\sim{}& A(X)(z-z_c)^{\frac{1}{2}+i\sqrt{\frac{N^2}{\bar{u}'^2}-\frac{1}{4}}}\\ &+B(X)(z-z_c)^{\frac{1}{2}-i\sqrt{\frac{N^2}{\bar{u}'^2}-\frac{1}{4}}},\end{aligned} \tag{16}$$

after applying the condition that $\phi \to 0$ as $X \to \pm\infty$. The upper boundary condition in this case is a radiation condition which specifies that there be no incoming wave energy from infinity and that the solution be finite as $z \to \infty$. By using group velocity arguments Booker and Bretherton (1967) showed that of the two solutions in (16) the one with the plus sign corresponds to an upward-propagating wave and the one with the negative sign to a downward-propagating wave. The radiation condition thus tells us that we must choose the one with the plus sign and, applying the condition at the source level, the solution is found to be

$$\phi^{(0)} \sim e^{-X^2} \frac{(z-z_c)^{\frac{1}{2}+i\sqrt{\frac{N^2}{\bar{u}'^2}-\frac{1}{4}}}}{(-z_c)^{\frac{1}{2}+i\sqrt{\frac{N^2}{\bar{u}'^2}-\frac{1}{4}}}}. \qquad (17)$$

The ratio $N^2/\bar{u}'^2$ is the Richardson number of the background flow (denoted as Ri), which is constant in this configuration, and gives a measure of the relative importance of the stratification to the shear. For atmospheric flows the Richardson number is large, in fact much larger than $1/4$, and so the square root $\gamma = \sqrt{\frac{N^2}{\bar{u}'^2} - \frac{1}{4}}$ is real. As is well-known from Booker and Bretherton's study, the z-dependent part of the solution (17) can be written as

$$(z-z_c)^{\frac{1}{2}+i\gamma} = \begin{cases} (z-z_c)^{\frac{1}{2}} e^{i\gamma \log(z-z_c)}, & z > z_c \\ (z-z_c)^{\frac{1}{2}} e^{i\gamma \log|z-z_c|} e^{\gamma\pi}, & z < z_c \end{cases} \qquad (18)$$

which shows that it is reduced across the critical level by a factor of $e^{-\gamma\pi}$. Since γ is large, this means that the wave packet amplitude is essentially reduced to zero above the critical level. The expressions in (18) also show that the vertical wavelength of the wave packet becomes increasing short as the packet approaches the critical level from below.

In the more general case where $\bar{u}$ is not a linear function of z, the solution of equation (13) can be obtained as a series in powers of $(z - z_c)$, where z_c is the altitude at which the critical level occurs. The first term in the series is the solution given in (17), so near the critical level where $(z - z_c) \ll 1$, the solution can be approximated by (17). The qualitative behaviour of the solution near the critical level is the same as that for the case represented by (17); the vertical wavelength decreases as the packet approaches the critical level from below, there is a phase change of $-\pi$ across the critical level, and the amplitude of the wave packet decreases by the same exponential factor.

The decrease in the wave packet amplitude at the critical level means that there is a corresponding decrease in the vertical flux of the horizontal wave momentum, which would be independent of height in the absence of a critical level (Eliassen and Palm, 1961). This means that there is a transfer of momentum to the mean flow from the wave packet, i.e., the packet is absorbed by the mean flow. Of course, to study this momentum transfer we need to use a nonlinear model (such as that described in Section 4.2).

3.2 *Time-dependent solution*

A more realistic representation of the solution is obtained by allowing the solution to vary with time. Our on-going investigation deals with the solution of the linear time-dependent problem defined by equation (7) with the boundary condition (9). Preliminary results show that in addition to the fast time variable t there is a slow time variable T which controls the long-time evolution of the wave packet. The appropriate definition of the slow time variable to found to be $T = \mu t$, where μ is the parameter that determines the horizontal extent of the wave packet (the width of the mountain). The linear time-dependent solutions derived in this manner can be used as the starting point of a nonlinear analysis, which is the eventual goal of this project.

4. NUMERICAL SIMULATIONS

4.1 *The numerical model*

In the numerical simulations we solve the governing time-dependent nonlinear equations (4)–(6) The numerical methods used for the simulations are based on those of Campbell and Maslowe (2003). The z derivatives are approximated using finite differences. In the x direction we take a Fourier transform of the equations, solve the transformed equations numerically and then invert the transform. In the nonlinear simulations a pseudo-spectral approximation is needed for transforming the nonlinear terms.

The numerical solutions are carried out on the rectangular domain $-50 < x < 50$, $0 < z < 10$. A lower boundary condition of the form (9) is applied at $z = 0$ to represent the effect of an isolated mountain. The lower boundary condition is kept fixed and the evolution of the wave packet is calculated starting from a zero initial condition. The Fourier transform of the lower boundary condition (9) is

$$\hat{\psi}(\kappa, z=0, t) = \mathcal{F}\{\psi(x, z=0, t)\} = \frac{\sqrt{\pi}}{\mu} e^{-\kappa^2/4\mu^2}. \qquad (19)$$

This is a Gaussian function, centered at the wavenumber (Fourier coefficient) $\kappa = 0$ and its width in Fourier space is determined by the parameter μ; for small μ it is high and narrow, for large μ it is low and wide. With this boundary condition, the Fourier spectrum of the solution is centered at the zero wavenumber. In the nonlinear simulations there are interactions between the various wavenumber components of the wavepacket and these give rise to a wave-induced mean flow as we shall see in Section 4.3.

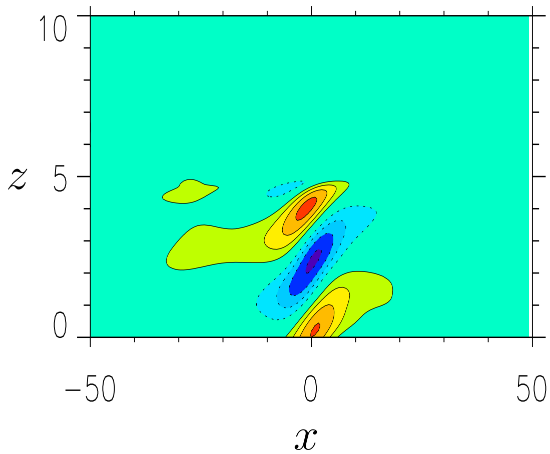


Figure 1: Linear simulation of a gravity wave packet forced by isolated topography: Perturbation streamfunction $\psi(x, z, t)$ at nondimensional time $t = 100$. The contour levels range from negative values of ψ (shown in blue) through zero (green) to positive values (red).

4.2 *Linear simulations*

Before proceeding to the nonlinear simulations, we first carry out a numerical solution of equations (4)–(6) with the nonlinear terms set to zero. The viscous and heat conduction terms are also set to zero in these linear computations.

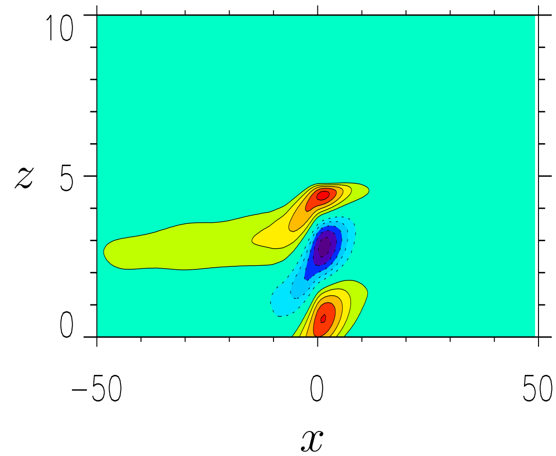
In the results shown here the nondimensional parameter μ is set to a value of 0.2, so that the horizontal extent of the mountain is within the interval $-10 < x < 10$. The background flow is set up so that the zero-wind line is at $z = 5$ nondimensional units. The mountain forces a wave packet that propagates up towards the critical level with an increasingly short vertical wavelength and is completely absorbed there, as predicted by the analytic solution. Figure 1 shows the streamlines (contours of the perturbation streamfunction ψ) at time $t = 100$ nondimensional units. By this time the solution has evolved in an almost steady state; the wave amplitude at a fixed point in space consists of small-amplitude oscillations in time about a fixed value.

4.3 *Nonlinear simulations*

In this section we describe the results of the numerical solution of the nonlinear equations (4)–(6). Figure 2(a) shows the perturbation streamfunction ψ at $t = 200$. There is still no transmission of wave activity above the critical level; the wave packet is completely absorbed by the mean flow. Momentum is deposited in the mean flow by the packet and the mean flow is gradually modified as shown in Figure 3.

Continuing the simulation shown in Figure 2(a) to

(a)



(b)

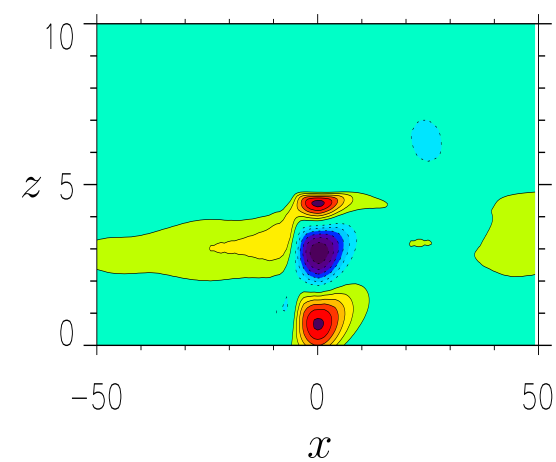


Figure 2: Nonlinear simulation of a gravity wave packet forced by isolated topography: Perturbation streamfunction $\psi(x, z, t)$ at nondimensional time (a) $t = 200$ and (b) $t = 400$. The contour levels range from negative values of ψ (shown in blue) through zero (green) to positive values (red).

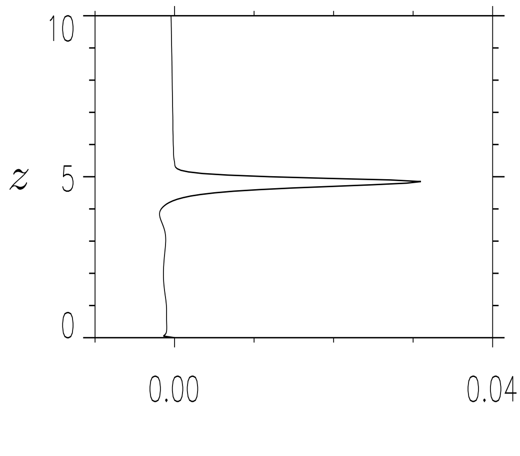


Figure 3: Nonlinear simulation of a gravity wave packet forced by isolated topography: Wave-induced horizontal mean velocity as a function of altitude at nondimensional time $t = 200$.

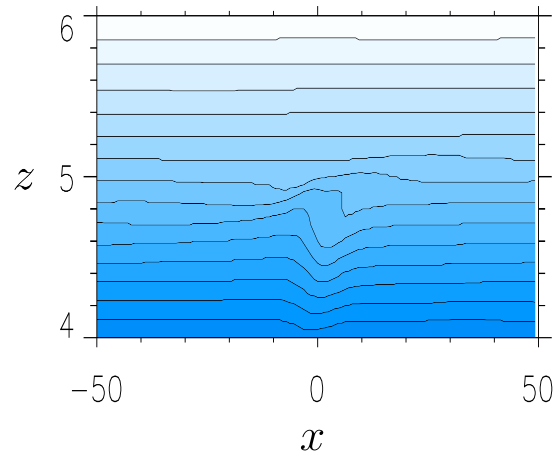


Figure 4: Nonlinear simulation of a gravity wave packet forced by isolated topography: Total density $\bar{\rho}(z)+\varepsilon\rho(x,z,t)$ at nondimensional time $t = 200$. Contour levels decrease with altitude (from dark blue through light blue to white).

a later time $t = 400$, we see that the streamlines are modified further in the region below the critical level, indicating the possibility of wave reflections (Figure 2 b). Contours of the density $\bar{\rho}(z)+\varepsilon\rho(x,z,t)$ at $t = 200$ in the vicinity of the critical levels are shown in Figure 4. At this time the density contours have started to overturn, indicating the onset of static instabilities.

5. CONCLUDING REMARKS

In this study, we used analytic and numerical methods to examine the evolution of an internal gravity wave packet forced by an isolated mountain. The Boussinesq approximation was made and that allowed us to simplify our governing equations into a single equation for the streamfunction of the wave packet. An analytic solution was derived the configuration in which the wave amplitude does not depend on time. In that case the governing equation for the amplitude of the packet is singular at the zero-wind level or critical level. As the packet propagates up to the critical level, its vertical wavelength decreases and at the critical level its amplitude is reduced to zero. The next steps in our analysis would involve including the effects of time-dependence and nonlinearity.

All our numerical solutions were time-dependent. In the linear configuration a quasi-steady was reached and the results were in agreement with the analysis in the sense that there was almost complete absorption of the wave packet at the critical level. In the nonlinear simulations, although there was still no transmission of the packet through the critical level, static instabilities began to develop at late time and the streamlines were modified below the critical level, suggestive of wave reflections.

The phenomenon of critical-level reflection in nonlinear gravity wave propagation problems is well-known and has been observed in earlier studies. In the last of a series of three articles, in which they extended the linear analysis of Booker & Bretherton into the nonlinear regime, Brown and Stewartson (1982) found evidence of gravity wave reflections at a nonlinear critical level; however, their analysis was limited to relatively early times. Campbell and Maslowe (2003) also observed wave reflections in their numerical simulations. The configuration they studied is similar to that described here, but their study focused on the case where the wave source spectrum is centered at some nonzero horizontal wavenumber, as in equation (10). This corresponds to a mountain range with multiple peaks. The wave packet then comprises a spectrum of waves (or Fourier coefficients) centered at a nonzero wavenumber $\kappa = k$ and the derivation of an analytic solution is considerably more complicated than in the configuration examined here.

With the present configuration the analytic solution procedure is simplified by the fact that the spectrum is centered at the zero wavenumber $\kappa = 0$ and, thus, we anticipate being able to derive an approximate analytic solution more readily. Once found the solution could be used to obtain insight into the results of the nonlinear numerical simulations.

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