P2.27 An Analytical Expression For Doppler Spectra Related to Terminal Velocity With Non-uniform Drop Size Distribution

Ming Fang[1*], Richard J. Doviak[2] and Pengfei Zhang[1]

[1]Cooperative Institute for Mesoscale Meteorological Studies, University of Oklahoma, Norman, OK
[2]NOAA/National Severe Storms Laboratory, Norman, OK

1. INTRODUCTION

The weather signal's autocorrelation function is (i.e. Eq. (1) of Fang and Doviak 2005)

$$R(\tau,\vec{r}_0)=\sum_k E[A_k^*(0)A_k(\tau)F_k^*(0,\vec{r}_0)F_k(\tau,\vec{r}_0)e^{-j4\pi v_k\tau/\lambda}] \quad (1),$$

where τ is lag time, A_k the complex amplitude of voltage echoed from the k^{th} particle, F_k the weight imposed on k^{th} particle due to the radar beam pattern, V_k the radial velocity of the k^{th} particle, and λ is the wavelength of electromagnetic wave transmitted by radar. " * " in (1) denotes the complex conjugate, and E denotes the expectation operation. $\vec{r}_0$, locates the center of V_6, and is a function of time if antenna is rotating. If the oscillation or/and wobbling of a hydrometer is independent from its radial velocity, starting from this equation, Fang and Doviak (2005) show that the integral form of the correlation function for a stationary beam can be written as (i.e. their Eq. (7))

$$R(\tau,\vec{r}_0)=\int_V I(\vec{r}_0,\vec{r})\eta(\vec{r})\rho_o(\tau,\vec{r})E_v[e^{-j4\pi v(\vec{r})\tau/\lambda}]dV \quad (2),$$

where

$$\rho_o(mT_s,\vec{r})\equiv\frac{E_\mu[A^*(0,\vec{r})A(\tau,\vec{r})]}{E_\mu[|A(0,\vec{r})|^2]} \quad (3),$$

E_v denotes an average over an ensemble of velocities, whereas E_μ denotes an average over the ensemble of hydrometer's back scattering cross sections. However, for a vertically pointed radar, a hydrometer's oscillation or/and wobbling could correlate to its radial velocity because the larger hydrometers could have larger oscillation and faster terminal velocity. Going through the steps from Eq. (1) to Eq. (7) given by Fang and Doviak (2005), one can rewrite Eq. (2) as

$$\overline{R(\tau,\vec{r},t)}=\int_V I(\vec{r}_0,\vec{r})\eta(\vec{r})E_v\left[\rho_o(\tau,\vec{r},t)e^{-j2kv_d(\vec{r},t)\tau}\right]dV \quad (4),$$

where we use the overbar to emphasize the autocorrelation function is an integral volumetric mean weighted by radar beam pattern and reflectivity. Because we are assuming a vertically pointed beam and the vertical wind component to be zero, the radial velocity in Eq. (2) reduces to the terminal velocity V_d in Eq. (4).

Starting from Eq. (4), this study will show that an analytical expression for the Doppler spectrum is related to the drop's terminal velocity and size distribution if there is a unique relationship between drop's diameter and its terminal velocity. The derivation does not require drop size distribution to be homogeneous. This generalized expression reduces to previously derived expression if drop size distribution is uniform.

2. THE EQUATION FOR DOPPLER SPECTRA RELATED TO TERMINAL VELOCITY

The corresponding correlation coefficient of Eq. (4) is

* Corresponding author address: Ming Fang, 120 David L Boren Blvd., Norman, OK 73072; email: ming.fang@noaa.gov

$$\rho_{od}(\tau,\vec{r}_0,t)=\frac{\overline{R(\tau,\vec{r}_0,t)}}{\overline{R(0,\vec{r}_0)}}=\frac{\int_{V_6} I(\vec{r}_0,\vec{r})\eta(\vec{r})E\left[\rho_o(\tau,\vec{r},t)e^{-j2kv_d(\vec{r},t)\tau}\right]dV}{\int_{V_6} I(\vec{r}_0,\vec{r})\eta(\vec{r})dV} \quad (5).$$

Fourier transforming the above equation over tau one obtains the weighted normalized Doppler spectrum of terminal velocity. That is

$$\overline{S_{nod}(v_d,\vec{r},t)}=\frac{\int_{V_6} I(\vec{r}_0,\vec{r})\eta(\vec{r})\widetilde{F}\left\{E\left[\rho_o(\tau,\vec{r},t)e^{-j2kv_d(\vec{r},t)\tau}\right]\right\}dV}{\int_{V_6} I(\vec{r}_0,\vec{r})\eta(\vec{r})dV} \quad (6),$$

where $\widetilde{F}$ denotes the Fourier transform over time lag domain. The part relating to Fourier transform in the numerator of the above equation is the unweighted normalized Doppler spectrum at $\vec{r}$, i.e. $S_{nod}(v,\vec{r},t)$ which relates to the hydrometers in a unit volume at $\vec{r}$. It has been shown that the power scattered by a hydrometer to the antenna is proportional to the backscatter cross section of the hydrometer, i.e. Eq. (3.24) of Doviak and Zrnić (1993). If $p(\vec{r},v_d,t)$ is the probability density of a drop having terminal velocity v_d at $\vec{r}$, $p(\vec{r},v_d,t)dV\int N(\vec{r},v,t)dv$ is the number of drops in elemental volume dV having a terminal velocity v_d, and $p(\vec{r},v_d,t)\sigma_b(v_d)dV\int N(\vec{r},v,t)dv$ is the corresponding total back scattering cross section of hydrometers with terminal velocity v_d in the elemental volume dV at $\vec{r}$, where $N(\vec{r},v,t)$ is DSD at $\vec{r}$ in velocity domain. The total back scattering cross section of all hydrometers in dV is $\eta(\vec{r})dV$. Thus, $S_{nod}(v,\vec{r})$ can be rewritten as

$$S_{nod}(v_d,\vec{r},t)=\frac{p(\vec{r},v_d,t)\sigma_b(v_d)\int N(\vec{r},v,t)dv}{\eta(\vec{r})} \quad (7).$$

In order to simplify the discussion, we assume the DSD is statistical stationary. That is, both p and N are independent of time. In the domain of drop size diameter, Eq. (7) can be rewritten as

$$S_{nod}(v_d,\vec{r})=\frac{p(\vec{r},D)\sigma_b(D)\left(\frac{dD}{dv_d}\right)\int N(\vec{r},D)dD}{\eta(\vec{r})} \quad (8),$$

where we have used the relation $p(\vec{r},D)dD=p(\vec{r},v_d)dv_d$ and assumed there is a unique relationship between drop's diameter and its terminal velocity. Thus, from Eqs. (6)-(8) the weighted normalized Doppler spectrum in terms of terminal velocity is

$$\overline{S_{nod}(v_d,\vec{r})} = \frac{\int_{V_6} I(\vec{r}_0,\vec{r})[p(\vec{r},v_d)\sigma_b(v_d)\int N(\vec{r},v)dv]dV}{\int_{V_6} I(\vec{r}_0,\vec{r})\eta(\vec{r})dV} \quad (9),$$

and in terms of drop's diameter is

$$\overline{S_{nod}(v_d,\vec{r})} = \frac{\int_{V_6} I(\vec{r}_0,\vec{r})\left\{p(\vec{r},D)\sigma_b(D)\left(\frac{dD}{dv_d}\right)\int N(\vec{r},D)dD\right\}dV}{\int_{V_6} I(\vec{r}_0,\vec{r})\eta(\vec{r})dV} \quad (10).$$

$N(\vec{r},D)dD$ represents the number of drops whose diameters fall in between $D + dD$ in a unit volume at $\vec{r}$. It is easy to see that $\int_{-\infty}^{\infty} N(\vec{r},D)dD$ is the total number of drops in a unit volume at $\vec{r}$. It is also easy to understand that the ratio of $N(\vec{r},D)dD$ to $\int_{-\infty}^{\infty} N(\vec{r},D)dD$, i.e. $\frac{N(\vec{r},D)dD}{\int_{-\infty}^{\infty} N(\vec{r},D)dD}$, is the probability that the diameter of a drop lies between $D + dD$. Thus, the normalized drop size distribution, i.e. $\frac{N(\vec{r},D)dD}{\int_{-\infty}^{\infty} N(\vec{r},D)dD}$, is a probability density function which defines the probability of a drop falling in unit interval around D at location $\vec{r}$ in drop's diameter domain. Thus we have

$$p(\vec{r},D) = \frac{N(\vec{r},D)}{\int_0^{\infty} N(\vec{r},D)dD} \quad (11).$$

Because

$$N(v_d,\vec{r}) = N(\vec{r},D)\left(\frac{dD}{dv}\right) \quad (12),$$

in velocity domain we have

$$p(\vec{r},v_d) = \frac{N(v_d,\vec{r})}{\int_0^{\infty} N(v_d,\vec{r})dv_d} \quad (13).$$

Substituting (11) into (10) for a uniform DSD, and therefore a uniform reflectivity, Eq. (10) reduces to

$$\overline{S_{nod}(v_d)} = \frac{\sigma_b(D)N(D)\left(\frac{dD}{dv_d}\right)}{\eta} \quad (14)$$

This is exactly the Eq. (8.77) given by Doviak and Zrnić (1993). However, if DSD is not uniform, the weighted normalized Doppler spectrum from a unit volume at $\vec{r}$ should be read as

$$S_{nod}(v_d,\vec{r}) = \frac{I(\vec{r}_0,\vec{r})N(\vec{r},D)\sigma_b(D)\left(\frac{dD}{dv_d}\right)}{\int_{V_6} I(\vec{r}_0,\vec{r})\eta(\vec{r})dV} \quad (15).$$

weather radar presents a volumetric weighted version of (15). In this case, the radar measured spectrum due to an inhomogeneous distribution of terminal velocities is

$$\overline{S_{nod}(v_d,\vec{r})} = \frac{\int_{V_6} I(\vec{r}_0,\vec{r})N(\vec{r},D)\sigma_b(D)\left(\frac{dD}{dv_d}\right)dV}{\int_{V_6} I(\vec{r}_0,\vec{r})\eta(\vec{r})dV} \tag{16}$$

Therefore, with the work presented this chapter, we have derived a general analytical Doppler spectrum expression for terminal velocity, and shown that the previously derived equation, i.e. Eq. (14), is a result of Eq. (16) when DSD is uniform.

3. CONCLUSIONS

This study obtains a generalized analytical expression for Doppler spectra related to the terminal velocity with non-uniform drop size distribution. If drop size distribution is uniform, this generalized equation reduces to a previous derived expression.

References

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