P1.4 ANALYTICAL-NUMERICAL SOLUTIONS FOR THE ONE DIMENSIONAL PBL TURBULENCE MODEL

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1. Introduction

Analytical and propagator numerical methods are elaborated for solution of Weng-Taylor turbulence model [1]. In the Weng-Taylor model the eddy viscosity coefficient nonlinearly depends on velocities and is defined from additional phenomenological consideration, which constitutes a closure of turbulence. In models of such type sharp vertical boundary layers cause difficulties for traditional numerical methods. In this work a new numerical method is proposed, which is based on analytical representation of Weng-Taylor model solutions. It is shown that these analytical solutions of constituted initial boundary value problem can be resolved by additional solutions of system of ordinary differential equations. This system of equations is solved analytically, by using polynomial type substitutions for generalized Lagrangian variables. The obtained numerical solution is compared to solution by using numerical propagator method [2].

2. Problem formulation

Weng-Taylor model equations for horizontal U and V velocity components, written here as the functions of the normalized vertical coordinate z , are

$$\frac{\partial U}{\partial t}=\frac{\partial}{\partial z}\left(K_m\frac{\partial U}{\partial z}\right)+Tf\cos(\alpha)(V-V_g), \quad (1)$$

$$\frac{\partial V}{\partial t}=\frac{\partial}{\partial z}\left(K_m\frac{\partial V}{\partial z}\right)+Tf\cos(\alpha)(U_g-U), \quad (2)$$

$$0<z<1,\ 0<t\le T,$$

where $f=10^{-4}$ (Hz) is the Coriolis force frequency and $U_g=10$ (m/s), $V_g=0$ (m/s). With the initial and boundary conditions:

$$U(0,z)=u_0(z), V(0,z)=v_0(z), 0\le z\le 1, \quad (3)$$

$$U(t,0)=0, V(t,0)=0,\ 0\le t\le 1, \quad (4)$$

$$U(t,1)=V_g, V(t,1)=0,\ 0\le t\le 1. \quad (5)$$

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The eddy viscosity coefficient K_m is defined from additional conditions, which constitutes a turbulence closure:

$$K_m=\frac{T}{L}(2l^2|\frac{\partial}{\partial z}\left((U^2+V^2)^{1/2}\right)|+\frac{\nu}{L}), \quad (6)$$

where $\nu=10^{-5}$ (m^2/s) is the molecular kinematics viscosity, $l=\kappa z$ is a mixing length scale in the simplest case, κ is von Karman constant, $L=890$ (m) is the depth of the turbulent layer and $T=16000$ (s) is the full time of calculations.

3. Problem solution

To solve the problem (1)-(6) we introduce the following two functions:

$$u(U,V)=\int_{U_0(0)}^{U}K_m dU_1, \quad (7)$$

$$\vartheta(U,V)=\int_{V_0(0)}^{V}K_m dV_1. \quad (8)$$

Since $\frac{du(U,V)}{dU}=K_m(U,V)$ and

$$\frac{\partial u(U,V)}{\partial t}=K_m(U,V)\frac{\partial U(t,z)}{\partial t},$$

$\frac{\partial u(U,V)}{\partial z}=K_m(U,V)\frac{\partial U(t,z)}{\partial z}$, then the equations becomes

$$\frac{\partial u(t,z)}{\partial t}=K_m(u,\vartheta)\frac{\partial^2 u(t,z)}{\partial z^2}+F_1(\vartheta), \quad (9)$$

$$\frac{\partial \vartheta(t,z)}{\partial t}=K_m(u,\vartheta)\frac{\partial^2 \vartheta(t,z)}{\partial z^2}+F_2(u), \quad (10)$$

where

$$F_1(\vartheta)\overset{def}{\equiv}Tf\cos(\alpha)(\vartheta-V_g),$$

$$F_1(u)\overset{def}{\equiv}Tf\cos(\alpha)(U_g-u).$$

Having introduced the Varshavsky integral transformation $h^{(u)}(u,\vartheta)=\int_0^u\frac{1}{K_m}du_1$ we obtain

$$h^{(u)}(u,\vartheta)=\int_0^u\frac{1}{K_m}du_1=$$

$$\int_0^u \frac{2\sqrt{u_1^2+\vartheta^2}}{\frac{2l^2T}{L}\left(u_1\frac{\partial u_1}{\partial z}+\vartheta\frac{\partial \vartheta}{\partial z}\right)+\frac{T\nu}{L^2}\sqrt{u_1^2+\vartheta^2}}du_1 =$$

$$\frac{2L^2}{T\nu}\int_0^u \frac{du_1}{1+\frac{2l^2L}{\nu}\frac{u_1\frac{\partial u_1}{\partial z}+\vartheta\frac{\partial \vartheta}{\partial z}}{\sqrt{u_1^2+\vartheta^2}}} =$$

$$\frac{2L^2}{T\nu}\left\{\frac{1}{2}\left(1+\frac{u^2}{2}\right)\left(1+\frac{u\vartheta^2}{2}\right)+u^2\vartheta+\left(\frac{2l^2T}{L}+\vartheta\right)u\right\}.$$

Similarly, if we consider the Varshavsky integral transformation $h^{(\vartheta)}(u,\vartheta)=\int_0^\vartheta \frac{1}{K_m}d\vartheta_1,$ we have

$$h^{(\vartheta)}(u,\vartheta)=\int_0^\vartheta \frac{1}{K_m}d\vartheta_1 =$$

$$\frac{2L^2}{T\nu}\left\{\frac{1}{2}\left(1+\frac{\vartheta^2}{2}\right)\left(1+\frac{\vartheta u^2}{2}\right)+\vartheta^2 u+\left(\frac{2l^2T}{L}+u\right)\vartheta\right\}.$$

Now, in order to make use of Biot variation principle (see [3]) we will introduce and calculate the following functions:

$$F^{(u)}(u,\vartheta)=\int_0^u \frac{u_1}{K_m}du_1=\frac{2L^2}{T\nu}\left\{\frac{\left(\frac{2l^2T}{L}+\vartheta\right)^2 u}{2}+\frac{1}{6}\left(1+\frac{u^2}{2}\right)\left(2+\frac{u^3\vartheta}{2}\right)+\frac{u^2\vartheta^2}{2}\right\},$$

$$F^{(\vartheta)}(u,\vartheta)=\int_0^\vartheta \frac{\vartheta_1}{K_m}d\vartheta_1=\frac{2L^2}{T\nu}\left\{\frac{\left(\frac{2l^2T}{L}+u\right)^2 \vartheta}{2}+\frac{1}{6}\left(1+\frac{\vartheta^2}{2}\right)\left(2+\frac{\vartheta^3 u}{2}\right)+\frac{u^2\vartheta^2}{2}\right\},$$

$$V^{(u)}\left((u,\vartheta)=q_1\right)=\int_0^{q_1(t)} F^{(u_1)}(u_1,\vartheta)du_1 = \int_0^{q_1(t)} du_1\int_0^{u_1}\frac{u_2}{K_m(u_2,\vartheta)}du_2,$$

$$V^{(\vartheta)}\left((u,\vartheta)=q_2\right)=\int_0^{q_2(t)} F^{(\vartheta_1)}(u,\vartheta_1)d\vartheta_1 = \int_0^{q_2(t)} d\vartheta_1\int_0^{\vartheta_1}\frac{\vartheta_2}{K_m(u,\vartheta_2)}d\vartheta_2,$$

where

$$u=C_3\left(1-\frac{z}{q_1(t)}\right)^2+F_1,\ C_3=const,$$

$$\vartheta=C_4\left(1-\frac{z}{q_2(t)}\right)^2+F_2,\ C_4=const.$$

After calculations of integrals in the expressions for the introduced functions $V^{(u)}(q_1)$ and $V^{(\vartheta)}(q_2)$ we obtain that

$$V^{(u)}(u,\vartheta)=\frac{7}{61}C_3^2q_1-\frac{1}{3}q_1^2q_2Tf\cos(\alpha)V_g, \quad (11)$$

$$V^{(\vartheta)}(u,\vartheta)=\frac{7}{61}C_4^2q_2+\frac{1}{3}q_1q_2^2Tf\cos(\alpha)U_g. \quad (12)$$

Now we can consider the following integrals and calculate them:

$H^{(u)}(q_1)=\int_z^{q_1} h^{(u)}dz$ and $H^{(\vartheta)}(q_2)=\int_z^{q_2} h^{(\vartheta)}dz.$

Indeed, having designated $\xi=1-\frac{z}{q_1}$ and $\eta=1-\frac{z}{q_2}$ we can write

$$H^{(u)}(\xi,\eta)=q_1\int_0^\xi h^{(u)}d\xi =$$

$$\frac{2L^2}{T\nu}\left\{\frac{4Tl^2}{3}\xi^2\eta+\frac{1}{40}\xi^5+\frac{1}{10}\eta^3\right\}C_3q_1,$$

$$H^{(\vartheta)}(\xi,\eta)=q_2\int_0^\eta h^{(\vartheta)}d\eta =$$

$$\frac{2L^2}{T\nu}\left\{\frac{4Tl^2}{3}\xi\eta^2+\frac{1}{40}\eta^5+\frac{1}{10}\xi^3\right\}C_4q_2.$$

It follows that

$$D^{(u)}(q_1)\overset{def}{\equiv}\frac{1}{2}\int_0^{q_1}\left(\frac{\partial H^{(u)}(q_1)}{\partial t}\right)^2dz =$$

$$\frac{12l^2L}{\nu}q_1^2(q_1')^3+\frac{1}{8}q_1(q_1')^2+\frac{1}{2}C_3q_1, \quad (13)$$

$$D^{(\vartheta)}(q_2)\overset{def}{\equiv}\frac{1}{2}\int_0^{q_2}\left(\frac{\partial H^{(\vartheta)}(q_2)}{\partial t}\right)^2dz =$$

$$\frac{12l^2L}{\nu}q_2^2(q_2')^3+\frac{1}{8}q_1(q_2')^2+\frac{1}{2}C_4q_2, \quad (14)$$

Now, following Biot variational principle, we can write two Lagrange-Biot equations:

$$\frac{\partial V^{(u)}}{\partial q_1}+\frac{\partial D^{(u)}}{\partial q_1'}=const, \quad (15)$$

$$\frac{\partial V^{(\vartheta)}}{\partial q_1}+\frac{\partial D^{(\vartheta)}}{\partial q_1'}=const. \quad (16)$$

Substituting the relevant expressions for $V^{(u)}$, $V^{(\vartheta)}$, $D^{(u)}$, $D^{(\vartheta)}$ from (11)-(14) in (15) and (16) we obtain the following system of two ODE:

$$\frac{7}{61}C_3^2+\frac{2}{3}q_2q_1Tf\cos(\alpha)V_g+\frac{36l^2L}{5\nu}q_1^2(q_1')^2q_1''+\frac{1}{2}q_1q_1'q_1''=0, \quad (17)$$

$$\frac{7}{61}C_4^2+\frac{2}{3}q_2q_1Tf\cos(\alpha)U_g+\frac{36l^2L}{5\nu}q_2^2(q_2')^2q_2''+\frac{1}{2}q_2q_2'q_2''=0. \quad (18)$$

Let us determine the analytical solution $\{q_1(t),q_1(t)\}$ of the system (17)-(18). Then the solution $\{u(t,z),\vartheta(t,z)\}$ of the reduced problem (9)-(10) is

$$u(t,z)=C_3\left(1-\frac{z}{q_1(t)}\right)^2+Tf\cos(\alpha)\left(q_1(t)-V_g\right),$$

$$\vartheta(t,z)=C_4\left(1-\frac{z}{q_2(t)}\right)^2+Tf\cos(\alpha)\left(U_g-q_2(t)\right).$$

Constants C_3 and C_4 can be found from the initial and boundary conditions (3)-(5).

In order to investigate features of proposed solutions for U and V we look here only at the begining stage of the boundary layer formation, when time t is relatively small. For this case the system of equation (17)-(18) for $q_1(t)$ and $q_2(t)$ can be solved by using asymptotic expansion around $t=0$. So $q_1(t)$ and $q_2(t)$ can be written as:

$$q_1(t)=A_0+A_1t+A_2t^2, \quad (19)$$

$$q_2(t)=B_0+B_1t+B_2t^2, \quad (20)$$

it is assumed that higher order coefficients in this expansion are small and therefore can be omitted.

Coefficients A_0 and B_0 can be resolved from (7)-(8) taking into account that boundary conditions $U(0,0)=0$ and $V(0,0)=0$, so, we also obtain that $u(0,0)=0$ and $\vartheta(0,0)=0$ too. Namely, for A_0 and B_0 we have:

$$A_0=\frac{C_4+U_gTf\cos(\alpha)}{Tf\cos(\alpha)},$$

$$B_0=\frac{V_gTf\cos(\alpha)-C_3}{Tf\cos(\alpha)}.$$

After substitution of $q_1(t)$ and $q_2(t)$ from (19)-(20) into the (17)-(18) we obtain nonlinear system for four equations which should be solved in order to find A_1, A_2, B_1 and B_2. This system reads:

$$A_0-A_0A_1A_2+\frac{36}{5}\frac{C_1}{C_2}A_0^2A_1^2A_2-\frac{7}{61}C_3^2-\frac{2}{3}V_gTf\cos(\alpha)A_0B_0=0, \quad (21)$$

$$A_1-A_1^2A_2-2A_0A_2^2+\frac{72}{5}\frac{C_1}{C_2}A_0A_1^3A_2-\frac{144}{5}\frac{C_1}{C_2}A_0^2A_1A_2^2-\frac{2}{3}V_gTf\cos(\alpha)\left(A_1B_0+A_0B_1\right)=0, \quad (22)$$

$$B_1-B_1^2B_2-2B_0B_2^2-\frac{72}{5}\frac{C_1}{C_2}B_0B_1^3B_2-\frac{144}{5}\frac{C_1}{C_2}B_0^2B_1B_2^2-\frac{2}{3}U_gTf\cos(\alpha)\left(A_1B_0+A_0B_1\right)=0, \quad (23)$$

$$2B_2^3+\frac{216}{5}\frac{C_1}{C_2}B_1^3B_2^2+\frac{576}{5}\frac{C_1}{C_2}B_0B_1B_2^3+\frac{2}{3}U_gTf\cos(\alpha)\left(A_2B_1+A_1B_2\right)=0, \quad (24)$$

where $C_1=\frac{2T\kappa^2}{L}$, $C_2=\frac{T\nu}{L^2}$.

In the considered case, when $V_g=0$, the system (21)-(24) splits into two independent ones in respect of coefficients A and B. Moreover, to obtain real solutions of the systems for A and B the absolute values of the coefficients C_3 and C_4 should be equal, $|C_3|=|C_4|$. Here for calculations we use the following values of $C_3=15.8$ and $C_4=-15.8$.

System (7)-(8) defines relations $U=U(u,\vartheta)$ and $V=V(u,\vartheta)$. To resolve these relations we rewrite (7)-(8) in the following form by subdividing all integration regions into sufficiently small parts and providing a respective integration in each part:

$$u(U,V)=\sum_i C_1\int_{U_i}^{U_{i+1}} z^2\frac{\partial}{\partial z}\left(U_1^2+V^2\right)^{\frac{1}{2}}dU_1+C_2U, \quad (25)$$

$$\vartheta(U,V)=\sum_i C_1\int_{V_i}^{V_{i+1}} z^2\frac{\partial}{\partial z}\left(U^2+V_1^2\right)^{\frac{1}{2}}dV_1+C_2V. \quad (26)$$

By using the Bonnet's second mean value theorem and substitution:

$$\frac{\partial}{\partial z}\left(U^2+V^2\right)^{\frac{1}{2}}=\frac{1}{2}\left(U^2+V^2\right)^{\frac{1}{2}}\frac{\partial}{\partial z}\ln\left(U^2+V^2\right), \quad (27)$$

the system (25)-(26) can be rewritten in the following form:

$$u(U,V)=\frac{1}{2}C_1\sum_i\frac{\partial}{\partial z}\ln\left(U^2+V^2\right)_i^{\frac{1}{2}}z_i^2\cdot$$

$$\int_{U_i}^{U_{i+1}}\left(U_1^2+V^2\right)^{\frac{1}{2}}dU_1+C_2U,\ U_i<U_{\xi_i}<U_{i+1}, \quad (28)$$

$$\vartheta(U,V)=\frac{1}{2}C_1\sum_i\frac{\partial}{\partial z}\ln\left(U^2+V^2\right)_i^{\frac{1}{2}}z_i^2\cdot$$

$$\int_{V_i}^{V_{i+1}}\left(U^2+V_1^2\right)^{\frac{1}{2}}dV_1+C_2V,\ V_i<V_{\xi_i}<V_{i+1}. \quad (29)$$

Providing here iteratively numerical calculations of (28)-(29) we approximately assumed $U_{\xi_i}=U_{i+1/2}$ and $V_{\xi_i}=V_{i+1/2}$.

Results of numerical calculations are shown in Fig.1, where wind module $\left(U^2+V^2\right)^{\frac{1}{2}}$ distributions are described in different time moments for the beginning stage of the boundary layer formation. The model is used to illustrate analytical method and a simple features of obtained solution. Extended analysis and more general numerical method for solution of nonlinear system of equations (17)-(18) and (28)-(29) need to be continued.

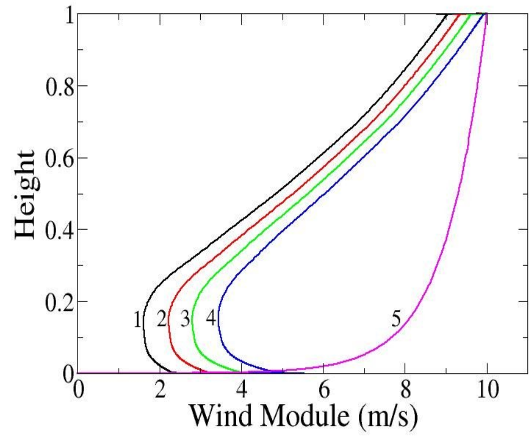

Fig.1 Wind module distributions in the different time moments: 1-t=0.0125, 2-t=0.0250, 3-t=0.0375, 4-t=0.005; 5-t=1, long time calculations (quasi steady-state solution) using propagator scheme.

Calculations of long time processes are provided by using propagator difference scheme, see Fig.1. It is shown in [2], that stability restrictions for the propagator scheme become more weaker in comparison to traditional semi-implicit difference schemes. In [2] it is proven that the scheme is unconditionally monotonic, it has truncation errors of the first order in time and of the second order in space. Propagator scheme is adopted for solution of problem (1)-(6) due to low order truncation error does not reflect the boundary layer formation in details. In Fig.1 only long time calculations (quasi steady-state solution) for wind module distribution are shown. Although, it should be noted that after properly chosen space grid mean values of von Karman constant and friction velocity, numerically calculated by using propagator scheme, can be obtained close to realistic. This allows considering that higher order propagator difference scheme can improve resolution in time and space, and will be more adopted for boundary layer calculations.

References

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