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A FIELD THEORETICAL MODEL OF THE STATIONARY ATMOSPHERIC VORTEX

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1 Introduction

The stationary two-dimensional approximation to the atmospheric vortex is a useful theoretical laboratory, making available results that can be compared with observations. There is another reason of interest for this approximation: the atmospheric vortex can be seen as a cuasi-coherent structure and it is known that a $2D$ fluid evolves at relaxation to states of self-organization that have these characteristic, a fact that is strongly supported by experiments and numerical simulations. A theoretical model of the asymptotic stationary states of fluids cannot be based on only the conservation laws (density, momentum, energy, etc.) since they allow for a large class of functions that could represent the final flow pattern. A natural approach is to first identify functionals of the fluid's state and apply variational procedures. The two-dimensional geometry of the fluid flow makes this possible. The $2D$ ideal incompressible fluid (Euler) can be represented as a discrete system of point-like vortices interacting by a long range potential [1]. The $2D$ atmosphere and plasma are equivalent with a discrete system of point-like vortices interacting by a short range potential [2]. These are well-known models that have been used in various applications. A fundamental property of these models consists of the fact that they formulate the dynamics in terms of matter, field and interaction. We then look again to the continuum limit of these models but preserving this structure. In both cases we obtain a classical field theory for the matter field (the density of point-like vortices) the gauge field (the long or short range potential) and interaction. The essential benefit of this approch is that it provides a Lagrangian density, whose integral on $2D$ space and time is the action functional.

2 The Euler fluid

For the $2D$ ideal fluid the Euler equation $d\omega/dt = 0$ is expressed for the discrete system of point-like vortices as $dr_k^i/dt = \varepsilon^{ij}\frac{\partial}{\partial r_k^j}\sum_{n=1,n\neq k}^{N}\omega_0 G\left(\mathbf{r}_k-\mathbf{r}_n\right)$, where $\mathbf{r}_i \equiv (x,y)$, ω_0 is the elementary vorticity carried by each point-like vortex and ε^{ij} is the antisymmetric tensor. The potential is long range $G\left(\mathbf{r},\mathbf{r}'\right) \approx -\left(2\pi\right)^{-1}\ln\left(\left|\mathbf{r}-\mathbf{r}'\right|/L\right)$ where L is the spatial extension of the flow. The continuum limit leads to a Lagrangian density expressed in terms of two fields: A_μ is the ("gauge") field representing the potential between vortices, ϕ is the complex scalar ("matter") field representing the increase or decrease of local vorticity (equivalently: increase/decrease of density of vortices). The fields A_μ and ϕ are 2×2 matrices with complex entries that belong to $su\left(2\right)$, as required by the spin-like nature of the point vortices. Looking for the extremum of the action $\mathcal{S} = \int d^2x dt \mathcal{L}$ we are guided by the observation that the asymptotic states of the fluid are coherent structures, possibly integrable. Since all known integrable structures (including solitons, instantons) are obtained at "self-duality" we look for this property of $\mathcal{S}$. In practical terms this means to write $\mathcal{S}$ as a sum of squares plus a topological term. Indeed $\mathcal{S}$ for the Euler fluid has this property (the topological term is zero) and $\mathcal{S}$ can be minimised by taking to zero the square terms. This leads to two equations (the "self-duality" equations) which, with an algebraic ansatz, lead to the *sinh*-Poisson equation [3]. The latter is known to describe the asymptotic structures reached by the Euler fluid at stationarity [4].

3 The $2D$ stationary atmospheric vortex

For the atmosphere the interaction between point-like vortices is short-range: $G \sim K_0\left(\left|\mathbf{r}_k-\mathbf{r}_n\right|/\rho_g\right)$ where K_0 is the modified Bessel function and ρ_g is the Rossby radius. The Lagrangian,

$$\mathcal{L} = -\mathrm{tr}\left[\left(D^\mu\phi\right)^\dagger\left(D_\mu\phi\right)\right] \qquad (1)$$

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$$-\kappa\varepsilon^{\mu\nu\rho}\mathrm{tr}\left(\partial_\mu A_\nu A_\rho+\frac{2}{3}A_\mu A_\nu A_\rho\right)$$
$$-V\left(\phi,\phi^\dagger\right)$$

similar to the Euler fluid case, consists of the kinetic term for the ϕ field, the Chern-Simons term of the A_μ field and the potential V for the self-interaction of the scalar field. The latter is essentially different from the Euler case: it provides to the system precisely the short-range required by the discrete model ($G\sim K_0$ above) by giving a "mass" to the A_μ-particle ("photon") [5], [6]. As sugested by the Euler model we look for self-duality and write the action $\mathcal{S}$ as a sum of squares and an additional term. However it is not possible to find for the latter a topological meaning, when the fields ϕ and A_μ belong to $su(2)$. This leaves a certain ambiguity in the separation into squares and the ground (called "vacuum") energy [7]. Taking to zero the square terms we derive the reduced set of equations [8]. Under the same algebraic ansatz we find

$$\Delta\psi+\frac{1}{2p^2}\sinh\psi\left(\cosh\psi-p\right)=0 \qquad (2)$$

where p is a positive constant and ψ is the streamfunction of the flow, *i.e.* $\mathbf{v}=-\nabla\psi\times\widehat{\mathbf{e}}_z$, with $\widehat{\mathbf{e}}_z$ the versor perpendicular on the plane. Physical quantities are normalised using the Rossby radius ρ_g and the Coriolis frequency f_0.

4 Numerical studies

Eq.(2) has been solved on spatial domains, $L=L^{phys}/\rho_g$ in the range $0.3<L<15$. The large number of numerical experiments allows us to obtain an approximative idea about the structure of the space of solutions and its neighborhood, *i.e.* the functions that are cuasi-solutions, verifying Eq.(2) within a lower precision. The strongly nonlinear character of Eq.(2) is manifested in the fact that some subset of solutions is accessed with difficulty by the integration procedure (with no intrinsic physical meaning). The trivial solution ($\psi=0$) is obtained in most of the cases. The nontrivial solutions are of three types: (1) smooth symmetric vortices; (2) highly localised vortices, with almost zero vorticity everywhere, except a small central region; (3) crystals of vortices. The smooth vortices are obtained with apparently arbitrary precision, while the strongly localised vortices arise in families of neighboring configurations of flow, for most of them the precision of verification of Eq.(2) is lower. We have defined an " error" functional to represent the degree of departure of a flow configuration from being an exact solution. Exploring a large set of results (solutions and "cuasi-solutions") we identify the structure of the space of functions around the extremum of the action (*i.e.* Eq.(2)). This shows the existence of degenerate directions, explaining the fact that an exact solution is connected along paths in function space with quasi-solutions with similar shape, and reaching at a limit the class of highly localised vortices. The physical meaning of such structure should be considered in depth, since a small external factor may drive the system along this path and this can represent an easier way for thermodinamical processes that lead to vorticity concentration.

5 Physical conclusions on cases relevant for the tropical cyclone

Systematically, the smooth solution of Eq.(2) has the morphology of the tropical cyclone [9], as shown in Fig.(1). We have identified two relationships between the main parameters: the maximum radial extension of the atmospheric vortex, $R_{\max}$, the radius of the eye-wall R_w and the maximum azimuthal wind $v_{\max}$:

$$v_{\max}(L)\simeq\frac{e^2}{2}\left[\alpha\exp\left(\frac{\sqrt{2}}{R_{\max}}\right)-1\right] \qquad (3)$$

$$\frac{R_w}{R_{max}}=\frac{1}{4}\left[1-\exp\left(-\frac{R_{max}}{2}\right)\right] \qquad (4)$$

valid for the region $0<L=R_{\max}/\sqrt{2}<6$, where e is the basis of natural logarithm and $\alpha=0.97$. They compare well with observations and are useful for approximative evaluation of one of the parameters, when two of them are known from observations.

The field theoretical formulation provides an interesting physical result. The states of the fields ϕ and A_μ are formally associated to the notion of particles, although we only need to think of them in terms of stationary vortex. The theory provides the masses of these particles and the result that the two kind of particles have equal masses. Or, in physical terms, one of the masses (of the field A_μ, the "photon") is the Rossby radius. The other mass (of the field ϕ, called the Higgs particle) is the spatial distance on which the field ϕ decays to zero, *i.e.* approximately the spatial extension of the atmospheric vortex. The equality means that the radial extension of the atmospheric vortex is approximately equal with the Rossby radius.

6 The $2D$ annular vortex

A configuration of flow consisting of a plane ring vortex is not an exact solution of the Eq.(2). However the search for the extremum of the action functional reveals an interesting state. Applying the standard Bogomolnyi procedure of writting the action as a sum of squares plus an additional term and looking for the condition that this additional term is topological (*i.e.* expressed in terms of the total vorticity in the field, which is a conserved quantity for the ideal case), we are led to assume a particular algebraic ansatz, where only one of the ladder generators of $su(2)$ is retained. The result is a new differential equation

$$\Delta\psi = \exp(\psi)\left[\exp(\psi) - 1\right] \tag{5}$$

for stationary states. The solutions of (5) consists this time precisely in vortices with annular shape, with spatial extension comparable with the Rossy radius. They are stable since their energy is the lower bound of the action at stationarity and this bound is topological (indestructible in the absence of dissipation). The solutions of Eq.(2) and of Eq.(5) are classes of functions that are accessible to the system when it starts its time evolution from different classes of initial conditions.

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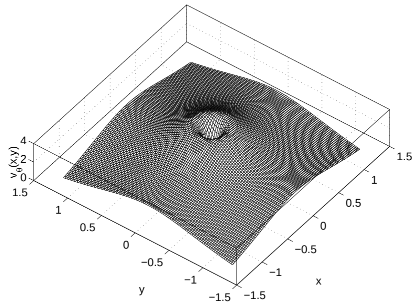

Figure 1: The azimuthal velocity of the vortex flow of the typical solution of Eq.(2).