3A.5 **Implications of a λ-μ Relationship for *Z-R* Relationships**

Paul L. Smith
South Dakota School of Mines and Technology, Rapid City, SD, USA

1. INTRODUCTION

Findings of systematic quadratic relationships between the size and shape parameters of gamma drop size distribution (DSD) functions fitted to observed raindrop spectra have been reported, though those findings are the subject of some debate (e.g., Zhang *et al*. 2001, 2003; Seifert 2005; Moisseev and Chandrasekar 2007). Problems have been noted with some of the data selection and statistical analysis procedures underlying those findings; in particular, the values of the DSD parameters have routinely been estimated with moment methods that can be subject to substantial biases and errors (Robertson and Fryer 1970; Smith *et al.* 2009). Nevertheless, similar findings continue to emerge from more careful analyses of observed raindrop size spectra. For purposes of the present discussion we take those findings as a point of departure, albeit with some reservations. Then analysis of the relationships between moments of the gamma DSD function under the constraint of such a relationship between its parameters suggests that the associated *Z-R* relationship may be linear, or nearly so.

2. ANALYSIS

Let M_i denote the *i*th moment of a gamma DSD. The DSD function can be written in terms of the total drop number concentration M_0 (zeroth moment) as

$$n(D) = M_0 \frac{\lambda^{\mu+1}}{\Gamma(\mu+1)} D^{\mu} \exp(-\lambda D) \qquad (1)$$

Here the other DSD parameters are λ (size) and μ (distribution shape); $\Gamma(x)$ denotes the gamma function. Then the *i*th moment of the distribution can be expressed as

$$M_i = \frac{M_0 \Gamma(\mu+i+1)}{\lambda^i \Gamma(\mu+1)} \qquad (2)$$

We can take any pair of moments and eliminate M_0 between them to obtain a relationship between the moments that depends only upon λ and μ. For example, relationships between the radar reflectivity *Z* (sixth moment M_6) and other moments M_i of the distribution take the form

$$M_6 = \frac{\Gamma(\mu+7)}{\lambda^{6-i}\Gamma(\mu+i+1)} M_i \qquad (3)$$

The coefficient of M_i depends only upon λ and μ, and a λ-μ relationship would permit reducing that to dependence upon a single parameter of the DSD function.

The rainfall rate can be approximated by the 3.67th moment of the DSD (with the appropriate leading coefficient K). The reported λ-μ relationships are quadratic expressions of the form

$$\lambda = a\mu^2 + b\mu + c \qquad (4)$$

Thus the relationships between M_6 (*Z*) and $M_{3.67}$ (*R*) can be written as

$$M_6 = \frac{K\Gamma(\mu+7)}{\Gamma(\mu+4.67)(a\mu^2+b\mu+c)^{2.33}} M_{3.67} \qquad (5)$$

They involve only the DSD parameter μ; if μ were constant in a given situation, M_6 would be directly proportional to $M_{3.67}$, i.e. *Z* would be directly proportional to *R*.

3. EMPIRICAL λ-μ RELATIONSHIPS

A variety of λ-μ relationships have been reported (Zhang *et al.* 2001; Brandes *et al.* 2003; Vivekanandan *et al.* 2004; Cao *et al.* 2008; Chu and Su 2008). With the exception of the one in Cao *et al.,* the relationships are quite similar (Fig. 1). For purposes of the analysis in the next section, we take the highest ("CS5L;" Chu and Su 2008) and lowest ("V04:" Vivekanandan *et al*. 2004) of the cluster of curves, along with the "Cao08" (Cao *et al*. 2008) curve, for comparisons.

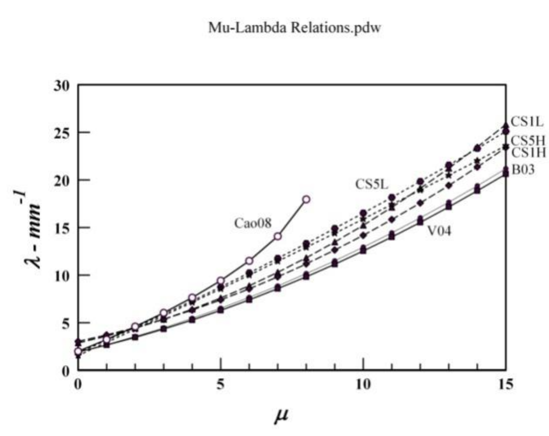

Fig. 1: Plots of several reported λ-μ relationships. The sources are Brandes et al. (2003; "B03"); Vivekanandan et al. (2004; "V04"); Chu and Su (2008; "CS1H, CS1L, CS5H, CS5L"); and Cao et al. (2008; "Cao08").

Corresponding author address: Paul L. Smith, Institute of Atmospheric Sciences, SDSM&T, 501 East Saint Joseph Street, Rapid City, SD 57701; e-mail: paul.smith@sdsmt.edu

4. NUMERICAL VALUES

Of course μ probably varies; both the numerator and denominator of (5) are monotonically increasing functions of μ. Taking the reported values of the parameters of the λ-μ relationships as discussed in Section 3 together with plausible values of μ leads to the values of the coefficients of $M_{3.67}$ in (5) (factor K omitted) plotted in Fig. 2. Over much of the range of plausible values of μ (perhaps 0 – 10) the coefficients of most of these M_6-$M_{3.67}$ relationships do not depend strongly upon the value of μ – which could suggest that the *Z-R* relationships may be nearly linear, a behavior postulated earlier by Jameson and Kostinski (2001). This suggests that the Jameson and Kostinski arguments should perhaps carry more weight than has generally been accorded to them.

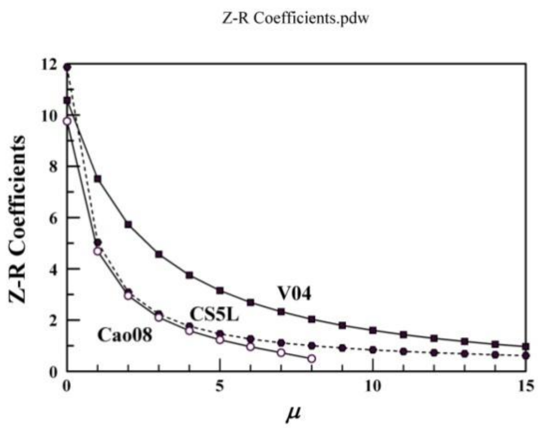

Fig. 2: Plot of the "Cao08", "CS5L" and "V04" coefficients (not including the factor K) for the Z-R relationship of Eq. (5) vs. values of the gamma shape parameter μ.

5. IMPLICATIONS FOR *Z-R* RELATIONSHIPS

Figure 2 suggests the possibility that the *Z-R* relationships may be nearly linear, but most of the reported *Z-R* relationships have exponents around 1.5. In the region of low values of μ in Fig. 1, say $\mu < 5$, there is more variation in the *Z-R* coefficients. This could be viewed in one of two ways:

(1) If μ varies systematically with *R*, then the exponent b in $Z = AR^b$ might well differ from unity. A plot of μ (values estimated from DSD observations by a moment method) versus *R* in Kozu and Nakamura (1991) shows considerable scatter; they indicate a weak dependence, with μ tending to increase with *R*. However, a similar treatment in Zhang *et al.* (2001) shows even greater scatter and indicates μ tending to decrease with increasing *R*. (The latter behavior could be a consequence of the known downward trend of the bias in the moment estimators for μ with increasing sample size, as the DSD sample size tends to increase with *R*; Smith *et al.* 2009). Comparable plots in Chu and Su (2008) also show considerable scatter, with any suggestion of a tendency for μ to vary systematically with *R* depending upon the combination of moments used to estimate the μ values. Overall, there is so far no convincing evidence of a systematic variation of μ with *R*.

(2) If μ (and hence also λ) varies more or less independently of *R*, it would imply the absence of a stable relationship between *Z* and *R*. This contrasts with the Testud *et al.* (2001) contention that the exponent *b* should be 1.5, independent of μ. Such behavior could help account for the absence of stable and widely applicable *Z-R* relationships.

The λ-μ and *Z-R* reports have generally come from separate analyses, focused either on fitting gamma functions to observed DSDs or determining *Z-R* relationships. It would be instructive to apply both types of analysis to the same set of data. Fitted values of μ (based on adequate sample sizes and reasonably accurate fitting procedures) could be plotted against the sample values of *R* to investigate this matter further. Lack of dependence of μ upon the values of the moment $M_{3.67}$ would support the idea that the *Z-R* relationships must be either nearly linear or highly variable.

Any validity of this whole story rests upon two critical assumptions:

- The gamma model is a valid description of the population DSD.
- The empirical λ-μ relationships are not artifacts of the analysis procedures.

Further verification of those assumptions is certainly needed.

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