P2.2 **Maximum Likelihood Estimation from a Left-Truncated Gamma Distribution**

Roger W. Johnson*, Donna V. Kliche, and Paul L. Smith
South Dakota School of Mines and Technology, Rapid City, SD, USA

Abstract

Past results of applying the maximum likelihood (ML) method to simulated raindrop samples demonstrated that ML estimators can yield more accurate estimates of raindrop size distribution parameters than moment method (MM) estimators when samples include the full range of drop sizes; this is true whether small or large samples are considered. However, in the absence of small drops in the samples (typical disdrometer minimum size thresholds are 0.3-0.5 mm) ML estimators ignoring this truncation problem show large bias; this bias does not decrease much with increasing sample size. Therefore, ML estimators require adaptation to deal with the problem of missing data on small drops. The present work provides the mathematical description of ML estimators modified for left-truncated distributions. This approach is tested via simulation with known gamma distributions. Our results show that modified ML estimators (c.f. Mallet and Barthes 2009) provide more accurate estimates than MM estimators even when samples are missing small drops due to instrument constraints.

1. INTRODUCTION

The traditional approach with experimental raindrop size data is to use the method of moments (MM) to estimate the parameters for the raindrop size distribution (DSD) functions. However, the moment method is known to be biased and can have substantial errors (Robertson and Fryer 1970; Smith and Kliche 2005; Smith *et al.* 2009). While the bias with the method of moments may be acceptably small for large sample sizes, the variability in these estimates may be considerable.

Superior alternative approaches to fitting the observed DSDs are available, and results of applying the maximum likelihood (ML) method to simulated raindrop samples demonstrated that ML estimators can yield more accurate estimates of DSD parameters than MM estimators when samples include the full range of drop sizes (Kliche 2007; Kliche *et al.* 2008). However, one important limitation of disdrometer instruments is the effect of truncating the observed size distributions at smaller drop diameters (typical disdrometer minimum size thresholds are 0.3-0.5 mm). In the absence of small drops in the samples ML estimators ignoring this truncation problem show large bias, and this bias does not decrease much with increasing sample size (Kliche 2007; Kliche *et al.* 2008). Consequently the ML estimators require some modifications to deal with situations where data on the small drops are lacking.

The goal of the present work is to provide the mathematical descriptions of the adaptations needed for both the ML and MM estimators in order to deal with the effects of missing small drops in the data. We focus on the gamma DSD function advocated by Ulbrich (1983) and Willis (1984), among others, since it can give an appropriate description of the natural variations of observed DSDs and the exponential distribution is a special case. This modified approach is then tested with simulated raindrop samples. We investigate how the modified ML estimators compare with the modified MM estimators and how the modifications to these methods improve parameter estimation when data on small drops are missing.

**Corresponding author address:* Roger W. Johnson, Dept. of Math and Computer Science, SDSM&T, 501 East Saint Joseph Street, Rapid City, SD 57701; e-mail: roger.johnson@sdsmt.edu

2. PARAMETER-FITTING PROCEDURES

From the statistical point of view, disdrometer measurements of raindrop sizes provide approximate descriptions of the populations from which they are taken. In particular, an analytical expression may be fit to the sample data to describe the underlying population of raindrops. The process begins by assuming that raindrop size has a distribution function which is a member of some family of distributions. For example, one may assume that some gamma distribution should reasonably describe the raindrop spectrum. Then, using the sample data, the particular distribution may be determined by estimating the unknown parameters of the assumed family of distributions.

Because it involves quantities having physical significance and is mathematically simple, the method of moments has been widely used to estimate these parameters. However, the bias and errors in the moment method lead to estimated parameters that often differ significantly from the true parameters of the population (e.g., Robertson and Fryer 1970; Smith and Kliche 2005; Kliche 2007; Kliche *et al.* 2008; Smith *et al.* 2009). The erroneous values can lead to wrong conclusions about the features of the DSDs being sampled.

Consequently, maximum likelihood (ML), advocated by Haddad *et al*. (1996, 1997), may be better suited to this problem. Maximum likelihood estimators, generally speaking, outperform other estimates, especially when the sample size is moderate to large (see, for example, Norden 1972, 1973). As noted in the introduction the ML method requires modification to deal with typical DSD observations where data on the small drops in the population are missing. Our approach employs a truncated form of the gamma DSD function.

The untruncated gamma distribution has a convenient representation in terms of the total drop number concentration N_T (Chandrasekar and Bringi 1987). The DSD adopted by Kliche *et al.* (2008) and Smith *et al.*

(2009) incorporated the mass-weighted mean diameter $D_m = (\mu + 4)/\lambda$, shown below:

$$n(D) = N_T \frac{(\mu+4)^{\mu+1}}{\Gamma(\mu+1)} \frac{D^{\mu}}{D_m^{\mu+1}} \exp\left[-(\mu+4) D / D_m\right] \qquad (1)$$

Here, the parameters are N_T, the shape parameter μ ($\mu > -1$) and D_m ($D_m > 0$), and Γ is the gamma function. This form can be recognized as the product of the mean total number concentration, N_T, and the gamma probability density function (PDF) of drop size. Equation (1) is similar to the one recommended by Chandrasekar and Bringi (1987), but we use D_m instead of their use of the median volume diameter D_0. When $\mu = 0$, the gamma DSD reduces to the exponential DSD.

The truncated gamma DSD distribution function is given by

$$n(D) = \tilde{N}_T \frac{\frac{\lambda^{\mu+1}}{\Gamma(\mu+1)} D^{\mu} \exp[-\lambda D]}{\left[1 - \frac{\gamma(\mu+1, \lambda D_{\min})}{\Gamma(\mu+1)}\right]}, \quad D > D_{\min} \qquad (2)$$

where $\tilde{N}_T$ is the number concentration for the truncated part of the DSD and

$$\gamma(a,x) \equiv \int_0^x t^{a-1} e^{-t}\, dt$$

is the *incomplete gamma function*. To see how (2) follows from (1) note, in general, that if $f(x)$ is the untruncated density, then $f(x)/[1 - F(D_{\min})]$ is the truncated density where $F(x) = P(X \le x)$ is the cumulative distribution function. Note that we don't see any observations below the cutoff D_{min} (note the distinction between D_m and D_{min}). Also, in the special case that D_{min} is zero, the incomplete gamma function γ takes on the value 0 and we reduce back to the untruncated case.

Note, by the way, that once estimates $\hat{D}_{\min}, \hat{\mu}, \hat{\lambda}$ of the parameters D_{min}, μ, λ have been determined,

$$\hat{p} = \int_0^{\hat{D}_{\min}} \frac{\hat{\lambda}^{\hat{\mu}+1}}{\Gamma(\hat{\mu}+1)} D^{\hat{\mu}} \exp[-\hat{\lambda} D]\, dD \qquad (3)$$

is an estimate of the proportion of missing drops.

2.1 THE MODIFIED MAXIMUM LIKELIHOOD METHOD FOR TRUNCATED GAMMA DSDS

The method of *maximum likelihood* (ML) is a traditional method used by statisticians to estimate the parameters of an assumed parametric model. The likelihood function represents a fundamental concept in statistical inference, and indicates how likely a particular population is to produce an observed sample. Kliche (2007) applied standard ML estimators to simulated samples of untruncated gamma DSD data and illustrated the improvement obtained over moment estimators; examples of her results appear in Kliche *et al.* (2008). ML in this untruncated case is somewhat biased but the bias decreases with increasing sample size (see, for example, Choi and Wette 1969).

Given an independent sample of observations D_1, D_2, …, D_c from the truncated gamma density

$$f(D) = \frac{\frac{\lambda^{\mu+1}}{\Gamma(\mu+1)} D^{\mu} \exp[-\lambda D]}{\left[1 - \frac{\gamma(\mu+1, \lambda D_{\min})}{\Gamma(\mu+1)}\right]}, \quad D > D_{\min}$$

the likelihood function L is given by the expression below:

$$L(\lambda, \mu, D_{\min}) = \prod_{i=1}^{C} f(D_i) = \left[1 - \frac{\gamma(\mu+1, \lambda D_{\min})}{\Gamma(\mu+1)}\right]^{-C} \frac{\lambda^{C(\mu+1)}}{[\Gamma(\mu+1)]^C} \left(\prod_{i=1}^{C} D_i\right)^{\mu} \exp\left(-\lambda \sum_{i=1}^{C} D_i\right)$$

when the observations D_1, D_2, …, D_c are all greater than D_{min} and zero otherwise. This leads to a maximum likelihood estimator for D_{min} of

$$\hat{D}_{\min} = \min(D_1, D_2, \ldots, D_C)$$

To find the maximum likelihood estimates of λ and μ note that maximizing the likelihood L is equivalent to maximizing

$$\frac{1}{C} \ln L(\lambda, \mu, \hat{D}_{\min}) = -\ln\left(1 - \frac{\gamma(\mu+1, \lambda \hat{D}_{\min})}{\Gamma(\mu+1)}\right) + (\mu+1)\ln\lambda - \ln\Gamma(\mu+1) + \mu\left[\frac{1}{C}\sum_{i=1}^{C} \ln D_i\right] - \lambda \bar{D} \qquad (4)$$

over λ and μ (which generalizes (A3) of Kliche *et al.* 2008). This maximization was accomplished by using the R software package (R Development Core Team 2009). In particular, the *optim( )* command of R was used along with the R function *likelihood* given in the Appendix. The *optim( )* command implements the non-linear optimization procedure discussed in Nelder and Mead (1965). Method of moments estimates for the untruncated case were used as a starting point for this iterative, numerical procedure.

2.2 THE MODIFIED MOMENT METHOD FOR TRUNCATED GAMMA DSDS

Various combinations of moments based on samples from the DSDs have commonly been used by atmospheric scientists to estimate the parameters of the underlying population distributions. For example, in the case of the gamma distribution, Szyrmer *et al*. (2005) used the zero moment, the 3rd moment, and the 6th moment in their fitting procedure; Smith (2003) suggested the combination of the 2nd moment, the 3rd moment, and the 4th moment; Ulbrich (1983), Kozu and Nakamura (1991), and Tokay and Short (1996) used the 3rd, 4th, and 6th moments, while Ulbrich and Atlas (1998) and Vivekanandan *et al.* (2004) used the 2nd, 4th, and 6th moments. It should be mentioned, by the way, that Vivekanandan *et al.* (2004) consider observations which come from a truncated gamma distribution.

The bias is stronger and the errors greater when higher order moments are used in calculating the parameters. When good sample values of the first moment are unavailable, as is often the case, the combination of 2nd, 3rd, and 4th moments typically gives the smallest bias and error.

The general form for the moments M_i of the untruncated gamma DSD function (1) can be written as

$$E(M_i) = N_T(\mu+1)(\mu+2)\cdots(\mu+i)\left[\frac{D_m}{\mu+4}\right]^i \tag{5}$$

where i is a non-negative integer and, for a sample of size C, the sample moments are

$$M_i \equiv \sum_{k=1}^{C} D_k^i$$

Setting $E(M_i) = M_i$ for i equal to 2,3,4 gives three equations in the three unknowns μ, λ, N_T (here we use $D_m = (\mu + 4)/\lambda$). Solving these gives the method of moments estimates

$$\hat{\mu} = \frac{4\alpha-3}{1-\alpha}, \quad \hat{\lambda} = \frac{M_3}{M_4}\frac{1}{1-\alpha},$$

$$\hat{N}_T = \frac{M_2^2}{M_4}\frac{\alpha}{(2-3\alpha)(1-2\alpha)} \tag{6}$$

where

$$\alpha \equiv \frac{M_3^2}{M_2 M_4}$$

Likewise, for the truncated gamma DSD (2) we have the following generalization of (5)

$$E(M_i) = \tilde{N}_T\,\frac{(\mu+1)(\mu+2)\cdots(\mu+i)}{\lambda^i}\,\frac{\left(1-\dfrac{\gamma(i+\mu+1,\lambda D_{\min})}{\Gamma(i+\mu+1)}\right)}{\left(1-\dfrac{\gamma(\mu+1,\lambda D_{\min})}{\Gamma(\mu+1)}\right)}$$

There are four unknown parameters to estimate in this truncated case – namely $\tilde{N}_T, \mu, \lambda$ and D_{min}. Consequently, we may set $E(M_i) = M_i$ for four values of i to obtain a system of four equations in four unknowns to estimate these. Instead, we go with the (somewhat conservative) maximum likelihood estimate of D_{min}

$$\hat{D}_{\min} = \min(D_1, D_2, \ldots, D_C) \tag{7}$$

and set $E(M_i) = M_i$ for i equal to 2,3,4 to estimate $\tilde{N}_T, \mu, \lambda$. In particular, we can solve these three equations by successfully minimizing the function

$$\mathrm{MM234}(\tilde{N}_T,\mu,\lambda) =$$

$$\left[1-\frac{\tilde{N}_T}{M_2}\frac{(\mu+1)(\mu+2)}{\lambda^2}\frac{\left(1-\dfrac{\gamma(\mu+3,\lambda\hat{D}_{\min})}{\Gamma(\mu+3)}\right)}{\left(1-\dfrac{\gamma(\mu+1,\lambda\hat{D}_{\min})}{\Gamma(\mu+1)}\right)}\right]^2 +$$

$$\left[1-\frac{\tilde{N}_T}{M_3}\frac{(\mu+1)(\mu+2)(\mu+3)}{\lambda^3}\frac{\left(1-\dfrac{\gamma(\mu+4,\lambda\hat{D}_{\min})}{\Gamma(\mu+4)}\right)}{\left(1-\dfrac{\gamma(\mu+1,\lambda\hat{D}_{\min})}{\Gamma(\mu+1)}\right)}\right]^2 +$$

$$\left[1-\frac{\tilde{N}_T}{M_4}\frac{(\mu+1)(\mu+2)(\mu+3)(\mu+4)}{\lambda^4}\frac{\left(1-\dfrac{\gamma(\mu+5,\lambda\hat{D}_{\min})}{\Gamma(\mu+5)}\right)}{\left(1-\dfrac{\gamma(\mu+1,\lambda\hat{D}_{\min})}{\Gamma(\mu+1)}\right)}\right]^2$$

(Note, by the way, from (3) that we may estimate N_T in (1) by taking the estimate of $\tilde{N}_T$ and dividing it by $(1-\hat{p})$.)

Once again, this optimization problem may be accomplished by using the R software package (R Development Core Team 2009). The *optim( )* command of R was used to minimize the R function *MM234* given in the Appendix. As before, method of moments estimates for the untruncated case were used as a starting point for this iterative, numerical procedure.

3. SIMULATION PROCEDURE

Comparisons of parameter-fitting procedures to evaluate biases and errors are readily done through computer simulation of repetitive sampling from known raindrop populations. The simulated gamma DSDs are represented as the product between the total drop number concentration, N_T, assumed to follow a Poisson distribution, and the corresponding probability density function (PDF) of drop size, as in (1). (We used the *rpois( )* and *rgamma( )* functions in the R package to generate these random observations.) In our simulations we took the mean number of drops in the samples to be the numerical value of N_T, and organized the results by the value of N_T. This approach can be interpreted as representing an instrument with a sample volume of 1 m^3 (independent of the drop size), or a sample volume of α m^3 with a mean drop concentration of N_T / α.

The size range for the computer-generated gamma raindrop populations is $0 < D < \infty$; we used about 1,000,000 drops for the simulated samples. For example, we drew 20,000 samples with N_T = 50 and 1,000 samples with N_T = 1,000.

Two distinct gamma populations were generated: one had shape parameter μ = 2 with scale parameter λ = 3 mm^{-1} and the other had shape parameter μ = 2 and scale parameter λ = 5 mm^{-1}. These two gamma densities are displayed, using a log scale, in Figure 1.

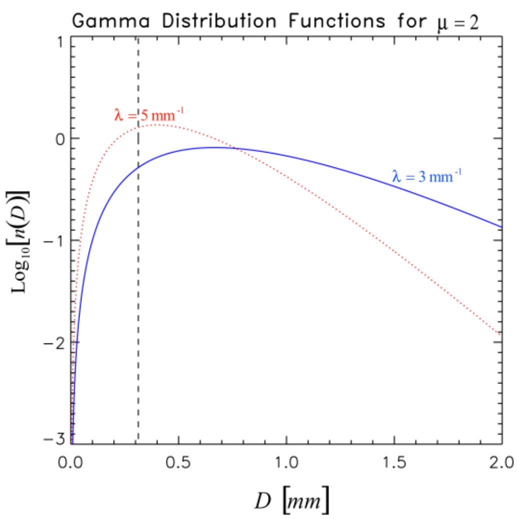

Figure 1. Plots of the two gamma densities considered in this study. The dashed vertical line indicates the data truncation point used in the simulations.

To investigate the effect of instrumental limitations at small drop sizes (disdrometers typically cannot respond to drop sizes < 0.3 mm or so), we withdrew from each generated sample the drops with sizes $D_i \le 0.313$ mm, which represents the lower threshold in the case of the Joss-Waldvogel disdrometer (JWD). In a gamma DSD with shape parameter μ = 2 and scale parameter λ = 3 mm^{-1}, 6.9% of the drops in the population have $D_i \le 0.313$ mm. Thus on average with N_T = 100 seven of the drops will be removed from each sample by imposing this threshold. For a gamma distribution having μ = 2 and λ = 5 mm^{-1}, about 20.8% of drops in the population have $D_i \le 0.313$ mm, so that on average with N_T = 100 about 21 drops will be removed from each sample.

4. COMPARISON OF ESTIMATORS

We begin this section by comparing the performance of the truncated maximum likelihood method (recall (4) above) with the ordinary maximum likelihood method which does not account for truncation (this amounts to maximizing (4) with $\hat{D}_{\min}$ set to zero – see (A3) of Kliche *et al.* (2008)). Boxplots of the various estimates of μ and λ appear in Figures 2 and 3, respectively, for the case μ = 2, λ = 5 mm^{-1}, N_T = 50. The truncated ML estimates, denoted ML(T) in the figures, of both μ and λ for this case show considerably less bias than the ordinary ML estimates. In particular, note that the median values of the truncated estimates of each parameter are very nearly equal to the population values while the ordinary estimates are nearly always overestimates. Figures 4 and 6 and Tables 1, 2, 3 and 4 demonstrate the trends with increasing sample sizes. As N_T increases these patterns persist for both μ = 2, λ = 3 mm^{-1} and μ = 2, λ = 5 mm^{-1}; we continue to see little bias for the truncated method and strong positive bias with the ordinary ML method. With both ML methods the amount of variability in the estimates decreases with increasing N_T. As the theoretical value of the mean squared error of an estimate is its variance plus the square of its bias, the sample root mean squared (RMS) error values shown for our ML estimates in Figures 5 and 7 are not surprising. In particular, note that for both μ and λ the RMS values for the truncated estimates become smaller and smaller as N_T increases - reflecting the small bias and decreasing variability. With the ordinary ML method the RMS values do decrease somewhat with increasing values of N_T, but apparently level off at positive values because of the continuing bias even with increasing values of N_T.

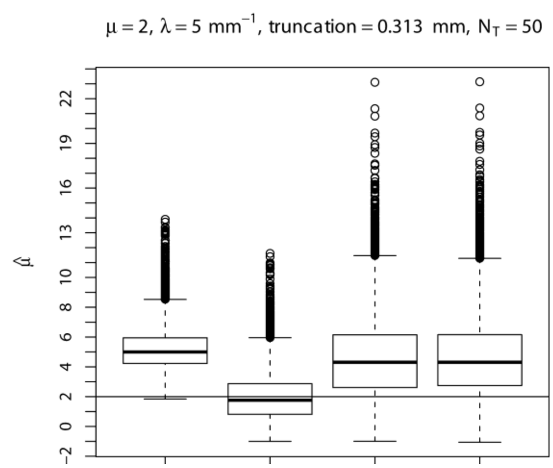

Figure 2. Comparative boxplots of estimates of μ in the case μ = 2, λ = 5 mm^{-1}, N_T = 50. ML and ML(T) denote ordinary and truncated ML estimators; MM234 and MM234(T) denote ordinary and truncated moment estimators based on sample moments 2, 3 and 4.

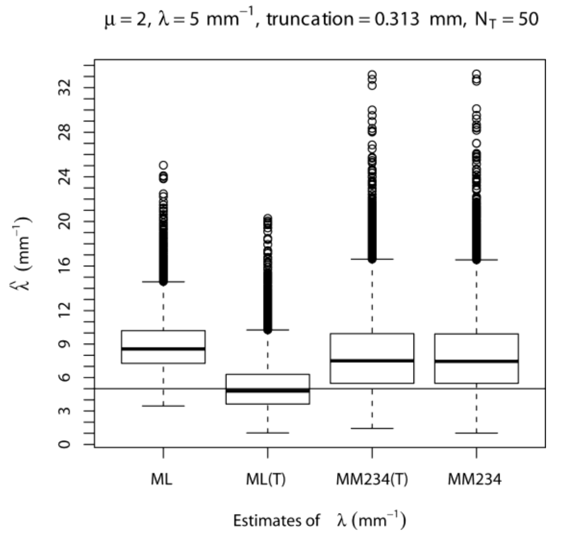

Figure 3. Comparative boxplots (as in Fig. 2) of estimates of λ in the case μ = 2, λ = 5 mm^{-1}, N_T = 50.

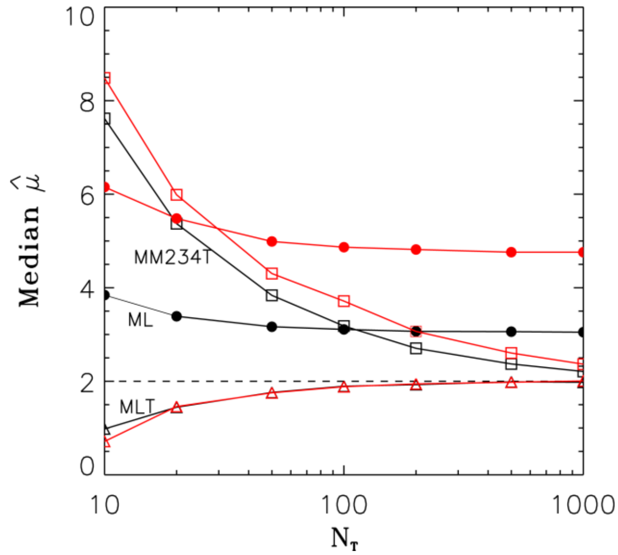

Figure 4. Median estimates of μ for both ML procedures and the truncated MM procedure as a function of N_T. Filled-in circles are used for the ordinary ML procedure, triangles are used for the truncated ML procedure and squares are used for the truncated MM procedure. The case μ = 2, λ = 5 mm^{-1} is shown in red, and the case μ = 2, λ = 3 mm^{-1} is shown in black.

The larger bias and larger variability for the ordinary ML method compared to the truncated ML method are less pronounced for the μ = 2, λ = 3 mm^{-1} case than for the μ = 2, λ = 5 mm^{-1} case. This is a consequence of a smaller fraction of drops being truncated in the former case. Compare Tables 1 and 2 with Tables 3 and 4 in this regard.

We now turn to the method of moment estimators of μ and λ using the second, third and fourth moments of drop sizes, with the ordinary method of moment estimators (denoted MM234 in the figures) given in (6) and the truncated method of moment estimators (denoted MM234(T)) given by minimizing (8) along with (7). The truncated method tends to slightly outperform the ordinary method – though the difference is not nearly as striking as that for the ML methods. In the boxplots in Figures 2 and 3 for the case μ = 2, λ = 5 mm^{-1}, N_T = 50, for example, we see comparable performance in the estimates. As N_T increases, whether for these values of μ and λ or μ = 2, λ = 3 mm^{-1}, the biases and variabilities of both methods decrease with slightly smaller biases and variabilities generally being associated with the truncated MM procedure. Figures 4 and 6 show how the median values of the truncated MM estimates approach the population values as N_T increases.

A few words comparing the ML estimates with the MM estimates are in order. Consider again Figures 2 and 3 involving the case μ = 2, λ = 5 mm^{-1}, N_T = 50. Here, for the modest sample size N_T = 50 we see that the truncated maximum likelihood procedure outperforms the method of moments procedures both in terms of bias and variability. This was generally seen to be true for a variety of values of μ and λ. Furthermore, while the bias for the truncated ML procedure and the two method of moments procedures each apparently go to zero for both μ and λ as N_T increases, the variabilities in the truncated ML estimates are substantially less than those for the method of moment procedures.

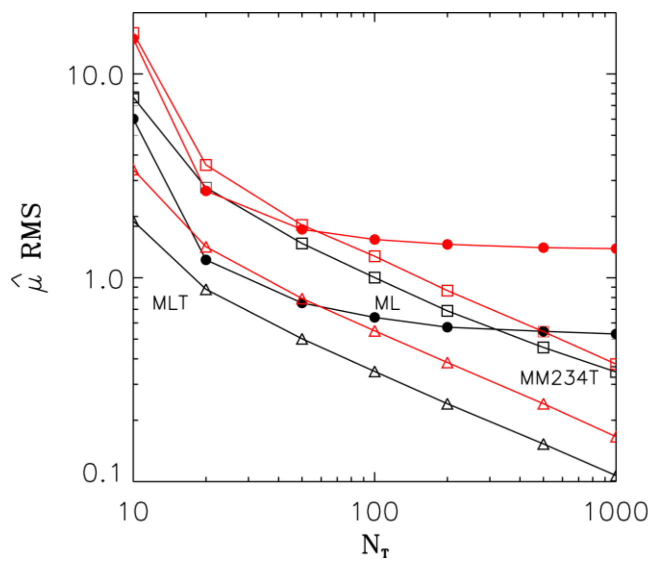

Figure 5. Root mean squared (RMS) error values of μ estimates for both ML procedures and the truncated MM procedure as a function of N_T. Filled-in circles are used for the ordinary ML procedure, triangles are used for the truncated ML procedure and squares are used for the truncated MM procedure. The case μ = 2, λ = 5 mm^{-1} is shown in red, and the case μ = 2, λ = 3 mm^{-1} is shown in black.

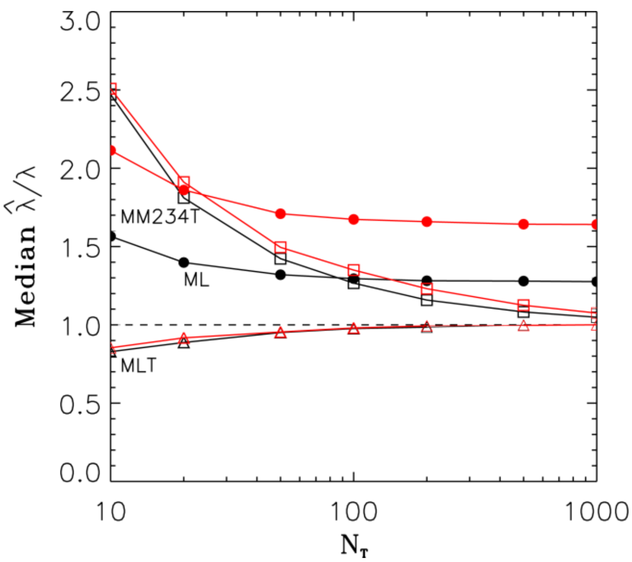

Figure 6. As Fig. 4, showing median estimates of $\hat{\lambda}/\lambda$ for both ML procedures and the truncated MM procedure as a function of N_T.

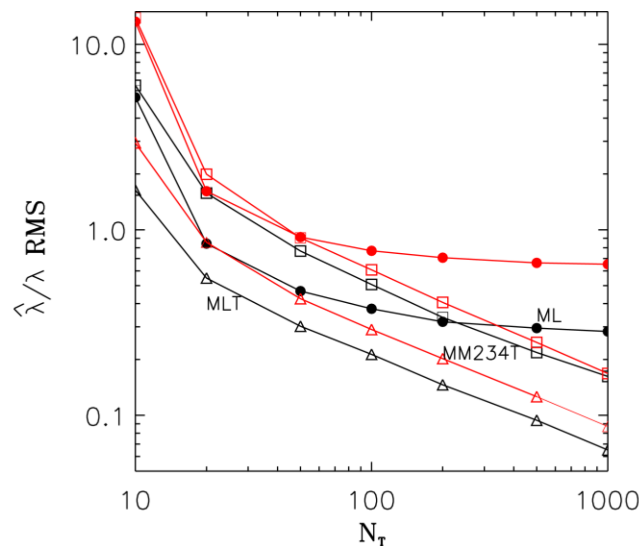

Figure 7. As in Fig. 5, root mean squared (RMS) error values of $\hat{\lambda}/\lambda$ estimates are given for both ML procedures and the truncated MM procedure as a function of N_T.

Table 1. Performance of μ estimators as a function of sample size, for both ML procedures and both MM procedures in the case μ = 2, λ = 3 mm^{-1}.

	ML	**ML(T)**	**MM234(T)**	**MM234**
N_T	Mean; Median; RMS	Mean; Median; RMS	Mean; Median; RMS	Mean; Median; RMS
10	5.408; 3.843 ; 6.019	1.692; 0.980; 1.899	9.917; 7.618 ; 7.678	10.110 ; 7.827; 7.656
20	3.747 ; 3.389 ; 1.224	1.686 ; 1.445; 0.878	6.029 ; 5.370 ; 2.770	6.015 ; 5.349 ; 2.768
50	3.273 ; 3.165 ; 0.752	1.853 ; 1.761 ; 0.502	4.048 ; 3.837 ; 1.474	3.926 ; 3.765 ; 1.461
100	3.156 ; 3.108 ; 0.639	1.934 ; 1.893 ; 0.347	3.289 ; 3.181 ; 1.002	3.172 ; 3.123 ; 0.999
200	3.084 ; 3.067 ; 0.572	1.952 ; 1.928 ; 0.241	2.750 ; 2.704 ; 0.688	2.694 ; 2.677 ; 0.703
500	3.067 ; 3.060 ; 0.546	1.997 ; 1.987 ; 0.153	2.383 ; 2.370 ; 0.455	2.376 ; 2.395 ; 0.471
1000	3.047 ; 3.049 ; 0.530	1.988 ; 1.978 ; 0.107	2.204 ; 2.209 ; 0.345	2.214 ; 2.231 ; 0.362

Table 2. Performance of λ estimators as a function of sample size, for both ML procedures and both MM procedures in the case μ = 2, λ = 3 mm^{-1}.

	ML	**ML(T)**	**MM234(T)**	**MM234**
N_T	Mean; Median; RMS	Mean; Median; RMS	Mean; Median; RMS	Mean; Median; RMS
10	6.545; 4.695 ; 5.177	3.438; 2.486 ; 1.637	9.967; 7.414 ; 6.001	10.080 ; 7.505 ; 5.922
20	4.603 ; 4.194 ; 0.842	3.000 ; 2.662 ; 0.549	6.167 ; 5.432 ; 1.576	6.142 ; 5.406 ; 1.575
50	4.077 ; 3.960 ; 0.467	2.969 ; 2.853 ; 0.302	4.508 ; 4.268 ; 0.768	4.409 ; 4.195 ; 0.760
100	3.945 ; 3.885 ; 0.375	2.988 ; 2.929 ; 0.213	3.925 ; 3.799 ; 0.507	3.833 ; 3.738 ; 0.505
200	3.866 ; 3.844 ; 0.319	2.978 ; 2.955 ; 0.146	3.522 ; 3.474 ; 0.337	3.473 ; 3.449 ; 0.343
500	3.845 ; 3.838 ; 0.295	3.005 ; 2.995 ; 0.094	3.266 ; 3.247 ; 0.218	3.250 ; 3.250 ; 0.225
1000	3.831 ; 3.827 ; 0.283	2.998 ; 3.000 ; 0.065	3.144 ; 3.145 ; 0.162	3.139 ; 3.153 ; 0.170

Table 3. Performance of μ estimators as a function of sample size, for both ML procedures and both MM procedures in the case $\mu = 2$, $\lambda = 5\ mm^{-1}$.

	ML	**ML(T)**	**MM234(T)**	**MM234**
N_T	Mean; Median; RMS	Mean; Median; RMS	Mean; Median; RMS	Mean; Median; RMS
10	9.629 ; 6.154 ; 14.871	2.329 ; 0.713 ; 3.397	12.550 ; 8.486 ; 15.884	13.090 ; 9.058 ; 15.426
20	6.172 ; 5.480 ; 2.668	2.032 ; 1.459 ; 1.419	6.975 ; 5.987 ; 3.577	7.310 ; 6.284 ; 3.680
50	5.176 ; 4.989 ; 1.727	1.926 ; 1.753 ; 0.791	4.521 ; 4.303 ; 1.815	4.583 ; 4.300 ; 1.825
100	4.951 ; 4.866 ; 1.541	1.965 ; 1.884 ; 0.549	3.746 ; 3.714 ; 1.275	3.675 ; 3.598 ; 1.247
200	4.852 ; 4.817 ; 1.458	1.981 ; 1.943 ; 0.384	3.164 ; 3.066 ; 0.865	3.173 ; 3.177 ; 0.914
500	4.784 ; 4.760 ; 1.405	1.997 ; 1.979 ; 0.241	2.662 ; 2.603 ; 0.544	2.708 ; 2.731 ; 0.612
1000	4.764 ; 4.759 ; 1.390	2.000 ; 2.006 ; 0.166	2.347 ; 2.365 ; 0.377	2.528 ; 2.551 ; 0.449

Table 4. Performance of λ estimators as a function of sample size, for both ML procedures and both MM procedures in the case $\mu = 2$, $\lambda = 5\ mm^{-1}$.

	ML	**ML(T)**	**MM234(T)**	**MM234**
N_T	Mean; Median; RMS	Mean; Median; RMS	Mean; Median; RMS	Mean; Median; RMS
10	16.920 ; 10.570 ; 13.238	7.074 ; 4.267 ; 2.939	20.310 ; 12.540 ; 13.870	20.750 ; 13.120 ; 13.252
20	10.580; 9.305 ; 1.614	5.648 ; 4.585 ; 0.856	11.330 ; 9.556 ; 1.993	11.630 ; 9.837 ; 2.043
50	8.913 ; 8.547 ; 0.913	5.115 ; 4.771 ; 0.425	7.962 ; 7.474 ; 0.904	7.943 ; 7.426 ; 0.906
100	8.536 ; 8.367 ; 0.770	5.057 ; 4.901 ; 0.290	6.981 ; 6.748 ; 0.608	6.810 ; 6.633 ; 0.592
200	8.374 ; 8.294 ; 0.707	5.030 ; 4.966 ; 0.202	6.309 ; 6.153 ; 0.406	6.207 ; 6.152 ; 0.420
500	8.252 ; 8.213 ; 0.663	5.011 ; 4.982 ; 0.126	5.722 ; 5.624 ; 0.247	5.660 ; 5.653 ; 0.270
1000	8.230 ; 8.206 ; 0.652	5.010 ; 5.001 ; 0.087	5.369 ; 5.376 ; 0.168	5.458 ; 5.447 ; 0.191

5. CONCLUSIONS FOR FITTING GAMMA DISTRIBUTIONS

When observing truncated data, the truncated maximum likelihood procedure is recommended over the two method of moments procedures given above. With small to moderate sample sizes the method of moments procedures have both larger bias and larger variability. With large sample sizes the bias of the method of moment procedure may be acceptably small, but we do not recommend method of moments estimates even in this situation because of much larger estimate variability.

The ordinary maximum likelihood procedure is not recommended for fitting DSDs in any situation where data truncation is thought to be a possibility. In this case the estimates of the shape and scale parameters are biased, with the amount of the bias increasing with the amount of truncation.

6. PUBLISHED WORK ON TRUNCATED GAMMA ML EQUATIONS

As this paper was being written the article of Mallet and Barthes (2009) appeared. These authors present a similar modification of maximum likelihood estimators for μ and λ to account for data truncation. In fact, Mallet and Barthes (2009) account for both lower and upper truncation of the sample. In this case the truncated density is not $f(x)/[1 - F(D_{\min})]$ as pointed out in Section 2 above but, using obvious notation, $f(x)/[F(D_{max} - F(D_{\min})]$.

While upper truncation may be readily incorporated into both (4) and (8) above, we note that the theoretical fraction of observations above the upper limit of $D = 8$ mm considered by Mallet and Barthes (2009) is quite small for the cases we considered above. In particular, in the case $\mu = 2$, $\lambda = 5\ mm^{-1}$, the theoretical fraction of drops above 8 mm is about 3.6×10^{-15}; in the case $\mu = 2$, $\lambda = 3\ mm^{-1}$, the theoretical fraction of drops above 8 mm is about 1.2×10^{-8}.

Mallet and Barthes (2009) provide few details about the implementation of their numerical procedures. Their work is formulated to deal specifically with data from surface disdrometers, while our expressions apply directly to volume samples. If the drop fall speeds are approximated by a power-law relationship $v_t(D) = \alpha D^{\beta}$, the ML(T) estimators based on (4) can be used with disdrometer data to estimate $\mu' = (\mu + \beta)$ and λ. For volume samples we assume a Poisson distribution and use the sample size, modified as necessary by the truncated fraction, as the ML estimator for N_T; for surface samples a more complicated estimator is required for either N_T or N_0^*.

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APPENDIX – R CODE

For the below, it is understood that

$$f = \hat{D}_{\min} \equiv \min(D_1, D_2, \cdots, D_C)$$
$$g = \frac{1}{C}\sum_{i=1}^{C} \ln D_i$$
$$h = \frac{1}{C}\sum_{i=1}^{C} D_i = \bar{D}$$

Also, the *pgamma* and *lgamma* functions, namely

$$pgamma(x,a) = \frac{1}{\Gamma(a)}\int_0^x t^{a-1}e^{-t}\,dt = \frac{\gamma(a,x)}{\Gamma(a)}$$

$$lgamma(x) = \ln\Gamma(|x|)$$

are standard functions in R.

a. Function *likelihood* – for use in finding the maximum likelihood estimates of μ, λ for data from a truncated gamma:

```
likelihood <- function(x){
 x1 <- x[1]        # mu
 x2 <- x[2]        # lambda
 if ((x1>-1) & (x2>0))
   {return(log(1.0-pgamma(f*x2,x1+1))-(x1+1)*log(x2)+lgamma(x1+1)-g*x1+h*x2)}
 else
   {return(Inf)}
}
```

b. Function *MM234* – for use in finding method of moments estimates of $\tilde{N}_T, \mu, \lambda$ for data from a truncated gamma:

```
MM234 <- function(x){
 x1 <- x[1]        # mu
 x2 <- x[2]        # lambda
 x3 <- x[3]        # N
 if ((x1>-1) & (x2>0))
  {return(   ( 1.0 - (x3/M2)*((x1+1)/x2)*((x1+2)/x2)*
                     (1-pgamma(f*x2,x1+3))/(1-pgamma(f*x2,x1+1))  )^2 +
                  ( 1.0 - (x3/M3)*((x1+1)/x2)*((x1+2)/x2)*((x1+3)/x2)*
                      (1-pgamma(f*x2,x1+4))/(1-pgamma(f*x2,x1+1))  )^2 +
                  ( 1.0 - (x3/M4)*((x1+1)/x2)*((x1+2)/x2)*((x1+3)/x2)*((x1+4)/x2)*
                      (1-pgamma(f*x2,x1+5))/(1-pgamma(f*x2,x1+1))  )^2   )}
else
  {return(Inf)}
}
```