P10.8 Investigations of Cai's Power Law for Strong Tornados

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1. INTRODUCTION.

For tornadic thunderstorms one can estimate the relationship between vorticity and the length scale using Doppler radar data. Case studies have indicated that there may be values of exponents that are strongly correlated with thresholds of tornado activity. In a recent paper, Cai (2005) defines the pseudovorticity by $\zeta_{pv} = \Delta V / L$ where $\Delta V = \left|(V_r)_{\max} - (V_r)_{\min}\right|$ is the difference between the maximum and minimum radial velocity of the mesocyclone (rotating updraft) and L is the distance between them. Cai filtered the data from the highest resolution to the smallest resolution determined by the diameter of the mesocyclone. He filtered the radar data to obtain data sets corresponding to different length scales (ε is the finest resolvable scale of the filtered radar data). By filtering, he obtained data points $(\ln(\varepsilon),\ \ln(\zeta))$. He plotted the points and obtained the best linear fit. Cai's study comparing mobile Doppler radar data from tornadic and non-tornadic storms indicates that the steeper slopes (smaller negative values) are indicative of tornadic storms. As those mesocyclones that produced tornados become stronger approaching tornadogenesis the slope of the line decreased. Cai found the threshold for strong tornados was slope m = -1.6. For tornadic mesocyclones, this suggests a power law of the form, $\zeta \propto r^{-b}$, where r is the radius of the vortex. Cai observed that the exponent can be thought of as measuring a fractal dimension associated with the vortex. For dimension associated with the vortex. For high-resolution mobile Doppler radar data, there has been some attempt to interpret this as a giving a power law for the drop-off of the velocity as a function of the radius of the vortex. Using Mathematica we revisited Serrin's model (Serrin 1972) and attempted to find solutions to the Navier Stokes equations in spherical coordinates, with r^b where *b* is not necessarily -1. We also considered solutions to the Euler equations as well. Recent studies of radar data (Markowski 2008) and numerical simulations [Straka 2007) have produced arching vortex lines in the rear flank of supercell storms. These appear to be correlated with tornado genesis. As more and more vortex lines enter this region, viscous interactions between neighboring vortex lines would lead to mergers. This should also lead to a strengthening of the vortex and increase in the vorticity as well. Several theories have been given for the production of the arching vortex lines (Markowski 2008; Straka 2007). One theory involves the production of vorticity along the edge of the rear flank gust front. These vortex lines are captured by the updraft and tilted vertically. As this tilting occurs, stretching creates a pressure drop in the near ground vortex that draws air down and pins the vortex to the ground. In a second theory the origin of the vorticity is in the shear between the mesocyclone updraft and the rear-flank downdraft. The vortex line is pulled in opposite directions, up by the updraft into the mesocyclone, and down to the ground on adjacent sides by the downdraft creating an arching vortex. It seems reasonable to assume a combination of these two processes should be present, with a reconnection of vortex lines produced by the two different processes. However we doubt tornados or the processes that give rise to them discriminate against vorticity because of its origin, therefore there may be other sources as well. We believe that the fractal dimension of 1.6 that Cai found in his study and have also been observed in several other studies comes from the roll-up in shear regions due to Kelvin-Helmholtz instability.

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2. TORNADOGENESIS

Observational analysis of videos of the tornadogenesis phase in large diameter tornados forming under low cloud bases, suggests the arching vortex lines which enter the developing tornado make a partial revolution about the ambient tornado vortex before 'kinking'. This 'kinking' is necessary to conserve energy as the vortex stretches and/or interacts with other vortices (Chorin 1994). In this process some energy is transmitted to much smaller scales, the so-called inertial range, beginning the Kolmogorov cascade to the viscous range and then dissipating as heat. But much energy is transmitted to the ambient vortex as kinetic energy of the flow and this increases the vorticity of the tornado. As more and more vortices successively enter the developing tornado this process repeats itself many times gradually increasing the vorticity of the tornado vortex. This raises the vorticity of the ambient vortex (assuming the vortex lines are produced uniformly) eventually achieving quasi-equilibrium with its environment. This process transfers energy from the smaller scales to the larger scales (cascades). As more vortex lines enter the ambient tornado vortex, large vortices tend to form; the stronger vortices at the core and slightly weaker vortices wrapping around them. This could manifest itself as multiple vortices or as a large single vortex.

In a study of Kelvin-Helmholtz instability, Baker and Shelly (1990), considered a thin vortex layer of thickness *h* in a two dimensional model. Above the layer and below the layer the fluid flowed in opposite directions. As the flow proceeded the region between the two layers rolled up into a double-branched spiral shaped vortex, with an approximately elliptical cross-section (Kirchhoff ellipse) at its core. They identified a relationship between the thickness of the vortex sheet, *h*, and the cross sectional area of the vortices, *A*. They found *A* scales like $0(h^{1.55})$ as *h* goes to 0, specifically, $A = 8.58\ h^{1.55}$. They observed the roll-up eventually changes the structure of the flow so that no new vorticity gets added to the core of the roll-up. They also comment that once the cores have formed they will interact with one another and in some cases form larger structures. From our perspective, the two dimensional model gives the cross sections of the vortex lines (tubes) that eventually arch and amalgamate together to form the tornado. For persistent shear layers, after a sequence of roll-up's, a sequence of new vortices would form. The new vortices would lie in a linear sequence in a thin vortex layer that would itself have local roll-ups just as the earlier layer did. Locally, the vortices from the first roll-up would wrap-around each other and roll-up into a new vortex. Thus forming a sequence of new-generation roll-up vortices made up of roll-up's of earlier generation roll-up vortices. By the Baker and Shelley result, the new roll-up vortices would satisfy the condition, *A* scales like $0(h^{1.55})$ were *h* is the thickness of the new layer. If the above process occurs repeatedly the result would be a self-similar fractal vortex. The vortex thus produced would be geometrically self-similar and would have the property that *A* scales like $0(h^{1.55})$. If the thin vortex layer bends, the layer locally looks like a plane and the above roll-up process will continue. This process creates structure in the cross-sections of the vortices that lie within the vortex layer. We assume that as the vortex layer arches and is lifted by the updraft and pinned or pulled down by the downdraft, these vortex lines within the vortex sheet become the arching vortex lines. As the vortex layer wraps up it begins to form the tornado vortex which gradually strengthens as the roll-ups continue. One can regard this process as an energy cascade from smaller to larger dimensions.

Another possible source for the fractal dimension of a tornadic vortex is the fractalization of vortex lines due to intense stretching by the updraft. While fractalization of vortex-lines has not been observed, turbulence theory suggests as the vortex lines are stretched they kink, becoming fractalized (Chorin 1994). This process acts along the axis of the vortex line.

If the above two processes occur simultaneously we may approximate this with a compound process by breaking the processes up into an alternating sequence, first the roll-up process then the stretching process, etc. We hypothesis that the above roll-up processes take place at many different scales, including within the tornado vortex itself.

3. VORTEX GASES The interaction of large numbers vortices in two and three-dimensional space has been studied by modeling the vortices as part of a vortex gas. This theory has its origins in the works of Helmholtz [H] and Kelvin [K] in the 1800's. The theory is the analogue of the classical statistical mechanics of gases, which attempts to explain the macroscopic behavior of gases by using the statistics of microscopic modeled behavior of molecules. In the vortex gas case the molecules are replaced by vortices. These could be arching vortex lines (tubes). Just as in the case of gases there is a notion of entropy and a notion of temperature. Onsager (1949) first suggested the notion of temperature for vortex gases. In this theory, negative temperatures are hotter than positive temperatures. Vortices with negative temperatures are smooth. Those with positive temperature are kinked. Those with infinite temperature are fractal. The infinite temperature is between positive and negative temperature. The closer a negative temperature is to zero the warmer it is. From this point of view, a tornadic vortex that begins with kinked up or fractal vortices in it and gradually over time forms into a cylindrical vortex with smooth vortices, would be heating up. Initially smooth vortices that kink up are cooling down. This is what happens to vortices that are stretched. The stronger (hotter) a vortex is, the more resistant it is to kinking up when it is stretched. For a discussion of these ideas see Chorin (1994, 1993). As smooth slender vortices enter the developing tornado vortex, they are being stretched and are cooling down. They kink up, and in doing so they lose energy in the form of kinetic energy to the mean flow of the developing tornado. This adds to the internal energy of the tornado. While the smooth slender vortex "cools" down the ambient tornado "heats" up. As this process repeats itself many times the tornado eventually achieves quasi-equilibrium with its environment.

Lions and Majda (2000) developed a three-dimensional equilibrium statistical mechanics theory for N vortex filaments that allows self-stretching. Each of N filaments has a parameterization of the form

$$(x_i(\sigma,t), y_i(\sigma,t)),$$

where σ parameterizes the asymptotic center of the filament. They found a Hamiltonian system governing the behavior of the vortices, consisting of several terms, which we simplify to two terms: a self-stretch term,

$$H_1$$

and an interaction term,

$$H_2$$

If we assume that there are small intervals of time and local regions in space during which equilibrium is achieved then we may use the Lions' and Majda's theory locally in space and time. Thus for the non-linear Hamiltonian we obtain two sets of equations: one part containing the stretching term,

$$H_1$$

and the other part containing an interaction term,

$$H_2$$

The governing equations for the stretching part are,

$$\frac{dx_i}{dt} = \frac{\partial H_1}{\partial y_i}, \; \frac{dy_i}{dt} = -\frac{\partial H_1}{\partial x_i}.$$

And for the interaction part,

$$\frac{dx_i}{dt} = \frac{\partial H_2}{\partial y_i}, \; \frac{dy_i}{dt} = \frac{\partial H_2}{\partial x_i}$$

The solution,

$$X_t$$

for the combined system,

$$\frac{dx_i}{dt} = \frac{\partial H}{\partial y_i}, \; \frac{dy_i}{dt} = \frac{\partial H}{\partial x_i}, \text{ where } H = H_1 + H_2$$

can be expressed in terms of the solutions for the stretching part,

$$X_{1,t}$$

and for the interaction part,

$$X_{2,t}$$

This can be done using the Trotter product formula,

$$X_t = \lim_{n\to\infty}(X_{1,t/n} \circ X_{2,t/n})^n (x_0, y_0),$$

where the power denotes composition and

$$(x_0, y_0)$$

is the initial condition. Naively, the roll-up discussed above is due to the

$$H_1$$

term, and the stretching is due to the

$$H_2$$

term.

The above limit of compositions can be thought of as the alternating sequence of roll-up process followed by stretching processes repeated many times. Idealy we would want to incorporate stretching due to the updraft and perhaps other effects in the as well in the above equation.

4.CAI'S POWER LAW

The connection between tornadoes and nearly continuously (periodically) produced arching vortex lines is that the vortex lines stir or pump the tornado and increase the vorticity. How frequently vortex lines are produced, their strength, and the stretching of the vortices determine the eventual strength of the tornado. This can be seen from the point of view of the vortex gas theory above. We assume the vortices are all of the same sign (rotation), as the arching vortex lines tend to segregate themselves with the positive and negative collections grouping together, the positive parts forming the cyclonic tornado.

We now give a heuristic argument to support Cai's power law for strong tornados,

$$\zeta \propto r^{-1.6}.$$

From Kelvin's Circulation Theorem vorticity times the cross-sectional area of a vortex tube is constant for Eulerian barotropic flows. Hence,

$$\zeta = \frac{C}{A}$$

were A the cross sectional area of the vortex. Recent numerical and radar studies of tornadic storms suggest that vortex lines produced on the rear flank gust front of a supercell thunderstorm (captured by the updraft) form arches to produce counter-rotating vortices. The vortex lines could also be produced by the shear between the mesocyclone and adjacent downdrafts, and then pulled in opposite directions by the updraft (into the mesocyclone and up) and downdrafts (to the surface). A result of a numerical study of Kelvin-Helmholtz instability by Baker and Shelley (1990), identifies a relationship between the thickness of the vortex sheet h and the cross sectional area of the vortices A. They found that A scales like $O(h^{1.55})$ as h goes to 0. Cai's paper suggests that the tornados are fractal. If the arching vortex lines combine to form the self-similar (fractal) vortex, the self-similarity of the vortex suggests the largest scales are similar to the smallest scales. Therefore, ζ scales like $O(h^{1.55})$.

5. GEOMETRIC SELF SIMILARITY

We have given an argument supporting Cai's power law for strong tornados. Tornados appear to have not only self-similarity of their vorticity, but they also appear to have geometric self-similarity. This appears in Doppler radar and reflectivity data (Bluestein 2000, Nova 2004) and in some high-resolution numerical simulations of tornadic storms (Nova, 2004; Adlerman 2002). The power law suggests a geometric self-similarity in the vortex structure as well. In the discussion above, the doubly branched spiral that results from the roll up in the thin vortex layer, that gives rise to the vortex lines that become the arching vortex lines, is similar in shape to the hook echo that is associated with the mesocyclone or tornado vortex. In two dimensions energy can cascade from smaller scales to larger scales, as tornados and the roll-up vortex tubes have a nearly two-dimensional structure one might expect the small scales to have a strong influence on the formation of the larger scales. The dimension of the geometric vortex can be thought of as 1.6, by the Baker and Shelley result. The initially horizontal vortex sheet containing the vortex lines tilts into the vertical. As the vortex lines arch into the vertical, the sheet then rolls up into the hook echo shaped object associated with the mesocyclone. As the vortex lines role along it they group together into the tornado vortex or pre-tornado vortex. There is also evidence of this in photos (Grazulis 1997, p. 1349). In a one-dimensional study of the roll up of vortex sheets, Chorin (1973) showed that vortex sheets consisting of cyclonically rotating vortices rolled up into a cyclonic vortex. (The roll up in Chorin (1973) should be interpreted as the roll up in Adlerman (2002).) Vortices were spaced along a segment representing the vortex sheet. The rolled up vortex sheet resembled the hook echo region of a supercell thunderstorm. With vortices of opposite sign grouped and placed in the different halves of the segment the vortex sheet rolled up into a cyclonic, anti-cyclonic couplet. This resembled the radar reflectivity couplets that suggest arching vortex lines. The initial arrangement of the vortices is linear along the vortex sheet, as the roll-up takes place the vortices fill out in a circle-bounded region. The circle-bounded region would be the region where it would be appropriate to use the vortex gas theory. In the linear region one might use a shift on a finite alphabet to study the vortex sheet. The coding of a fractal into the shift on a finite alphabet is used in the study of fractal dimension as presented by Edgar (1992). This is a common way to study dynamical systems. In Edgar's book he studies the fractal dimension of the boundary of the Heighway dragon fractal, the dimension is ~1.52. This fractal has a crude resemblance to radar reflectivity image of the hook echo region of a supercell, the hooks on it representing the successive vortices in the vortex sheet, as in (Nova 2004; Chorin 1994; Adlerman 2002). On can think of a 2 dimensional radar image as a Poincare section of the supercell dynamical system. One often studies the dynamical systems associated with Poincare sections using discrete dynamical systems. These systems often take the form of shifts on finite alphabets, like the system studied in Edgars book. There are several other "dragon" fractals that have a resemblance to the radar reflectivity image of the hook echo region of a supercell thunderstorm.

6. SUCTION SPOTS

The vortex line theory we have used here has been used to study the interaction of pairs of cyclonically rotating vortices in the half-plane. The paths of the pairs of interacting vortices (Marchioro 1994, p. 53) form the same type of pattern as the tracks of overlapping suction spots moving through fields as observed by Fujita and others from the air (Grazulis 1997, p. 1379). One can identify the two counter-rotating (anti-cyclonic) mirror vortices in the other half plane as the other ends of the arching vortex lines from the original pair. These would be rotating in the in the opposite orientation.

7. CONCLUSION We believe that the fractal dimension of 1.6 that Cai found in his study and has also been observed in several other studies as well comes from the roll-up in shear regions due to Kelvin-Helmholtz instability. The discrepancy between the 1.6 of Cai and 1.55 of Baker and Shelley could be due to additional contributions from stretching of the vortex and/or fractalization due to the Kolmogorov cascade in the inertial range. We also believe that the model discussed above can unify and explain many of the different phenomena associated with tornados and tornadogenesis (Davies-Jones).

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