P6.8 A NEW LOOK AT EDDY DIFFUSIVITY

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1. Introduction

Despite the well-established equations of motion for Newtonian fluids, complete description of atmospheric and oceanic motion remains an elusive goal due to a vast range of scales involved. In practice, one is only interested in dynamics and kinematics at certain "large" scales in time and space, and hence it is more economical to consider suitably "averaged" equations.

Since averaging relinquishes information at small scales, the effects of unresolved eddies on the resolved motion must be estimated and ultimately parameterized. For tracer transport, this operation is commonly expressed in the flux-gradient relationship

$$\overline{\mathbf{v}'q'} \equiv -\mathbf{K}\nabla\overline{q}\,, \qquad (1.1)$$

where **v** and q are the advecting velocity and advected scalar, bar and prime denote an unspecified Eulerian mean (spatial or temporal) and deviation from the mean (i.e., eddy), and **K** is a second-order tensor.

If the lhs of (1.1) denotes unresolved eddy flux, which is to be modeled by the rhs, then the problem is one of *flux closure*: it amounts to expressing **K** in terms of the mean (i.e., resolved) quantities of the flow. Alternatively, (1.1) can be seen as a diagnostic relation for **K**, given the (resolved) eddy flux and the mean scalar gradient. In this paper, we are primarily concerned about this latter situation in the context of mixing diagnostic.

Even though (1.1) has formal similarity to the Fickian model of molecular diffusion, it by itself does not constrain the nature of eddy flux in any way, diffusive or otherwise: for an arbitrary eddy flux and mean gradient, one can always find **K** that satisfies (1.1). Furthermore, **K** is not unique, since any $\mathbf{K}'$ that satisfies $\mathbf{K}'\nabla\overline{q} = \mathbf{0}$ can be added to **K** without affecting (1.1).

It is easily verified that if **K** is symmetric and positive definite, then the eddy flux is down the mean gradient and in that sense the transport is diffusive. **K** is then a measure of local *eddy diffusivity*. However, as it is well known, for **K** to be *always* downgradient with respect to a given mean state, the flow must satisfy quite stringent conditions, even if q is a passive scalar. Among the exceptional cases wherein **K** is known to be diffusive are: (i) when the amplitude of spatial eddy is small and growing in time (Plumb 1979); and (ii) homogeneous turbulence with (nearly) constant mean gradient (McIntyre 1993). These types of flows satisfy an important prerequisite for the eddy diffusivity concept: separation of scales between eddy and mean gradient (the former through the smallness of eddy amplitude and the latter through the greatness of the variation scale of the mean gradient). This permits the eddy diffusivity to be defined *locally*.

In reality, there is never a clear scale separation in large-scale transport of the atmosphere and ocean. The size of eddy is often comparable to, and even exceeds, the scale of the mean gradient. That is, coherent eddies are embedded in a very inhomogeneous environment. Under such circumstances, **K** fails to be local: the nature of transport may be irreversible, but it involves excursion of fluid parcels over a long distance so the *local* gradient has little bearing on the eddy flux. As a result, the flux can be up or down the gradient, mitigating the utility of eddy diffusivity as a diagnostic of mixing.

The above point is illustrated in Fig.1. A tracer, initially only a function of y, is subjected to advection by a steady 2D flow, $\Psi = \cos x \cos y$, and a constant subgrid-scale diffusion D. As time goes on, the tracer contours are wrapped around the streamlines, and an increasingly complex geometry emerges (Figs.1b-d). [An analytic solution for the inviscid case was given by

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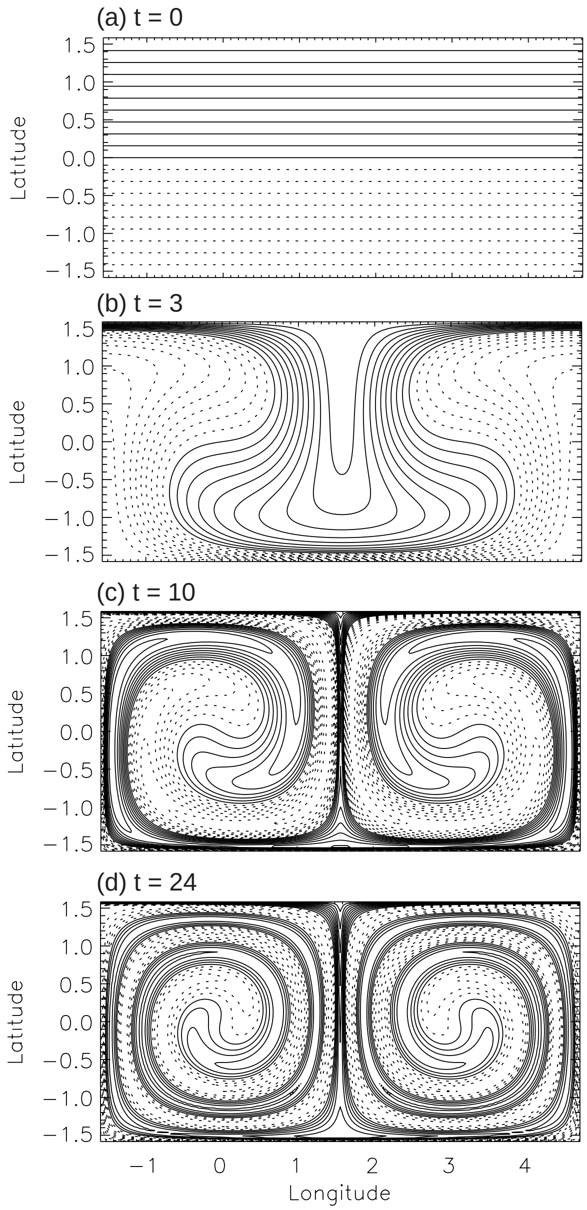

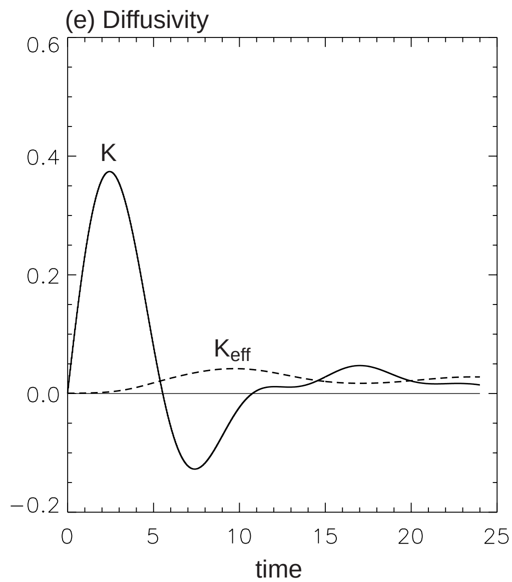

Figure 1 (a)-(d) Passive tracer field advected by steady flow Ψ=cos(x)cos(y) and diffused by constant subgrid difffusion D = 0.0005. The initial condition is a uniform gradient in y, and the concentration is fixed at $y = -0.5\pi, 0.5\pi$ to maintain domain-averaged gradient. (a) t = 0 (b) t = 3 (c) t = 10 (d) t = 24. (e) Time dependence of domain-averaged eddy diffusivity. Solid curve: conventional eddy diffusivity (K). Dashed curve: effective eddy diffusivity (K_{eff}). Although the tracer is irreversibly wrapped into filaments, K swings into negative values due to nonlocal effect. K_{eff} is by definition positive, and increases appreciably as tracer cascades to smaller scales. After the memory of initial condition is lost, the two diffusivities attain similar magnitude, which is about two orders of magnitude larger than the subgrid diffusivity D.

Warn and Gauthier (1989) in terms of Jacobi elliptic functions.] The solid curve in Fig. 1e shows the time dependence of eddy diffusivity *K*, defined as domain-averaged northward eddy flux divided by the domain-averaged southward gradient. This denominator is fixed by prescribed tracer concentrations at the zonal boundaries. Initially *K* is positive and grows, whilst the tracer is dispersed meridionally (Fig.1b). However, as the tongue of tracer wraps around, *K* turns negative at $t \approx 6$. This suggests that transport is reversible to some degree, perhaps not surprisingly considering the periodicity of the flow. On the other hand, change in tracer geometry is unmistakably irreversible. Therefore, this example demonstrates that eddy diffusivity does not necessarily capture the essence of mixing.

One way of avoiding the difficulty of nonlocalness and achieving tighter integration of diffusivity and mixing is to abandon (1.1) and redefine diffusivity with respect to moving fluid mass. In particular, Nakamura (1996) used area enclosed by tracer contour as a coordinate of 2D mixing. The area is invariant under advection by nondivergent flows, so the “transport” in the area coordinate is attributed to nonconservative processes, including unresolved mixing (ultimately molecular diffusion). Nakamura showed that the role of eddy stirring is to enhance tracer gradients and perimeter length of area, thereby enhancing *effective diffusivity*. Effective diffusivity is positive by construction (i.e., the flux is always downgradient) and large where the geometry of tracer contour is complex and small where it is smooth—a more truthful representation of mixing than the traditional eddy diffusivity.

While mathematically rigorous and useful in certain applications (e.g., Nakamura and Ma 1997, Allen et al. 1999, Haynes and Shuckburgh 2000ab, Allen and Nakamura 2001), the Lagrangian-mean nature of effective diffusivity imposes limitations of its own. The area coordinate makes no reference to geographical locations, so it is difficult to speak of "local" diffusivity (or mixing) with respect to the surface of the Earth. Also, effective diffusivity is an average on a closed tracer contour and it says nothing about the variation of diffusivity along the contour.

In view of this, it seems worth revisiting (1.1) and re-formulating Eulerian-mean equation in search for a better measure of local eddy diffusivity *and* mixing.

2. Synopsis and summary

Due to limitation of space, only a brief outline of formalism is presented below. A more complete discussion will be found in Nakamura (2001). We shall decompose eddy flux in (1.1) into along-gradient and cross-gradient components and single out the along-gradient component (that is, component normal to the isosurface of $\overline{q}$). The scalar eddy diffusivity in this dimension is

$$K = \frac{(\nabla\overline{q})^T \mathbf{K} \nabla\overline{q}}{|\nabla\overline{q}|^2}. \tag{2.1}$$

Using advection-diffusion equation for incompressible flow with constant molecular diffusivity D, it is readily shown

$$K + D = K^* + K_{eff}; \tag{2.2a}$$

$$K^* \equiv \frac{\frac{\partial}{\partial t}\overline{q'^2} + \nabla\cdot\overline{\mathbf{v}q'^2} - D\nabla^2\overline{q'^2}}{2|\nabla\overline{q}|^2}; \tag{2.2b}$$

$$K_{eff} \equiv D\frac{\overline{|\nabla q|^2}}{|\nabla\overline{q}|^2}. \tag{2.2c}$$

K^* represents dispersion of tracer contours about their mean positions. The second term of the numerator of (2.2b) includes triple eddy correlation, which is the cause of the nonlocal effect when the amplitude of eddy is large. Hence K^* can take either sign. In contrast, K_{eff} is positive by definition and constructed only from local quantities.

(2.2c) clearly shows that K_{eff} represents the effect of molecular diffusion, but its effectiveness is amplified by the factor $\overline{|\nabla q|^2}/|\nabla\overline{q}|^2$. This ratio, which we call *mixing efficiency*, roughly measures the average square wavenumber of eddies against the square wavenumber of the mean, and hence quantifies complexity of local tracer geometry. When there is no eddy ($q = \overline{q}$), K_{eff} reduces to D, while $K = K^* = 0$. K_{eff} is analogous to the effective diffusivity in the area coordinate (Nakamura 1996), in that it represents small-scale diffusion amplified by the geometrical complexity of tracer.

We argue that K_{eff} is a useful measure of local eddy mixing. The dashed curve in Fig.1e shows the time dependence of domain-averaged K_{eff} associated with the mixing event shown in Figs.1a-d. K_{eff} increases with time as stirring creates increasingly complex tracer geometry. Although the magnitude of K_{eff} is much smaller than $|K|$ or $|K^*|$, it is still about two orders of magnitude larger than the subgrid diffusion D.

Figure 2 shows an example of *mixing efficiency diagnostic*, using the tropopause level potential vorticity and nitrous oxide simulated by GFDL SKYHI GCM for the month of March. In both spatial and temporal means, the diagnostic shows very clearly the existence of minimum mixing efficiency (i.e., mixing barriers) along the axis of jetstreams. The barrier locations are more zonally localized in the Northern Hemisphere than in the Southern Hemisphere.

3. References

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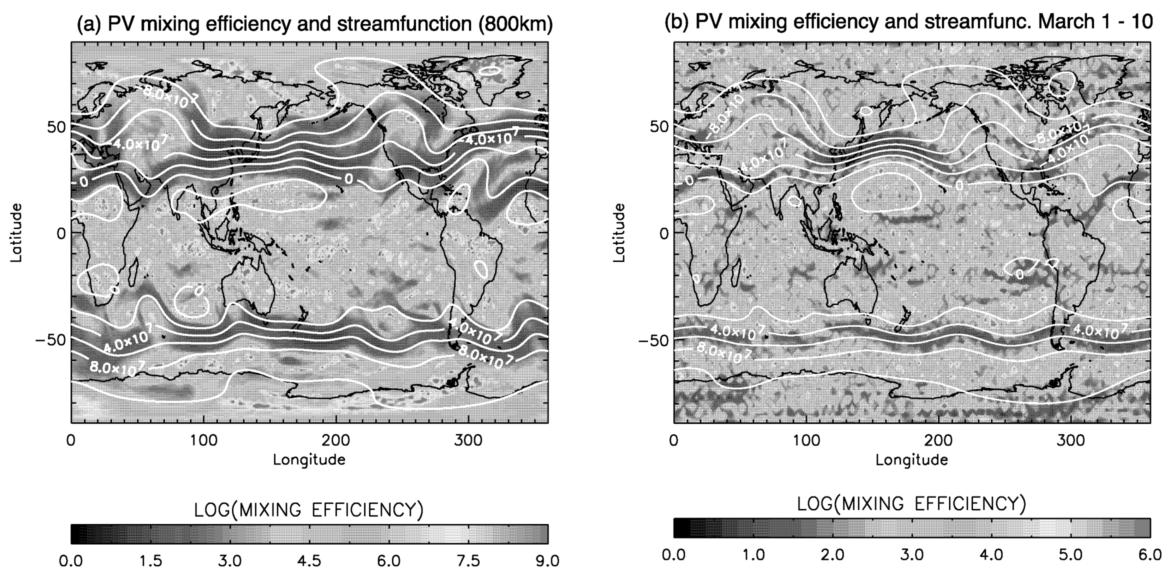

Figure 2 Natural log of PV mixing efficiency (in shades) and streamfunction (solid contours) on 320K isentropic surface of SKYHI GCM. (a) with spatial average applied to instantaneous fields at 12:00 UTC 1 March. At each longitude-latitude grid, average is taken over surrounding grids within radius R = 800km. (b) same as (a) but with temporal average over a 10-day period starting from March 1.