6.7 The Statistical Dynamics of Geophysical Flows

T.J. O'Kane and J.S. Frederiksen
Department of Mathematics, Monash University, Clayton, Victoria, Australia.
CSIRO Division of Atmospheric Research, Aspendale, Victoria, Australia.
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A. Introduction

Although the topography has long been recognized as playing a major role in determining ocean atmosphere circulation, aspects of this interaction remain poorly understood. Recent parametrizations [7] have been used to argue that for ocean circulations the important interaction is not the topographic gravity drag but the interaction between turbulent vorticies and the topography. Although there exist tractable closure theories for flow over topography these have been for ensembles of random topography with zero mean value. These models unfortunately say little about the effect of the mean topography of the ocean or atmosphere on the structures of the mean flows nor can they comment on the problem of parametrizing the effects of the interaction of subgrid-scale turbulent eddies with the mean topography.

In this context we have developed a non-Markovian closure model consisting of direct interaction closure equations for barotropic flow over mean (single realization) topography, with and without non-Gaussian restarts. The closure, established on the basis of a quasi-diagonal direct interaction approximation(QDIA), is compared with ensemble averaged direct numerical simulations (DNS) for severely truncated two-dimensional Navier-Stokes flows. The model incorporates equations for the mean vorticity, vorticity covariance and response functions [4], and is formulated for discrete spectra relevant to flows on the doubly periodic domain. This procedure allows the unambiguous comparison between the closure and the DNS as well as allowing for the incorporation of all interactions both local and non-local. A significant computational efficiency is gained via the periodic truncation of the potentially long time-history integrals where the closure and meanfield equations are restarted using both two and three-point cumulants as new non-Gaussian initial conditions in both the mean and fluctuating fields. The closures and DNS are compared in 80-day integrations employing typical meteorological time and space scales in inviscid, viscous decay and forced dissipative experiments.

email: Terence.OKane@maths.monash.edu.au
email: Jorgen.Frederiksen@dar.csiro.au

The system demonstrates the required conservation laws in the inviscid and forced dissipative systems ie. kinetic energy and potential enstrophy, as well as satisfying the canonical equilibrium relation.

B. Barotropic flow over topography

The evolution equation for two dimensional flow over a fixed topography on a periodic f-plane ($0 \leq x \leq 2\pi$) ,($0 \leq y \leq 2\pi$) with no explicit large-scale flow is simply the nondimensional barotropic vorticity equation which takes the form

$$\frac{\partial \zeta}{\partial t} = -J(\psi, \zeta + h) - \nu_0(k)k^2\zeta + f^0 \qquad (1)$$

where f^0 is the bare forcing. The vorticity is given in terms of the stream function. We make the assumption that the variation in the bottom topography is small and write the vorticity equation in spectral form as

$$\begin{aligned}(\frac{\partial}{\partial t} + \nu_0(k)k^2)\zeta_{\mathbf{k}}(t) = \\ \sum_{\mathbf{p}}\sum_{\mathbf{q}} \delta(\mathbf{k}+\mathbf{p}+\mathbf{q}) \\ \times [K(\mathbf{k},\mathbf{p},\mathbf{q})\zeta_{-\mathbf{p}}\zeta_{-\mathbf{q}} + A(\mathbf{k},\mathbf{p},\mathbf{q})\zeta_{-\mathbf{p}}h_{-\mathbf{q}}] \\ + f_{\mathbf{k}}^0 \qquad (2)\end{aligned}$$

where $k = (k_x^2 + k_y^2)^{1/2}$ and $\zeta_{-\mathbf{k}} = \zeta_{\mathbf{k}}^*$ are conjugate.

The interaction coefficients are governed by the following relationships

$$A(\mathbf{k},\mathbf{p},\mathbf{q}) = -(p_x q_y - p_y q_x)(p^2 - q^2)/p^2 \qquad (3)$$

$$\begin{aligned}K(\mathbf{k},\mathbf{p},\mathbf{q}) &= \frac{1}{2}[A(\mathbf{k},\mathbf{p},\mathbf{q}) + A(\mathbf{k},\mathbf{q},\mathbf{p})] \\ &= \frac{1}{2}(p_x q_y - p_y q_x)(p^2 - q^2)/p^2q^2 \qquad (4)\end{aligned}$$

where

$$K(\mathbf{k},\mathbf{p},\mathbf{q}) + K(\mathbf{p},\mathbf{q},\mathbf{k}) + K(\mathbf{q},\mathbf{k},\mathbf{p}) = 0 \qquad (5)$$

by cyclic symmetry.

For an ensemble of flows satisfying eqn(2) we may express the vorticity in terms of the ensemble mean $\langle\zeta_{\mathbf{k}}\rangle$ and the deviation from the ensemble mean $\hat{\zeta}$ i.e.

$$\zeta_{\mathbf{k}} = \langle \zeta_{\mathbf{k}} \rangle + \hat{\zeta}. \tag{6}$$

hence we may write equations for the ensemble mean and the deviation with forcing $f^0_{\mathbf{k}} = \langle f^0_{\mathbf{k}} \rangle + \hat{f}^0_{\mathbf{k}}$. The single time cumulant is defined as

$$C_{-\mathbf{p}-\mathbf{q}}(t,s) = \langle \hat{\zeta}_{-\mathbf{p}}(t) \hat{\zeta}_{-\mathbf{q}}(s) \rangle. \tag{7}$$

The derivation of the closure equations can be found in Frederiksen [4] and we only present numerical results here. The closure is achieved via a generalisation of an approach due to Kraichnan (1964) [10] allowing the off-diagonal 2-point cumulant ie

$$C_{\mathbf{k}}(t,t') \equiv C_{\mathbf{k}-\mathbf{k}}(t,t') \tag{8}$$

to be represented in terms of diagonal cumulant and response functions and via a similar argument the off-diagonal response function

$$R_{\mathbf{kl}}(t,\acute{t}) = \langle \frac{\partial \hat{\zeta}_{\mathbf{k}}(t)}{\partial \hat{f}_{\mathbf{l}}(\acute{t})} \rangle \tag{9}$$

The most important integral invariants in the system are those that are quadratic in ψ, namely the total kinetic energy and the potential enstrophy. The existence of the quadratic invariants implies stability for the stationary state [1], conditions for which can be readily derived via canonical equilibrium theory.

C. QDIA and CUQDIA closures versus DNS

DNS and closure results are presented for $k_{max} = 3$ with circular truncation of the whole wavenumber space, corresponding to the system having 28 components. Although $k_{max} = 3$ represents severely truncated 2-D turbulence there remain sufficient degrees of freedom such that the systems are mixing [11]. As noted by Frederiksen et al(1994) [5] the use of discrete spectra allows not only the incorporation of all interactions both local and nonlocal but also enables the detection of systematic errors due to the closures with no ambiguity due to different formulations with discrete and continuous spectra. We have used typical meteorological time and space scales. The space scale used is the earths radius $r = 6.37122 \times 10^6 m$ and the time scale of $\sqrt{2}\Omega^{-1}$ where $\Omega = 7.292 \times 10^{-5} s^{-1}$ is the earths angular velocity. The nondimensional time step we have used is 1.11375 corresponding to $1/8^{th}$ of a day.

The size of the timestep is determined largely by the numerical stability of the DNS calculation but also in part due to the need for good energy conservation in the closure equations. The forced dissipative experiments have a random forcing given by

$$F^0_{\mathbf{k}} = 2\hat{\nu} k^2 C^{eq}_{\mathbf{k}} \tag{10}$$

with

$$C_{\mathbf{k}}(t,t) = C^{eq}_{\mathbf{k}} = \frac{k^2}{a + bk^2} \tag{11}$$

where a and b are parameters determined by the energy and potential enstrophy for the inviscid system. The corresponding condition for canonical equilibrium for inviscid flow over topography on the doubly periodic domain is given by

$$\langle \zeta_{\mathbf{k}}(t) \rangle = \langle \zeta^{eq}_{\mathbf{k}} \rangle = \frac{-bk^2 h_{\mathbf{k}}}{a + bk^2} \tag{12}$$

$$= -b h_{\mathbf{k}} C^{eq}_{\mathbf{k}} \tag{13}$$

The required choice of random forcing in the mean field term to ensure that the closures will asymptote to the equilibrium solutions eqns(11 and 13) in the forced viscid case are given by

$$\bar{f}^0_{\mathbf{k}} = \langle f^0_{\mathbf{k}} \rangle = \hat{\nu}_0(\mathbf{k}) k^2 \langle \zeta^{eq}_{\mathbf{k}} \rangle \tag{14}$$

$$= -\frac{1}{2} b h_{\mathbf{k}} F^0_{\mathbf{k}}. \tag{15}$$

The closure equations have Gaussian initial conditions, with the subsequent restarts using non-Gaussian initial conditions via the two- and three-point cumulant terms. The DNS results are averaged over 5000 realizations with the real and imaginary parts of $\zeta_{\mathbf{k}}(t)$ at $t = 0$ having a joint Gaussian distribution.

In the case where the initial mean-field $\langle \zeta_{\mathbf{k}}(0) \rangle$ is zero the resulting mean-field evolution is due entirely to the topography (figure 1). We note that the mean-field develops relatively quickly and is almost at full strength after an integration period of only 10 days. The restart procedure is employed at every 20 timesteps. The comparison between the DNS and CUQDIA shows good agreement with both systems evolving to a stationary state. Between $T = 20 (t = 160)$, and $T = 50 (t = 400)$ days (timesteps) the CUQDIA exhibits slight erroneous oscillations as seen previously in the isotropic homogeneous calculations of Frederiksen et al (1994) [5] and the original work of Rose (1985) [13]. The oscillations are to some extent dependent on step size however are largely due to the restart time T. A doubling of the restart time significantly reduces such oscillations however this is at the expense of computation time. In the absensce of topography an initially non-zero mean-field dissipates as the energy is transfered to the closure equations with the result that as $t \rightarrow \infty$ we are left with a purely fluctuating field.

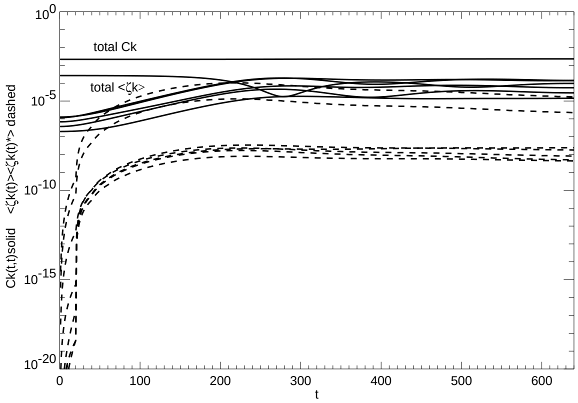

FIG. 1. Due to degeneracy we have displayed only the first 6 spectral components and the total. Restarts every 20 timesteps.
$h_{\mathbf{k}} = 1.0e - 4 \times C_{\mathbf{k}}(0,0)$
$\langle \zeta_{\mathbf{k}}(0) \rangle = 0$
$C_{\mathbf{k}}(0,0) = \frac{k^2}{a+bk^2} = C^{eq}_{\mathbf{k}}$
$a = 4.842 \times 10^4$
$b = 2.511 \times 10^3$

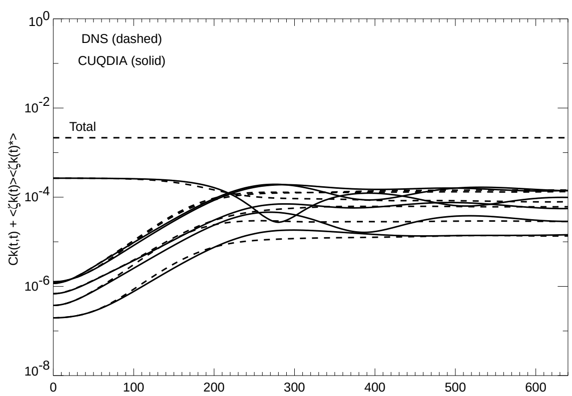

FIG. 2. Comparison of the total field with parameters as for figure1 with DNS.

Holloway and Hendershott [8], Bretherton and Haidvogel (1976) [2] and later Carnevale and Frederiksen [3] pointed out that the enstrophy cascade associated with turbulent eddies in two-dimensions implies that for initially turbulent flow above topography the flow will tend to approach a stable state of minimum enstrophy for a given kinetic energy. In the case where the topography is random this minimum enstrophy state has steady streamlines proportional to the topography on the larger scales but with some smoothing of the finer features. Herring [6] also considered the problem via a simple extension of Kraichnan's isotropic DIA [9] and a heuristic theory based on the test-field model developed by Leith [12] finding also that the vorticity was strongly locked to the topography at large scales.

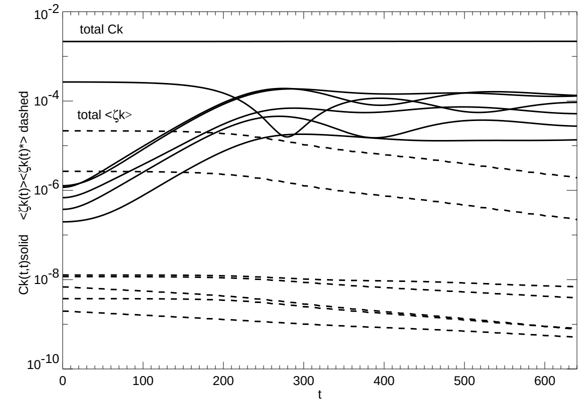

FIG. 3. Restarts every 20 timesteps.
Topography $h_{\mathbf{k}} = 0$
Mean-field $\langle \zeta_{\mathbf{k}}(0) \rangle = 1.0e - 2 \times C_{\mathbf{k}}(0,0)$
Twice enstrophies $C_{\mathbf{k}}(0,0) = \frac{k^2}{a+bk^2} = C^{eq}_{\mathbf{k}}$
$a = 4.842 \times 10^4$
$b = 2.511 \times 10^3$

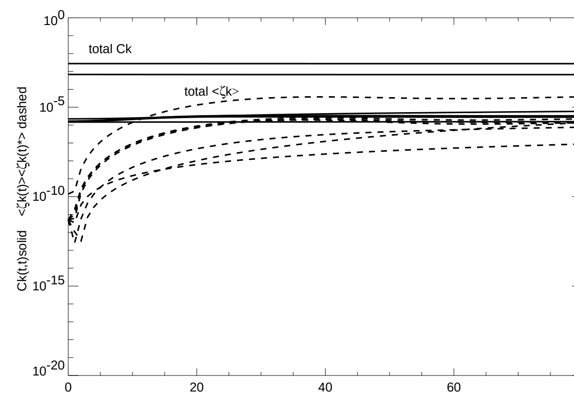

FIG. 4. Without restarts.
Viscosity 3.378×10^{-4}
Twice enstrophies forcing $F^0_{\mathbf{k}} = 2\hat{\nu} k^2 C^{eq}_{\mathbf{k}}$
Mean-field forcing$\langle f^0_{\mathbf{k}} \rangle = \hat{\nu} k^2 \langle \zeta^{eq}_{\mathbf{k}} \rangle$
Topography $h_{\mathbf{k}} h_{-\mathbf{k}} = \sqrt{\frac{a+bk^2}{(kb)^2}}$
Mean-field $\langle \zeta_{\mathbf{k}}(0) \rangle = C_{\mathbf{k}=(1,1)}(0,0)$
Twice enstrophies as before ie $C_{\mathbf{k}}(0,0) = \frac{k^2}{a+bk^2} = C^{eq}_{\mathbf{k}}$ except that $C_{\mathbf{k}=(1,0)}(0,0)$ has been perturbed by a factor of 100.
$a = -5.969 \times 10^5$
$b = 7.444 \times 10^5$

Our results are in good agreement with the idea that the turbulence evolves into a steady (in this case stationary) state in which the flow on the larger scales is along contours of constant $h_{\mathbf{k}}$.

In the viscid forced case (figures 4-6) the mean-field is seen to undergoe a reversal (sign change) in order to reach equilibrium. This is a direct result of the choice of initial conditions and the equilibrium condition eqn(13). As the mean-field in figure 4 is relatively weak the reversal occurs early in the evolution. In figure 5 we have an enhanced initial $\langle \zeta_{\mathbf{k}}(0) \rangle$ and now see a drammatic difference in the time each of the spectral components changes sign.

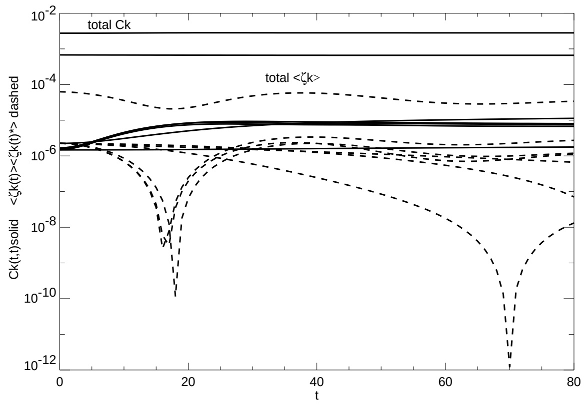

FIG. 5. Conditions as for the previous figure however the strength of the initial mean-field has been significantly enhanced ie Mean-field $\langle \zeta_{\mathbf{k}}(0) \rangle = \sqrt{C_{\mathbf{k}=(1,1)}(0,0)}$

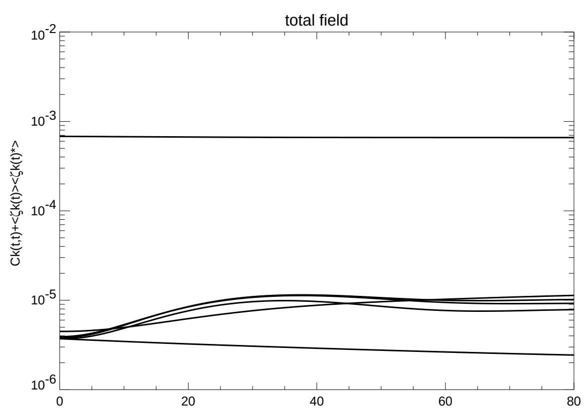

FIG. 6. The total field is displayed with parameters as for figure 5.

D. Conclusion

We have demonstrated the development of a tractable closure model for inhomogeneous flow over mean (single realization) topography. Although the results presented are for low resolution severly truncated Navier-Stokes flow the model obeys canonical equilibrium conditions, and energy conservation requirements. A formal restart procedure has also been developed in the hope of reducing the numerical task. The model run at high resolution offers a tool for the study of the eddy topographic force, eddy viscosity, and stochastic backscatter and this is our current focus.

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