6.3 WIND-INDUCED CONSTRAINTS ON THE MASS AND BUOYANCY TRANSPORTS IN A SIMPLE ADIABATIC MODEL OF THE OCEAN CIRCULATION

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1. INTRODUCTION

In physical oceanography, two streamfunctions are of particular practical importance. These are: 1) the barotropic streamfunction Ψ, related to the vertically-integrated mass transport $\mathbf{U} = \int_{-H}^{0} \mathbf{u}\, dz$ by $\mathbf{U} = \hat{\mathbf{z}} \times \nabla\Psi$; and 2) the meridional overturning streamfunction Φ, related to the zonally-integrated meridional mass transport $(L\langle v\rangle, L\langle w\rangle)$ by

$$L\langle v\rangle = -\Phi_z, \qquad L\langle w\rangle = \Phi_y, \tag{1}$$

where $\langle . \rangle$ is the zonal average. [1] Ψ is important because its determination is a prerequisite to that of the total pressure field, while the importance of Φ lies in its strong link to the oceanic poleward heat transport. These quantities are thus central to the issue of ocean climate.

The main theoretical challenge is to express Ψ and Φ in terms of the external wind and buoyancy forcing. The simplest mathematical model to study Φ and Ψ is arguably composed by the viscous/diffusive planetary geostrophic equations (PGE), which contain as particular cases most mathematical models underlying all large-scale ocean circulation theories:

$$f\hat{\mathbf{z}} \times \mathbf{u} + \nabla p = A_H \Delta \mathbf{u} + A_V \mathbf{u}_{zz} \tag{2}$$

$$p_z - b = 0 \tag{3}$$

$$\mathrm{div}\mathbf{u} + w_z = 0 \tag{4}$$

$$\mathbf{u}.\nabla b + w b_z = \mathcal{Q}, \tag{5}$$

where $\mathbf{u} = (u, v)$ is the horizontal velocity; w is the vertical velocity; $b = g(\rho_0 - \rho)/\rho_0$ is the buoyancy (g being the gravitational acceleration, ρ the density, and ρ_0 a reference density); $p = P/\rho_0 + gz$ is the pressure divided by the reference density plus the geopotential per unit mass; A_H and A_V are the horizontal and vertical turbulent eddy viscosities respectively. $\mathcal{Q}$ represents diabatic effects. In practice, the form $\mathcal{Q} = K_H \Delta b + K_V b_{zz}$ is often assumed, with K_H and K_V horizontal and vertical turbulent diffusivities respectively.

In the past decades, much progress in the theoretical study ofΨ was achieved by so-called wind-driven circulation theories (see Hendershott (1986) and ref. therein). The well-known elliptical problem for Ψ:

$$-A_H \Delta^2 \Psi + \beta \Psi_x = \mathrm{curl}\tau, \tag{6}$$

with $\beta = df/dy$, is amenable to boundary layer methods. One thus write $\Psi = \Psi_I + \Psi_M$, with Ψ_I an interior part, and Ψ_M a western boundary layer part. The classical result for Ψ_I is:

$$\Psi_I = \frac{1}{\beta} \int_L^x \mathrm{curl}\tau \, dx'. \tag{7}$$

In contrast, a similar theory linking Φ to the external wind and buoyancy forcing has yet to be developed. In view of the considerable theoretical difficulties involved, much of our current knowledge of Φ comes from sensitivity experiments performed with coarse resolution General Circulation Models (GCM). Such experiments show Φ particularly sensitive to the diapycnal (vertical) diffusivity K_V. For this reason, there is currently much debate about oceanic diapycnal mixing, because observations suggest that K_V can vary from very small values in the open ocean to large ones near coastal boundaries and bottom topography. Observations thus suggest to seek for a theory of Φ with diabatic effects occurring only within surface and meridional boundary layers, the interior being adiabatic. This idea was investigated recently by means of numerical GCMs running with a vanishing K_V everywhere except near boundaries (Marotzke 1997, Samelson 1997).

In this paper, we seek to develop an analytical model based on the above idea that is sufficiently simple to be analytically tractable. In section 2, the consideration of the zonally-averaged equations

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[1] For simplicity, we restrict our discussion to the case of a rectangular ocean basin, defined by $0 \le x \le L$, $0 \le y \le D$, and $-H \le z \le 0$.

of motion reminds the reader that Φ is largely controlled by the East-West pressure difference. Section 3 establishes the link between Ψ and Φ through the pressure field. In section 4, the issues of thermocline theories and the determination of the density field are briefly reviewed. A method to construct a closed solution satisfying the ideal fluid thermocline theory in the interior is also proposed. In sections 5 and 6, the method is applied to the case of the ventilated thermocline model, for which the meridional overturning streamfunction and buoyancy transports are computed. The latter are shown to be strongly controlled by the wind forcing.

2. ZONALLY-AVERAGED EQUATIONS

A natural starting point for a theoretical study of Φ is the zonally-averaged equations (2-5), given by

$$-f\langle v\rangle + \delta p = A_H \delta u_x + A_H \langle u\rangle_{yy} + A_V \langle u\rangle_{zz} \quad (8)$$

$$f\langle u\rangle + \langle p\rangle_y = A_H \delta v_x + A_H \langle v\rangle_{yy} + A_V \langle v\rangle_{zz} \quad (9)$$

$$\langle p\rangle_z - \langle b\rangle = 0 \quad (10)$$

$$\langle v\rangle_y + \langle w\rangle_z = 0 \quad (11)$$

$$\langle v\rangle\langle b\rangle_y + \langle w\rangle\langle b\rangle_z = \langle Q\rangle - \langle b'v'\rangle_y - \langle b'w'\rangle_z \quad (12)$$

where for an arbitrary function $\mathcal{G}$,

$$\delta\mathcal{G} = [\mathcal{G}(L,y,z) - \mathcal{G}(0,y,z)]/L, \quad (13)$$

while a primes denotes departing from zonal average.

However, this system is not closed. The main difficulty is to parameterize the term δp in (8) in terms of the resolved quantities. This term is particularly important because scaling shows that below the surface Ekman layer, at leading order, $\langle v\rangle \approx \delta p/f$. Given that from (1), one has

$$\Phi = -L\int_{-H}^{z} \langle v\rangle\, dz' \approx -\frac{L}{f}\int_{-H}^{z} \delta p\, dz', \quad (14)$$

it follows that the knowledge of δp is actually central to the theoretical determination of Φ. Thus, unlike Ψ, Φ critically depends upon the stratification through the hydrostatic approximation (3).

In the past decade, there have been attempts to "close" the system (8-12). For instance, Wright and Stocker (1991) suggested a relationship of the form $\delta p \propto \langle p\rangle_y$, neglecting in a first approximation the other unknowns $A_H\delta u_x$, $A_H\delta v_x$, $\langle b'v'\rangle$, and $\langle b'w'\rangle$. This closure is supported by three-dimensional numerical integrations, but like any other proposed closure, it lacks a rigorous theoretical justification.

In theory, it is not clear whether the East-West pressure difference can be computed without actually solving the whole three-dimensional structure of the density field. That horizontal variations are important can be demonstrated in the case of the surface pressure. Indeed, the following section makes it clear that its determination is closely linked to that of the barotropic streamfunction Ψ.

3. THE PRESSURE FIELD

As is well-known, the integration of the hydrostatic approximation (3) over depth from the surface allows to express the pressure field in terms of the buoyancy field up to the unknown surface pressure p_s, i.e., $p = p_s + \int_z^0 b(x,y,\zeta)d\zeta$. Mathematically, the determination of p_s is closely related to that of Ψ. To see it, first integrate the momentum equations (2) over depth, and replace $\mathbf{U}$ by its expression in terms of Ψ. The resulting equations are as follows:

$$-f\Psi_x + \Theta_x = -A_H(\Delta\Psi)_y + \tau_x, \quad (15)$$

$$f\Psi_y + \Theta_y = A_H(\Delta\Psi)_x + \tau_y \quad (16)$$

where $\Theta = \int_{-H}^0 p\,dz = Hp_s + \int_{-H}^0\int_z^0 b(x,y,\zeta)d\zeta dz$. To determine p_s, it suffices to determine Θ. This is done by first integrating (15) over longitude from the eastern boundary. This yields:

$$\Theta = f\Psi + \int_L^x (\tau_x - A_H(\Delta\Psi)_y)\, dx' + \Theta(L,y), \quad (17)$$

where $\Theta(L,y)$ is the value of Θ along the eastern boundary. The latter is determined by integrating (16) over latitude, so that

$$\Theta(L,y) = \Theta(L,y_0) + \int_{y_0}^{y} (\tau_y + A_H(\Delta\Psi)_x)\, dy'. \quad (18)$$

These expressions show that Θ, like Ψ, can be expressed as the sum of an interior plus a western boundary layer part. At leading order, the latter are given by

$$\Theta_I \approx f\Psi_I + \int_L^x \tau_x dx' + \int_{y_0}^{y} \tau_y dy' + \Theta(L,y_0) \quad (19)$$

$$\Theta_M \approx f\Psi_M. \quad (20)$$

The function Θ is often referred to as the Sverdrup function, and plays a fundamental role in thermocline theories. The complete determination of $\langle p\rangle$ from (10) requires $\langle p_s\rangle$. The latter is given by $\langle p_s\rangle = H^{-1}\langle\Theta\rangle - \frac{1}{H}\int_{-H}^0\int_z^0 \langle b\rangle(y,\zeta)d\zeta dz \approx H^{-1}f\langle\Psi\rangle + (HL)^{-1}\int_0^L\int_L^x \tau_x dx' dx + H^{-1}\Theta(L,y) - \frac{1}{H}\int_{-H}^0\int_z^0 \langle b\rangle(y,\zeta)d\zeta dz$. This formula establishes the connection between Φ and Ψ.

4. THERMOCLINE THEORIES, WELL-POSEDNESS AND BOUNDARY CONDITIONS

What controls the east-west difference of the density field? Obviously, one way to answer this question is by knowing what controls the three-dimensional structure of the density field. This issue has been addressed in the past decades by so-called thermocline theories. So far, two main mathematical models have been proposed: 1) The ideal fluid thermocline (IFT) theory by Welander (1959); and 2) the advective-diffusive thermocline (ADT) theory by Robinson and Stommel (1959). Both models use the geostrophic and hydrostatic approximations, as well as the continuity equation (4). They only differ in their approximation of the full conservation equation for the buoyancy (5), the IFT theory using

$$\mathbf{u}.\nabla b + w b_z = 0, \qquad (21)$$

compared with $wb_z = K_v b_{zz}$ for the ADT theory. These theories neglect terms that are important near boundaries, so that they are only valid in the ocean interior. It follows that boundary conditions must be specified for the interior solution. So far, however, this problem has not received a satisfactory solution. This makes it difficult to know how to extend the interior solution to the lateral boundaries, so that δp, and thus Φ, can be computed.

To shed some light on this issue, let us assume that the interior density field satisfy the IFT equations. The latter, as shown by Welander (1971), can be reduced to the the following nonlinear PDE:

$$J(M_z, M_{zz}) + \frac{\beta}{f} M_x M_{zzz} = 0, \qquad (22)$$

where M is such that $u = -M_{yz}/f$, $v = M_{xz}/f$, $p = M_z$, $w = \beta M_x/f^2$ and $b = M_{zz}$. Owing to this simplification, boundary conditions for the interior solution can be imposed on M alone. Among these boundary conditions, the wind forcing imposes:

$$\frac{\beta}{f^2} M_x = w_E, \qquad \text{at} \qquad z = 0 \qquad (23)$$

$$\frac{\beta}{f^2} M_x = 0, \qquad \text{at} \qquad z = -H \qquad (24)$$

$$M_y(L, y, 0) - M_y(L, y, -H) = \tau_y. \qquad (25)$$

Eq. (23) is the classical Ekman pumping condition for the vertical velocity at the basis of the surface Ekman layer. Eq. (24) is simply $w = 0$ at the bottom. Eq. (25) comes from (16) considered along the eastern boundary, neglecting the viscous term. There is some consensus to also impose the surface buoyancy, but arguably this is somewhat artificial because the latter should actually be determined as part of the solution. Furthermore, this is mathematically justified only in places where the Ekman pumping is negative, since otherwise causality principles are violated. Nevertheless, when this is possible, this condition further imposes

$$M_{zz} = b_s(x, y) \qquad \text{at} \qquad z = 0. \qquad (26)$$

By themselves, (23-26) are not sufficient to single out a function M among all those satisfying Eq. (22). The difficulty of specifying further boundary condition for b can be traced back to the difficulty of solving the boundary layer structure of the full problem (2-5) with an imposed functional form $\mathcal{Q} = \mathcal{Q}(b)$ for the diabatic term in the r.h.s of (5). To obtain a tractable problem, we investigate the opposite approach which consists in imposing the functional form of the buoyancy near the boundaries, so that the diabatic term of the r.h.s. of (5) becomes diagnostic. More specifically, given one particular function M satisfying (22) and the boundary conditions (23-26), we assume that the validity of the interior solution for $b = M_{zz}$ extends to the boundaries as well. This is equivalent to state, if we decompose the total velocity field into a geostrophic and ageostrophic parts, i.e., $\mathbf{u} = \mathbf{u}_G + \mathbf{u}_A$, that the equation $\mathbf{u}_G.\nabla b + w_G.b_z$ is valid everywhere. As a result, the diabatic term $\mathcal{Q}$ becomes an entirely diagnostic function of the buoyancy and ageostrophic velocity as follows:

$$\mathcal{Q} = \text{div}(\mathbf{u}_A b) + (w_A b)_z. \qquad (27)$$

Since the ageostrophic terms are important only within the boundary layers, the approach has the desired properties of confining the diabatic effects to the boundary layers only. Such a model is thus consistent with the idea that enhanced mixing takes place near boundaries. Note that by construction the solution satisfies all the boundary conditions on the velocity field, so that an arbitrary wind forcing can be imposed. This can not be done, however, for the buoyancy forcing since diabatic effects are entirely diagnostic through (27). It follows that the meridional mass and buoyancy transports in this model are entirely constrained by the specification of the wind forcing and the density surface. This is in this sense that the title should be understood.

5. A CLOSED LPS SOLUTION

An idealized example illustrating the preceding ideas can be constructed within the framework of the ventilated thermocline model of Luyten et al. (1983). This model is based on the IFT model, but with the continuous stratification replaced by a stackered set of a few homogeneous density layers overlying a resting abyss. The geometry and particular notations are depicted in Fig. 1.

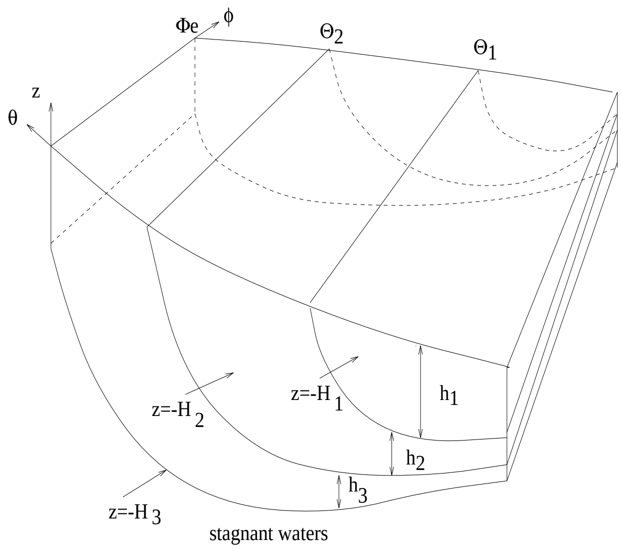

Figure 1: Geometry and notations for LPS model

A particular solution of the LPS model is as follows:

$$H_3(\phi,\theta) = \left(\frac{2\Theta(\phi,\theta)}{\mathcal{G}_3(\theta)}\right)^{1/2} \qquad (28)$$

where $\mathcal{G}_3(\theta)$ is a function that depends uniquely upon latitude. Its expression in the regions $\mathcal{R}_3(\theta_2 \leq \theta \leq \theta_N)$, $\mathcal{R}_2(\theta_1 \leq \theta \leq \theta_2)$, and $\mathcal{R}_3(\theta_S \leq \theta \leq \theta_1)$ is

$$\mathcal{G}_3 = \begin{cases} \gamma_3, & (\mathcal{R}_3), \\ \gamma_3 + \gamma_2(1 - f/f_2)^2, & (\mathcal{R}_2), \\ \gamma_3 + \gamma_2(1 - f/f_2)^2 + \\ \gamma_1(1 - f/f_1)^2(1 - f/f_2^*)^2, & (\mathcal{R}_1), \end{cases} \qquad (29)$$

where the parameter f_2^* ($f_2^* > f_2$) is defined by:

$$f_2^* = \left(1 + \frac{\gamma_3}{\gamma_2}\frac{1}{1 - f_1/f_2}\right) f_2. \qquad (30)$$

The layer thicknesses H_1 and H_2 are expressed as a function of H_3 by $H_2(\phi,\theta) = (1 - f/f_2)H_3(\phi,\theta)$ and $H_1(\phi,\theta) = (1 - f/f_1)(1 - f/f_2^*)H_3$. These expressions represent a concise version of those derived by LPS. For simplicity, the above functional relationships are assumed to hold near the western and eastern boundaries, in places of the homogeneous potential vorticity pool and shadow zone discussed by LPS.

6. MERIDIONAL TRANSPORT OF MASS AND BUOYANCY

The meridional overturning streamfunction Φ is determined by using (14), the pressure in each layer being given by

$$p_1 = \gamma_1 H_1 + \gamma_2 H_2 + \gamma_3 H_3 + (\gamma_1 + \gamma_2 + \gamma_3 - g)z$$

$$p_2 = \gamma_2 H_2 + \gamma_3 H_3 + (\gamma_2 + \gamma_3 - g)z$$

$$p_3 = \gamma_3 H_3 + (\gamma_3 - g)z$$

where $\gamma_i = g(\rho_{i+1} - \rho_i)/\rho_0$ is the reduced gravity associated with the layer i. One verifies that these expressions are continuous across the layer interfaces at $z = -H_1$, $z = -H_2$ and $z = -H_3$. Accordingly, the east-west pressure difference is either a constant or a linear function of depth, so that Φ varies either linearly or quadratically with depth. An example of what can be obtained is depicted in Fig. 2.

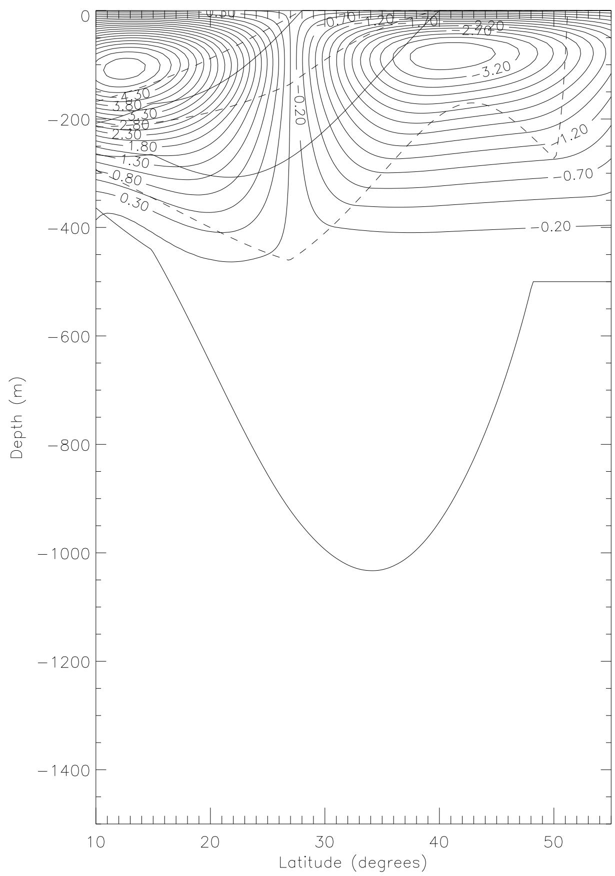

Figure 2: An example of meridional overturning streamfunction in a LPS type model. Superimposed are the isolines for the stratification at the eastern and western boundaries.

In order to compute the diagnostic meridional buoyancy transport from the formula (27), we make the assumption that $\mathcal{Q}$ can be expressed as the vertical divergence of a vertical buoyancy flux, i.e., $\mathcal{Q} = \partial_z \mathcal{F}_q$. As a result, the meridional buoyancy transport must schematically behave as depicted in Fig. 3. Physically, the mass conversion taking place within the Ekman layer must be locally balanced by an incoming surface buoyancy flux. This behavior is consistent with the formula for the meridional heat transport derived by Klinger and Marotzke (2000) for the meridional subtropical gyre in a continuously stratified ocean:

$$H(y) = \rho C_p L \int_{T_0}^{T_S(y)} \frac{\tau_x(T')}{\rho f} dT'$$

where C_p is the heat capacity, and T_S the surface temperature, with the isotherm $T = T_0$ outcropping at the latitude of zero wind stress.

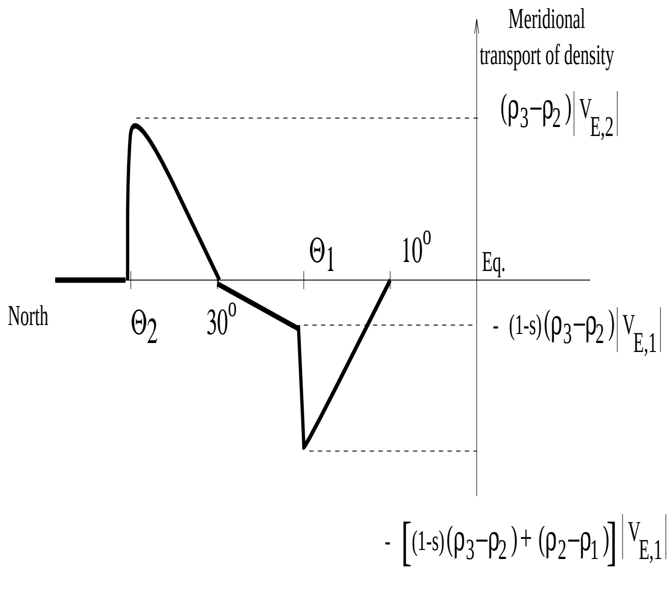

Figure 3: Schematic buoyancy transport associated with the meridional overturning streamfunction of Fig. 2. In this example, the zonal component of the wind stress vanishes at the latitudes $10°N$ and $30°N$. s is a parameter lower than unity depending on the stratification only, while V_E denotes the Ekman transport.

7. DISCUSSION

An analytically tractable model for the meridional overturning streamfunction and meridional buoyancy transport based on the ventilated thermocline theory has been constructed. By construction, diabatic effects are confined solely to the surface and meridional boundaries. This model is thus consistent with an ocean with enhanced mixing near boundaries, as observations suggest. However, because of the idealization of a resting abyss, the model can only model the upper wind-driven overturning cells. As a result, Φ must be of the form

$$\Phi(y, z) \approx G(y, z) \frac{\langle \tau_x \rangle}{f}$$

where $G(y, z)$ is a function of the stratification only. Despite its idealization, this model predicts a meridional buoyancy transport consistent with that obtained in the subtropical gyre by Klinger and Marotzke (2000) in a coarse resolution GCM. Further work is obviously needed to address the more general case.

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