P1.12 DISTINGUISHED LIMITS, MULTIPLE SCALES ASYMPTOTICS, AND NUMERICS FOR ATMOSPHERIC FLOWS

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1. INTRODUCTION

Atmospheric flow theory relies to a large extent on scale analyses and perturbation methods. By exploring the smallness of, e.g., the Froude, Mach, and Rossby numbers $(\mathrm{Fr}, \mathrm{M}, \mathrm{Ro})$, and of various length and time scale ratios these analyses yield simplified asymptotic equations describing the dynamics on specific flow scales. Two observations concerning the use of such simplified equations in numerical simulation codes motivate the present work:

- These simplifications generally arise because of singular degeneracies of the governing equations as the non-dimensional parameters vanish.
- Simplifications that may be well justified for large scale flow simulations cannot be maintained when numerical grid resolutions are increased, and smaller scale phenomena are to be described.

A prominent motivation for adopting simplified equation systems are the expected gains in computational efficiency, which result from reductions in the numbers of dependent variables and equations. However, the very (singular) behavior that makes these simplifications possible in the first place can also lead to severe difficulties in solving the underlying more complete original equations: Numerical truncation errors from the discretization of dominant, nearly balanced terms become comparable with the next order terms that make up the "interesting"dynamics. Prominent examples are spurious winds over steep topography, and unphysical sound wave generation in simulations based on fully compressible dynamics.

With continously increasing available computational capacities higher and higher grid resolutions are becoming affordable. As a consequence, a single computation today already bridges two to three orders of magnitude of resolved horizontal scales. A typical computational run simultaneously includes large scale synoptic dynamics, essentially following quasi-geostrophic theory, medium scale wave phenomena that can be well captured by the hydrostatic primitive equations, and meso-scale non-hydrostatic fluctuations. The resulting trend in meteorological modelling is to move away from simplified model equations that can describe only a limited range of scales and to adopt more comprehensive sets of equations. The current production code, LM, of the *Deutscher Wetterdienst* (German Weather Forecast Service) in fact uses the full three-dimensional compressible flow equations, Doms and Schättler (1999).

The present paper describes a strategy to address the following two issues associated with this trend:

- The scale-related singular dominant balances, such as quasi-geostrophy, near-hydrostatics, and near-anelasticity, are not automatically well represented in numerical solutions of equation sets that do not presume them explicitly.
- The prevailing near-singular solution behavior can severely affect numerical accuracy and efficiency.

These issues arise from undesired interactions of numerical truncation errors with the singular balances that characterize atmospheric flows. Our approach to addressing them proceeds as follows:

1. We consider the full three-dimensional compressible flow equations in the same form we use for numerical discretization, e.g., in conservation form, in primitive variables, or in a vorticity-based form.
2. Exploiting the smallness of the Rossby, Froude, and Mach numbers we introduce a general multiple scales asymptotic ansatz that captures the most relevant atmospheric scale ratios and limit regimes.

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3. The asymptotic analysis reveals the mathematical structure of the scale-related dominant balances.
4. The results are used as guidelines in the construction of numerical discretizations that avoid the pollution of physical effects by singular amplification of truncation errors.

The rest of this paper is organized as follows. Section 2. summarizes the generalized multiple scales asymptotic ansatz, section 3. justifies this approach by showing that it allows us to recover many, if not most, of the well-established simplified model equations for atmospheric flow dynamics, section 4. discusses the problem of hydrostatic balances in compressible flow simulations, summarizes the construction of an asymptotics-motivated "well-balanced scheme", and provides first successful results.

2. SCALINGS, DISTINGUISHED LIMITS, AND MULTIPLE SCALES

A quite convincing agreement regarding several characteristic orders of magnitude in atmospheric flows, (Gill (1982),Pedlosky:86 (1987),Zeytounian (1991)). Typical atmospheric flow velocities are $|\boldsymbol{v}| \sim u_{\rm ref} = 10\,{\rm m/s}$, the atmospheric scale height is $h_{\rm sc} \sim \ell_{\rm ref} = 10\,{\rm km}$, the atmospheric pressures and density are $\rho \sim \rho_{\rm ref} = 1.0\,{\rm kg/m^3}$ and $p \sim p_{\rm ref} = 10^5\,{\rm kg\,m/s^2}$, a characteristic Brunt-Väisälä frequency is $N \sim N_{\rm ref} = 10^{-2}\,{\rm s^{-1}}$, and the gravitational acceleration and earth rotation frequency are $g \sim 10\,{\rm m/s^2}$ and $\Omega \sim 10^{-4}\,{\rm s^{-1}}$. Based on these estimates, the characteristic Mach, Froude, and Rossby numbers are

$$
\begin{array}{lcccc}
\mathrm{M} & = & \dfrac{u_{\rm ref}}{\sqrt{p_{\rm ref}/\rho_{\rm ref}}} & \approx & 1/33 \\
\mathrm{Fr} & = & \dfrac{u_{\rm ref}}{\sqrt{g\ell_{\rm ref}}} & \approx & 1/33 \\
\mathrm{Fr}^{\rm stab.} & = & \dfrac{u_{\rm ref}}{\ell_{\rm ref}N} & \approx & 1/7 \\
\mathrm{Ro} & = & \dfrac{u_{\rm ref}}{2\Omega\ell_{\rm ref}} & \approx & 5
\end{array}
\qquad (1)
$$

With these definitions the dimensionless govering equations for three-dimensional, inviscid, dry atmospheric flows read

$$
\begin{array}{rcl}
\rho_t + \nabla\cdot(\rho\boldsymbol{v}) & = & 0 \\
\boldsymbol{v}_t + \boldsymbol{v}\cdot\nabla\boldsymbol{v} + \frac{\boldsymbol{\Omega}\times\boldsymbol{v}}{\mathrm{Ro}} + \frac{\nabla p}{\mathrm{M}^2\rho} & = & -\frac{\boldsymbol{k}}{\mathrm{Fr}^2} \\
p_t + \boldsymbol{v}\cdot\nabla p + \gamma p\nabla\cdot\boldsymbol{v} & = & 0
\end{array}
\qquad (2)
$$

where $\rho, \boldsymbol{v}, p$ are the density, (three-dimensional) flow velocity, and pressure, respectively, $\Omega, \boldsymbol{k}$ are unit vectors pointing in the direction of the earth rotation axis and in the vertical direction,respectively, and γ is the (constant) isentropic exponent of the considered ideal gas.

These equations will suffice as a basis for the subsequent discussions. Notice that parameterizations of subgrid scale effects can be handled straightforwardly, provided they are non-singular in the limit regimes considered below.

Various familiar simplifying approximations of theoretical meteorology can be justified as asymptotic limits where one of the above non-dimensional parameters vanishes. Thus the limit $\mathrm{M} \to 0$ yields the anelastic approximation, $\mathrm{Fr} \to 0$ induces hydrostatic balance, $\mathrm{Fr}^{\rm stab.} \to 0$ implies strongly stratified flow with restricted vertical motions, and $\mathrm{Ro}\cdot\ell_{\rm ref}/L_{\rm ref} \to 0$ implies (quasi-) geostrophy, when $L_{\rm ref}$ is a typical horizontal synoptic scale.

We obtain a unified mathematical representation of these various limits and a systematic approach to studying multi-scale interactions by introducing the following *distinguished limits*:

$$
\mathrm{Fr}^{\rm stab.} \sim \sqrt{\mathrm{Fr}} \sim \sqrt{\mathrm{M}} \sim \frac{1}{\mathrm{Ro}} \sim \varepsilon \ll 1 \qquad (3)
$$

This ansatz, which reduces the number of degrees of freedom in the set of non-dimensional parameters to only one, is pragmatically justified by the concrete numbers given in (1).

Let $\mathbf{U}(\boldsymbol{x}, z, t; \varepsilon) = (\rho, \boldsymbol{u}, w, p)(\boldsymbol{x}, z, t; \varepsilon)$ denote solutions of (2). Here $\boldsymbol{u}, w$ and $\boldsymbol{x}, z$ denote the horizontal and vertical flow velocities and co-ordinates, respectively. Then our generalized asymptotic ansatz reads

$$
\mathbf{U} = \sum_i \varepsilon^i \mathbf{U}^{(i)} \qquad (4)
$$

where all expansion functions $\mathbf{U}^{(i)}$ have multiple-scales representations

$$\mathbf{U}^{(i)}(\frac{t}{\varepsilon}, t, \varepsilon t, \varepsilon^2 t, \frac{\boldsymbol{x}}{\varepsilon}, \boldsymbol{x}, \varepsilon\boldsymbol{x}, \varepsilon^2\boldsymbol{x}, \frac{z}{\varepsilon}, z) \qquad (5)$$

Clearly, a comprehensive analysis of solutions for this very general ansatz is too involved to be presented here, and—most likely—it would also be too complex to be useful. However, there are three different lines of research, based on simplified versions of (5), which are worthy of being explored:

- Judicious specializations of (5), which include only one rescaled representative each for $t, \boldsymbol{x}$, and z, allow us to recover various classical results of theoretical meteorology. Thus (3)–(5) provides a unified mathematical framework for meteorological scaling analysis.
- By construction, the ansatz in (5) is a natural basis for the analysis of scale interactions, provided that more than one rescaled representative for $t, \boldsymbol{x}$, and z is accounted for.
- By revealing the mathematical structure of the various scaling regimes in atmospheric flows, a multiple-scales analysis also provides valuable hints regarding the construction of uniformly accurate and efficient numerical techniques for solving the full three-dimensional compressible flow equations.

In the rest of this paper we concentrate on the first and last items of this list.

3. RECOVERING CLASSICAL RESULTS

3.1 Stratified and nearly-neutral meso-scale flows

Klein (2000) considers a specialization of (5) for meso-scale flows, with horizontal scales comparable with the pressure scale height. In the present terminology, this amounts to

$$\mathbf{U}^{(i)} = \mathbf{U}^{(i)}(t, \boldsymbol{x}, z)\,. \qquad (6)$$

Specifically, t measures variations on the $20\,\mathrm{min}$ time scale, and $\boldsymbol{x}, z$ are both non-dimensionalized by the reference length of $h_{\mathrm{sc}} \sim 10\,\mathrm{km}$. We will not repeat the detailed derivations from the original reference here, but rather quote the key results.

Consistent with the derivations in Ogura and Phillips (1962) and Zeytounian (1991), two quite different regimes of vertical stratification must be distinguished. If the (dry) atmospheric stratification is consistent with the above estimate of the Brunt-Väisälä frequency of $0.01\,\mathrm{s}^{-1}$ or, equivalently, with $\mathrm{Fr}^{\mathrm{stab}} \sim \sqrt{\mathrm{Fr}}$, then the potential temperature $\theta = p^{1/\gamma}/\rho$ has an expansion

$$\theta = \theta_\infty + \varepsilon^2\theta^{(2)}(t, \boldsymbol{x}, z) + o(\varepsilon^2) \qquad (7)$$

with $\theta_\infty \equiv \mathrm{const.}$ and $\partial\theta^{(2)}/\partial z > 0$. In this regime, the governing equations from (2) reduce to decoupled flow in horizontal layers (unless parameterizations of vertical turbulent fluxes are accounted for). To leading order, the atmosphere is hydrostatically balanced as regards the pressure and temperature distributions,

$$\begin{aligned} p^{(0)}(z) &= \left(1 - \tfrac{\gamma-1}{\gamma}\, z\right)^{\frac{\gamma}{\gamma-1}}, \\ \rho^{(0)}(z) &= p^{(0)}(z)^{\frac{1}{\gamma}}\,. \end{aligned} \qquad (8)$$

and the flow field is determined by decoupled incompressible flow equations for layers $z = \mathrm{const.}$

$$\begin{aligned} \boldsymbol{u}_t + \boldsymbol{u}\cdot\nabla_\parallel\,\boldsymbol{u} + \nabla_\parallel\,\pi^{(2)} &= 0 \\ \nabla_\parallel\cdot\boldsymbol{u} &= 0 \\ w^{(0)} &= 0 \end{aligned} \qquad (9)$$

Here $\nabla_\parallel = (\partial_x, \partial_y, 0)$ denotes the horizontal gradient components, and $\pi^{(2)} \equiv p^{(2)}/\rho_0(z)$.

Remark: The inclusion of (parameterized) small scale heat sources induces vertical motions of mass elements towards new layers of neutral stratification. In this case the vertical velocity is non-zero and given through an algebraic relation involving the heat source strength and the potential temperature stratification. See Botta, Klein, and Almgren (1999) and Klein (2000) for further details.

Remark: The effect of a large scale, horizontally homogeneous heat loss, e.g., due to radiative cooling, is expected to yield a quite different relation for vertical motions. This regime is currently being analyzed.

3.2 Quasi-geostrophic flows

Here we consider a specialization of (5) that involves large horizontal scales and associated much longer advective time scales. The Ansatz reads

$$\mathbf{U}^{(i)} = \mathbf{U}^{(i)}(\varepsilon^2 t, \varepsilon^2 \boldsymbol{x}, z)\,. \tag{10}$$

We let $(\tau, \boldsymbol{\xi}) = (\varepsilon^2 t, \varepsilon^2 \boldsymbol{x})$, $\boldsymbol{\xi} = (\xi, \eta, 0)$, and $\nabla_{\boldsymbol{\xi}} = (\partial_\xi, \partial_\eta, 0)$ and from the expansion obtain automatically the $\boldsymbol{\beta}$–plane approximation

$$\boldsymbol{\Omega} = \boldsymbol{\Omega}_0(1 + \beta\eta)\,. \tag{11}$$

The leading order pressure and density are given by homentropic hydrostatics as in (8). The potential temperature obeys an expansion

$$\theta = 1 + \varepsilon^2 \Theta^{(2)}(z) + \varepsilon^3 \theta^{(3)}(\boldsymbol{\xi}, z, \tau) + \dots\,, \tag{12}$$

and the higher order pressure and potential temperature perturbations are related via $p^{(1)} \equiv 0$ and

$$\frac{\partial \pi^{(i)}}{\partial z} = -\theta^{(i)} \quad (i = 2, 3), \tag{13}$$

where $\pi^{(i)} = \pi^{(i)}/\rho^{(0)}$. Vertical velocities are consistently small, $w^{(i)} \equiv 0$ for $(i = 0, 1, 2)$, and third order entropy perturbations are advected according to

$$\theta^{(3)}_\tau + \boldsymbol{u}^{(0)} \cdot \nabla\theta^{(3)} + w^{(3)} \frac{\mathrm{d}\Theta^{(2)}}{\mathrm{d}z} = 0\,. \tag{14}$$

The horizontal velocity $\boldsymbol{u}^{(0)}$ is computed through the vertical vorticity component $\zeta^{(0)} = \nabla_{\boldsymbol{\xi}} \times \boldsymbol{u}^{(0)}$, which obeys the pressure-vorticity relation

$$\zeta^{(0)} + \nabla^2 \pi^{(3)} = 0\,. \tag{15}$$

Taking the curl of the horizontal momentum equation one obtains the vorticity evolution equation

$$\zeta^{(0)}_\tau + \boldsymbol{u}^{(0)} \cdot \nabla_{\boldsymbol{\xi}} \zeta^{(0)} + \beta v^{(0)} = -\nabla_{\boldsymbol{\xi}} \cdot \boldsymbol{u}^{(1)} \tag{16}$$

Noticing further that $v^{(0)} = (\partial_\tau + \boldsymbol{u}^{(0)} \cdot \nabla)\,\eta$, that mass conservation implies

$$\nabla \cdot \boldsymbol{u}^{(1)} = -\frac{1}{\rho^{(0)}} \frac{\partial}{\partial z}(\rho^{(0)} w^{(3)})\,, \tag{17}$$

and eliminating $w^{(3)}$ using (14) one finds

$$(\partial_\tau + \boldsymbol{u}^{(0)} \cdot \nabla)\, q = 0 \tag{18}$$

where q is the potential vorticity, given by

$$q = \zeta^{(0)} + \beta\eta + \frac{1}{\rho^{(0)}} \frac{\partial}{\partial z} \left(\frac{\rho^{(0)}}{\mathrm{d}\Theta/\mathrm{d}z}\, \theta^{(3)} \right). \tag{19}$$

Thus we have shown that the specialization (10) of the general multiple scales ansatz in (5) reproduces the classical quasi-geostrophic theory.

3.3 Further results

The authors have studied a further specializations of (5) and found the following correspondences:

$\boldsymbol{U}^{(i)}(t, \boldsymbol{x}, z)$	Anelastic/pseudo-incompr. flow
$\boldsymbol{U}^{(i)}(t, \varepsilon\boldsymbol{x}, z)$	Large scale internal gravity waves
$\boldsymbol{U}^{(i)}(\varepsilon^2 t, \varepsilon^2 \boldsymbol{x}, z)$	Quasi-Geostrophic Theory

We expect the following relationships to hold as well:

$\boldsymbol{U}^{(i)}(\frac{t}{\varepsilon}, \boldsymbol{x}, \frac{z}{\varepsilon})$	Small scale internal gravity waves
$\boldsymbol{U}^{(i)}(\varepsilon t, \varepsilon^2 \boldsymbol{x}, z)$	Rossby adjustment
$\boldsymbol{U}^{(i)}(\varepsilon t, \frac{\xi(\varepsilon^2 \boldsymbol{x})}{\varepsilon}, z)$	Semi-geostrophic flows

4. A WELL-BALANCED SCHEME FOR NEARLY HYDROSTATIC FLOWS

As mentioned earlier, numerical weather prediction and climate modelling are adopting more comprehensive equation sets that are valid over increased scale ranges. Motivated by our co-operation with the *Deutscher Wetterdienst* we are developing asymptotics-based numerical techniques for the full three-dimensional compressible flow equations in atmospheric flow regimes. Key requirements are conservation of mass, momentum, and energy, acceptable accuracy even for relatively coarse grids, and computational efficiency. Here we sketch our very first successful developments and results.

We adopt a Godunov-type conservative compressible flow solver which, by construction, conserves mass, momentum, and total energy up to machine roundoff errors, LeVeque (1990). For low Mach numbers the pressure gradient and the gravitational acceleration dominate the momentum balance.

In most numerical scheme the gravitation term is discretized as a volume force while the pressure gradient appears through flux- or finite differences. As a consequence there is no mechanism that guarantees a balance of these terms also on a discrete level. In fact, attempts to reproduce the hydrostatic state of an atmosphere at rest in the vicinity of a high mountain leads to spurious vertical motions that can easily reach velocities of $50\,\mathrm{cm/s}$ and more, unless special care is taken to account for near-hydrostasy.

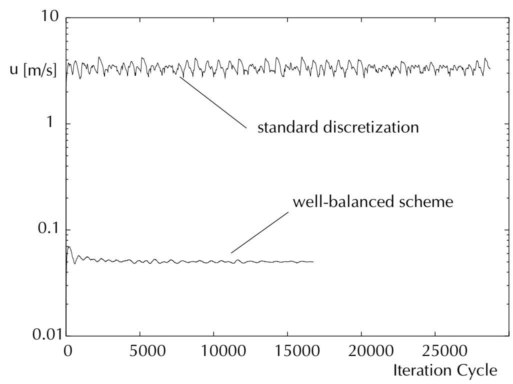

Fig. 1: Development of spurious winds near a $3000\,\mathrm{m}$ mountain

Figure 1 shows the maximum vertical velocities vs. the iteration cycle in two sample computations for such a simulation. Of the 32 vertical grid levels about 8 cover the lower $3000\,\mathrm{m}$, and of 128 lateral grid points the mountain is resolve by about 40. The two computations differ in the discretization of the gravitational source term and the vertical pressure gradients. While both schemes exhibit second order convergence with grid refinement, their actual performance for realistic resolutions differs by two orders of magnitude.

The conference contribution will explain the asymptotics-motivated construction of the well-balanced scheme, which was also used to simulate the lee-wave generation over topography as depicted qualitatively in Figure 2.

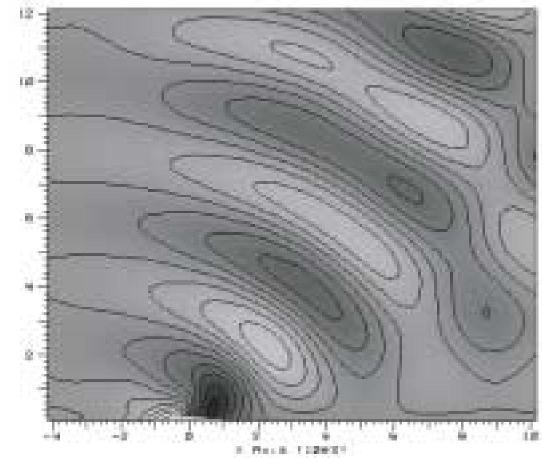

Fig. 2: Lee-wave generation over topography

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