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ENERGY CONSERVING LOW-ORDER MODELS FOR POTENTIAL VORTICITY DYNAMICS AND CONVECTION WITH SHEAR

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1. INTRODUCTION

Potential vorticity dynamics and thermal convection are two areas where low-order models (LOMs) have traditionally been an effective research tool (Saltzman 1959, 1962, Lorenz 1963a, 1963b, 1982, Charney and DeVore 1979, Treve and Manley 1982, Legras and Ghil 1985, Howard and Krishnamurti 1986, De Swart 1988, Thiffeault and Horton 1996). LOMs are commonly obtained by the Galerkin method: fluid dynamical fields are expanded in series of time-independent orthogonal functions satisfying the boundary conditions, then the series are truncated, and a finite system of ODEs for the coefficients in these expansions (the LOM) is derived from the original fluid dynamical equations by using the orthogonality properties. The method, unfortunately, does not provide criteria for selecting modes, so that truncations are arbitrary, which often results in LOMs that violate fundamental conservation properties of the original equations and exhibit unphysical behavior. Instead of ad hoc truncations, we follow a systematic, modular procedure that results in physically sound LOMs (Gluhovsky 1982, Gluhovsky and Agee 1997, Gluhovsky and Tong, 1999).

We construct LOMs in the form of coupled 3-mode systems known in mechanics as Volterra gyrostats. The Volterra (1899) gyrostat,

$$\begin{aligned}\dot{v}_1 &= p v_2 v_3 + b v_3 - c v_2\,,\\ \dot{v}_2 &= q v_3 v_1 + c v_1 - a v_3\,,\\ \dot{v}_3 &= r v_1 v_2 + a v_2 - b v_1\,; \quad p+q+r=0,\end{aligned} \tag{1}$$

where p, q, r, a, b, c are constants, also describes certain fluid dynamical situations (Gluhovsky, 1982; Gluhovsky and Tong, 1999), and the simplest gyrostat in a forced regime,

$$\begin{aligned}\dot{v}_1 &= -q v_2 v_3 \qquad\qquad - \gamma_1 v_1 + f,\\ \dot{v}_2 &= \quad q v_3 v_1 - a v_3 \;\; - \gamma_2 v_2,\\ \dot{v}_3 &= \qquad\qquad a v_2 \;\; - \gamma_3 v_3,\end{aligned} \tag{2}$$

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becomes, after a linear change of variables (Gluhovsky 1982), the celebrated Lorenz (1963) model for 2D Rayleigh–Bénard convection.

In LOMs having the form of coupled gyrostats, linear gyrostatic terms (linear terms in Eqs. (1)) occur due to various factors peculiar to geophysical fluid dynamics, such as stratification, rotation, and topography. When such models are expanded by increasing the order of approximation or by adding new physical mechanisms, they still have the structure of coupled gyrostats. This structure ensures energy conservation (throughout the paper "conservation" means "conservation in the absence of forcing and dissipation"). This property of fundamental importance is sometimes missing in LOMs (a vivid example is the well-known Howard – Krishnamurti (1986) model of convection with shear). In addition, the gyrostatic form permits optimal modes selection.

Our approach is illustrated below with three important LOMs. One is an extension of the Charney – DeVore (1979) model of a barotropic atmosphere over topography. The other is the modification of the Howard – Krishnamurti (1986) model that restores conservation of energy and of total vorticity to the original model thus permitting a coupled gyrostats form for it. In both cases, the "minimal" LOM is extracted that possesses conservation properties and demonstrates the effect of interest: vacillation between high-index and low-index regimes and tilting of convection cells, respectively. Both LOMs describe two-dimensional flows. The third LOM is a model for 3D Rayleigh-Bénard convection.

2. A LOM FOR THE QUASI – GEOSTROPHIC POTENTIAL VORTICITY EQUATION

A LOM was developed by Charney and DeVore (1979) that describes the evolution of two zonal flow modes and two Rossby waves and contained 6 modes. DeSwart (1988) included two more Rossby waves and discovered in his 10-mode system chaotic vacillations between two distinct regions in the phase space that he identified as high-index and low-index regimes. Earlier, Legras and Ghil (1985) demonstrated transitions between different regimes in a 25-mode system. DeSwart argued in terms of the attractor dimension that transitions should exist even in an 8-mode system. However, the standard mode selection procedure prevented him from choosing the set of 8 modes necessary to obtain the desired effect. By converting DeSwart's system into coupled gyrostats we were able to single out an 8-mode subsystem of coupled gyrostats (Eqs. (3)) that indeed exhibits the chaotic vacillations (Figure 1). In Eqs. (3) (and elsewhere), vertical bars are used to organize the model into a superposition of gyrostats; modes v_1 and v_4 correspond to two zonal flow modes, while the six other modes represent three Rossby waves. The four gyrostats involving modes $\mathrm{v}_1, \ldots, \mathrm{v}_6$ correspond to the original Charney and DeVore (1979) model. The model takes into account the effects of rotation and topography. Accordingly, linear gyrostatic terms with coefficient a are due to rotation; those with coefficient b are due to topography.

$$
\begin{aligned}
\dot{\mathrm{v}}_1 &= b_1\mathrm{v}_3 - C\mathrm{v}_1 + F_1\\
\dot{\mathrm{v}}_2 &= q_1\mathrm{v}_3\mathrm{v}_1 - a_1\mathrm{v}_3 - C\mathrm{v}_2 + q_3\mathrm{v}_4\mathrm{v}_6 + q_5\mathrm{v}_5\mathrm{v}_8 - q_5\mathrm{v}_6\mathrm{v}_7\\
\dot{\mathrm{v}}_3 &= -q_1\mathrm{v}_1\mathrm{v}_2 + a_1\mathrm{v}_2 - b_1\mathrm{v}_1 - C\mathrm{v}_3 - q_3\mathrm{v}_4\mathrm{v}_5 - q_5\mathrm{v}_5\mathrm{v}_7 - q_5\mathrm{v}_6\mathrm{v}_8\\
\dot{\mathrm{v}}_4 &= -C\mathrm{v}_4 + F_2 + p_3\mathrm{v}_6\mathrm{v}_2 + b_3\mathrm{v}_6 - p_3\mathrm{v}_5\mathrm{v}_3\\
\dot{\mathrm{v}}_5 &= q_2\mathrm{v}_6\mathrm{v}_1 - a_2\mathrm{v}_6 - C\mathrm{v}_5 - r_3\mathrm{v}_3\mathrm{v}_4 + p_5\mathrm{v}_8\mathrm{v}_2 + b_5\mathrm{v}_8 - p_5\mathrm{v}_7\mathrm{v}_3\\
\dot{\mathrm{v}}_6 &= -q_2\mathrm{v}_1\mathrm{v}_5 + a_2\mathrm{v}_5 - C\mathrm{v}_6 + r_3\mathrm{v}_2\mathrm{v}_4 - b_3\mathrm{v}_4 - p_5\mathrm{v}_7\mathrm{v}_2 - b_5\mathrm{v}_7 - p_5\mathrm{v}_8\mathrm{v}_3\\
\dot{\mathrm{v}}_7 &= q_4\mathrm{v}_8\mathrm{v}_1 - a_4\mathrm{v}_8 - C\mathrm{v}_7 - r_5\mathrm{v}_2\mathrm{v}_6 + b_5\mathrm{v}_6 - r_5\mathrm{v}_3\mathrm{v}_5\\
\dot{\mathrm{v}}_8 &= -q_4\mathrm{v}_1\mathrm{v}_7 + a_4\mathrm{v}_7 - C\mathrm{v}_8 + r_5\mathrm{v}_2\mathrm{v}_5 - b_5\mathrm{v}_5 - r_5\mathrm{v}_3\mathrm{v}_6
\end{aligned}
\qquad (3)
$$

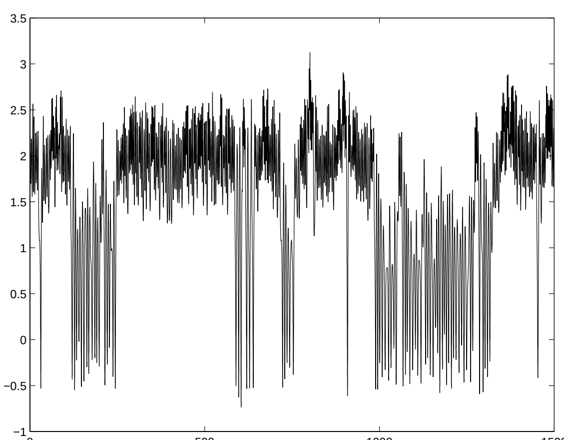

Figure 1. Time series of v_1- component of LOM (3) demonstrating chaotic vacillations between high-index and low-index regimes.

3. THE MODIFIED HOWARD-KRISHNAMURTI (1986) MODEL OF CONVECTION WITH SHEAR

The dimensionless Boussinesq equations for 2-D Rayleigh – Bénard convection are

$$
\begin{aligned}
\frac{\partial}{\partial t}\nabla^2\Psi &= \sigma\nabla^4\Psi + \sigma\frac{\partial\theta}{\partial x} + \frac{\partial\Psi}{\partial x}\frac{\partial\nabla^2\Psi}{\partial z} - \frac{\partial\Psi}{\partial z}\frac{\partial\nabla^2\Psi}{\partial x},\\
\frac{\partial\theta}{\partial t} &= \nabla^2\theta + Ra\frac{\partial\Psi}{\partial x} + \frac{\partial\Psi}{\partial x}\frac{\partial\theta}{\partial z} - \frac{\partial\Psi}{\partial z}\frac{\partial\theta}{\partial x},
\end{aligned}
\qquad (4)
$$

where Ψ is the stream function, θ is the deviation from the equilibrium vertical temperature profile, σ is the Prandtl number, and Ra is the Rayleigh number.

To arrive at the modified Howard – Krishnamurti model that possesses appropriate conservation properties, the following Galerkin expansions are used:

$$\Psi = A(t)\sin\alpha x \sin z + B(t)\sin z + C(t)\cos\alpha x \sin 2z + \\ + G(t)\sin 3z, \qquad (5)$$

$$\theta = D(t)\cos\alpha x \sin z + E(t)\sin 2z + F(t)\sin\alpha x \sin 2z + \\ + H(t)\sin 4z.$$

Substituting expansions (5) into Eqs. (4) results in a LOM that, by a linear change of variables, may be transformed into the following system of six coupled gyrostats, where variables $x_1 - x_8$ correspond to *E*, *D*, *A*, *C*, *B*, *F*, *H*, and *G*, respectively, in expansions (5).

$$\begin{array}{lll|r|l|l|r|l|l|}
\dot{x}_1 = & -x_1 + f_1 & & -x_2x_3 & & & & & \\
\dot{x}_2 = & -\alpha_2 x_2 & & +x_3x_1 - x_3 & & -cx_5x_6 & & & +(c/3)x_8x_6 \\
\dot{x}_3 = & -\alpha_3 x_3 & & x_2 & +px_4x_5 & & & +Px_4x_8 & \\
\dot{x}_4 = & -\alpha_4 x_4 & & & +qx_5x_3 & & +x_6 & +Qx_8x_3 & \\
\dot{x}_5 = & -\alpha_5 x_5 & & & +rx_3x_4 & & & & \\
\dot{x}_6 = & -\alpha_6 x_6 & & & & +cx_5x_2 & +dx_4x_7 - x_4 & & -(c/3)x_8x_2 \\
\dot{x}_7 = & -\alpha_7 x_7 + f_2 & & & & & -dx_6x_4 & & \\
\dot{x}_8 = & -\alpha_8 x_8 & & & & & & +Rx_3x_4 & \\
 & & & \text{I} & \text{II} & \text{III} & \text{IV} & \text{V} & \text{VI}
\end{array} \qquad (6)$$

The gyrostats composing model (6) are particular cases of system (1). Gyrostats I & IV are equivalent to system (2) (we call it the Lorenz gyrostat), gyrostats II & V are equivalent to the Euler gyroscope (a gyrostat without linear terms), and gyrostats III & VI are the degenerative Euler gyroscopes. Taken alone (i.e. with $\dot{x}_5 = \dot{x}_8 = 0$), the latter are just simple linear oscillators, but coupled here with other gyrostats possessing time varying modes x_5 and x_8, they are nonlinear subsystems.

It may easily be checked that in any system of coupled gyrostats the sum of squares of all modes, representing an energy quantity (see discussion in Gluhovsky and Tong (1999)), is conserved. Thus, the failure to convert a LOM into coupled gyrostats form may signal violation of energy conservation. This is the case with the Howard – Krishnamurti (1986) model that is based on the terms with coefficients *A* to *F* in expansions (5), which cannot be converted into coupled gyrostats. Its modification by Thiffeault and Horton (1996), who added the $H(t)\sin 4z$ term in Eqs. (5), conserves energy and can be converted (Gluhovsky and Tong, 1999) into a system of coupled gyrostats.

Hermiz et al. (1995) noticed that the Howard – Krishnamurti (1986) model also lacks total vorticity conservation due to an insufficient number of shearing modes, which can be remedied by adding the second shearing mode $G(t)\sin 3z$ in expansion (5). The same is true for the Thiffeault and Horton (1996) version of the model. However, as Thiffeault and Horton (1996) pointed out, the Hermiz et al. (1995) model still lacks energy conservation, which could be corrected by again including the $H(t)\sin 4z$ mode in expansion (5).

Model (6) incorporates the suggestions of both Hermiz et al. (1995) and Thiffeault and Horton (1996), thereby ensuring both total energy and total vorticity conservation. The latter implies the existence of a linear integral of motion (total vorticity), $I = B + 3G$, which is equivalent to the conservation of $Rx_5 - rx_8$ in LOM (6). LOM (6) also has tilted roll solutions like the original Howard-Krishnamurti (1986) model.

As in the case of the LOM discussed in section 2, presenting a system in a gyrostatic form permits to identify a subsystem still possessing the desired effect (tilting in this case). Such a subsystem has to contain enough modes to exhibit the effect of interest, while maintaining a coupled gyrostats structure to have conservation properties. With this in view, we reduced system (6) to the following subsystem of three gyrostats composed of modes $x_i = X_i,\ i = 1, ..., 5;\ x_8 = X_6$:

$$\begin{array}{ll|r|l|l|}
\dot{X}_1 = & -\gamma_1 X_1 + f_1 & -X_2X_3 & & \\
\dot{X}_2 = & -\gamma_2 X_2 & +X_3X_1 - X_3 & & \\
\dot{X}_3 = & -\gamma_3 X_3 & +X_2 & +pX_4X_5 & +PX_4X_6 \\
\dot{X}_4 = & -\gamma_4 X_4 & & +qX_5X_3 & +QX_6X_3 \\
\dot{X}_5 = & -\gamma_5 X_5 & & +rX_3X_4 & \\
\dot{X}_6 = & -\gamma_6 X_6 & & & +RX_3X_4
\end{array} \qquad (7)$$

In Eqs. (7), $\gamma_i = \alpha_i, i = 1, ..., 5;\ \gamma_6 = \alpha_8$, and other coefficients are from Eqs. (6).

Figure 2 demonstrates that the tilting of convection cells is preserved in LOM (7). Its subsystem composed of modes $X_1, ..., X_5$ (thus equivalent to the first two gyrostats in Eqs. (7)) was studied by Aoyagi et al. (1997). Their model, however, does not conserve the total vorticity. In contrast, LOM (7) does conserve total vorticity, thanks to the linear integral $Rx_5 - rx_6$ (similar to the linear integral in LOM (6)). As a system of coupled gyrostats, LOM (7) also conserves energy. Thus, LOM (7) is the smallest LOM exhibiting spontaneous vertical shear while respecting all conservation laws.

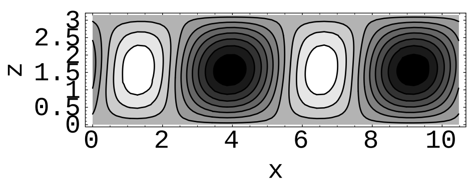

Figure 2. Contour plot of the stream function for a steady state solution of LOM (7) showing the tilting of convection cells.

4. A LOW-ORDER MODEL FOR 3D RAYLEIGH-BÉNARD CONVECTION

Instead of the stream function formulation employed in Section 3 for the 2D case, here we use the following nondimensional system based on the equation for the vorticity $\varsigma = \nabla \times \boldsymbol{v}$:

$$\frac{\partial \varsigma}{\partial t} = (\varsigma \cdot \nabla)\boldsymbol{v} - (\boldsymbol{v} \cdot \nabla)\varsigma + \nu \nabla^2 \varsigma + \left(\hat{\boldsymbol{x}} \frac{\partial \theta}{\partial y} - \hat{\boldsymbol{y}} \frac{\partial \theta}{\partial x} \right),$$
$$\frac{\partial \theta}{\partial t} = -\boldsymbol{v} \cdot \nabla \theta + v_z + \kappa \nabla^2 \theta, \quad \nabla \cdot \boldsymbol{v} = 0, \qquad (8)$$

All variables in Eqs. (8) are dimensionless forms of **v** (velocity), ς (vorticity), θ (temperature deviation from stable profile), ν (kinematic viscosity), and κ (thermal conductivity). The following Galerkin expansions were used:

$$v_x = x_1(t)\sin ax \cos z + w(t)\sin ax \cos ay \cos 2z,$$
$$v_y = y_1(t)\sin ay \cos z + w(t)\cos ax \sin ay \cos 2z,$$
$$v_z = -ax_1(t)\cos ax \sin z - ay_1(t)\cos ay \sin z - aw(t)\cos ax \cos ay \sin 2z, \qquad (9)$$
$$\theta = \theta_{101}(t)\cos ax \sin z + \theta_{011}(t)\cos ay \sin z + \theta_{112}(t)\cos ax \cos ay \sin 2z + \theta_{002}(t)\sin 2z + \theta_{004}(t)\sin 4z.$$

These expansions result in a LOM that, by a linear change of variables, was transformed into system (10), where variables $x_1 - x_8$ correspond to θ_{002}, θ_{101}, x_1, θ_{011}, y_1, w, θ_{112}, and θ_{004}, respectively, in expansions (9).

LOM (10) is a system of six coupled gyrostats: three Lorenz gyrostats (I, II, and III), one Euler gyroscope (IV), and two degenerative Euler gyroscopes (V and VI). Lorenz gyrostats I and II compose a 5-mode LOM that is a 3D analog of the Lorenz (1963) model. These two gyrostats describe dynamics in the (x, z) and (y, z) planes, respectively.

$$\begin{array}{ll|l|l|l|l|l|l}
\dot{x}_1 = & -x_1 + f_1 & -x_2 x_3 & -x_4 x_5 & & & & \\
\dot{x}_2 = & -\alpha_2 x_2 & +x_3 x_1 - x_3 & & & & -(1/2)x_5 x_7 & \\
\dot{x}_3 = & -\alpha_3 x_3 & x_2 & & & -(\beta/2)x_5 x_6 & & \\
\dot{x}_4 = & -\alpha_4 x_4 & & +x_5 x_1 - x_5 & & & & -(1/2)x_3 x_7 \\
\dot{x}_5 = & -\alpha_5 x_5 & & +x_4 & & -(\beta/2)x_6 x_3 & & \\
\dot{x}_6 = & -\alpha_6 x_6 & & & +\beta x_7 & +\beta x_3 x_5 & & \\
\dot{x}_7 = & -\alpha_7 x_7 & & & +2\beta x_6 x_8 - \beta x_6 & & +(1/2)x_2 x_5 & +(1/2)x_4 x_3 \\
\dot{x}_8 = & -4x_8 + f_2 & & & -2\beta x_7 x_6 & & & \\
 & & \text{I} & \text{II} & \text{III} & \text{IV} & \text{V} & \text{VI}
\end{array} \qquad (10)$$

Das et al. (2000) studied a model based on expansion (9) without the $\theta_{004}(t)\sin 4z$ term. However, we found that their LOM, similar to that of Howard and Krishnamurti (1986), possessed trajectories going to infinity, again because it does not conserve energy. In contrast, our model (10) as a system of coupled gyrostats conserves energy and has only bounded solutions.

5. SUMMARY AND CONCLUSIONS

In this paper, LOMs of atmospheric circulations are developed in the form of coupled Volterra gyrostats. Restricting LOMs to this form

1) ensures energy conservation, thus preventing unphysical behavior often observed in LOMs based on *ad hoc* truncations;
2) helps to single out the most effective interactions and to design LOMs of optimum size;
3) allows a modular implementation of the Galerkin technique using gyrostats as basic building blocks.

These advantages of our approach were demonstrated with three important LOMs: the Charney – DeVore (1979) model of a barotropic atmosphere over topography, a modification of the Howard – Krishnamurti (1986) model of convection with shear, and a model of 3D convection.

Another area where LOMs proved useful is turbulence: the so-called shell models of turbulent cascade originated by Obukhov (1971) and Lorenz (1972). Gluhovsky and Tong (1999) presented a coupled gyrostats shell model exhibiting Kolmogorov spectral behavior in a chaotic regime.

As noted by Brown and Chua (1992), "there is a pressing need for new nonlinear techniques that employ a building block approach whereby simple well-understood components are used to construct models of complex dynamical systems". We believe that coupled gyrostats could play the role of the above building blocks in problems of atmospheric dynamics and turbulence.

6. ACKNOWLEDGMENTS

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