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Consequences of Nonlinearities on the Low-Frequency Behavior of an AGCM

Judith Berner*

National Center for Atmospheric Research,† Boulder, Colorado and
Meteorologisches Institut der Universität Bonn, Germany

Grant Branstator

National Center for Atmospheric Research,† Boulder, Colorado

1. Introduction

An exceptionally long Atmospheric General Circulation Model (AGCM) integration is used to estimate the mean evolution of the geopotential heights Φ at 500hPa in EOF-phase space. These mean tendencies exhibit nonlinear characteristics as described in detail in Berner 1999. It is the aim of this paper to show that although the mean motion describes only a very small portion of the total motion, it contains essential information about the evolution of the $\Phi_{500\mathrm{hPa}}$. To quantify to what degree the detected nonlinearities produce different behavior than a linear system, linear and nonlinear stochastic models are built using the mean tendency fields derived from the GCM dataset for the deterministic part. These models are fitted in low-order phase space. Therefore the noise component represents the interaction of the truncated model with other modes, as well as the residual motion not described by the mean tendencies. The nonlinearities are found to markedly affect the Probability Density Function (PDF) of such models even when they are highly truncated. Indeed, the nonlinearities produce non-Gaussian distributions very similar to those found in the GCM generated data. In conclusion, it is remarkable how well a highly truncated stochastically forced *nonlinear* model captures the low-order statistics and dynamics of the GCM.

* *Corresponding author address:* Judith Berner, Advanced Study Program, NCAR, P.O. Box 3000, Boulder, CO 80307-3000.
WWW: http://www.cgd.ucar.edu/~berner/
E-mail: berner@ucar.edu

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2. Data

The dataset in this study consists of simulated geopotential heights Φ at 500hPa. The AGCM used was developed at the National Center for Atmospheric Research and is known as CCM0B[1]. Sixty runs, each of 5×10^4 d duration, were performed. To focus on internal variability each of these integrations was forced by the same time-independent January boundary conditions (perpetual January mode), but was started from a different initial condition. The dataset consists of 12 hourly sampled data. To limit the degrees of freedom and yet concentrate on high-amplitude circulation features, an Empirical Orthogonal Function (EOF) analysis was performed. In this investigation we limit out attention to projections of the data onto the leading four EOFs.

3. Nonlinear signatures

a. *Probability Density Function*

The PDF of the $\Phi_{500\mathrm{hPa}}$ exhibits modest, but significant departures from bivariate Gaussianity in the form of radial "ridges", though no more than one maximum is found (Fig. 1a). These ridges become local density maxima in the *mutual information density* (MID) $c(x,y)$ (Fig. 1b), that is obtained by dividing the 2D PDF $f(x,y)$ by the product of its marginal distributions $f(x)$ and $f(y)$, taking the logarithm of this quantity and multiplying it with $f(x,y)$:

$$c(x,y) = f(x,y)\log\frac{f(x,y)}{f(x)f(y)}\,. \qquad (1)$$

[1]Community Climate Model — Version 0

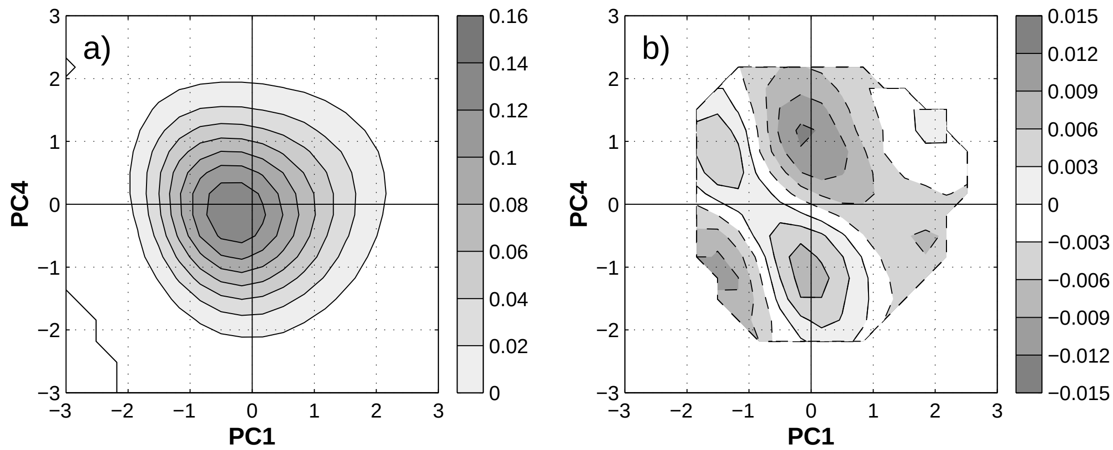

FIG. 1: a) PDF of the $\Phi_{500\mathrm{hPa}}$ unfiltered CCM0 data (sample size: 6×10^6). The data is projected onto the 2D phase space spanned by EOF1 and EOF4. The data is normalized so that each PC has a standard deviation of one. b) Mutual information density of the PDF in a) obtained by dividing the 2D PDF by the product of its marginal distributions, taking the logarithm of this quantity and and multiplying it with the PDF (see text).

The integral over the mutual information density

$$C = \iint f(x,y) \log \frac{f(x,y)}{f(x)f(y)} \, dx \, dy \tag{2}$$

is a well known quantity in information theory and referred to as *mutual information* (Shannon and Weaver 1949). We will use the mutual information density as a measure to assess how well the PDF of various stochastic models captures the density inhomogeneities of the GCM's PDF.

b. Mean tendencies

Nonlinear behavior is even more evident in the mean phase space tendencies (Fig. 2a). Using a least squares fitting procedure, the mean trajectories can be decomposed a linear part and a residual part, which is nonlinear at least within the 2D analysis. We will hence refer to it as nonlinear motion. In some subspaces these nonlinear tendencies together with the trajectories they imply produce distinct signatures of more than one equilibrium point (Fig. 2b).

However, the speed of the nonlinear motion compared to the total motion is very small and does not exceed 3% at any location in the 2D phase space (not shown). Although the nonlinear contribution is small, we will demonstrate that it is not negligible at all, but contains essential information about the statistics of the $\Phi_{500\mathrm{hPa}}$. This goal is achieved by building linear and nonlinear stochastic models using the mean tendency fields derived from the GCM dataset for the deterministic part and comparing the behavior of these stochastic models to the GCM's behavior. Our criterion for a good stochastic model will be its ability to match the PDF of the GCM as closely as possible.

4. The Stochastic Approach

We assume that the system can be modeled by a Markov Process, that consists of a damped deterministic part and is excited by a forcing that is white in time but might be spatially dependent. The change of the PDF $W(\vec{x},t)$ with time for such a process is given by the Fokker-Planck Eq. (FP-Eq.):

$$\begin{aligned} \frac{\partial W(\vec{x},t)}{\partial t} = & -\frac{\partial}{\partial x_i} D_i^{(1)}(\vec{x})\, W(\vec{x},t) \\ & + \frac{\partial}{\partial x_i \partial x_j} D_{ij}^{(2)}(\vec{x})\, W(\vec{x},t)\,;\ i,j = \{1,2\}\,, \end{aligned} \tag{3}$$

where $D_i^{(1)}$ and $D_{ij}^{(2)}$ denote the drift- and diffusion coefficients. The drift coefficient $D_i^{(1)}$ reflects the deterministic part of the Markov Process plus a drift component that is caused by the noise. The diffusion coefficient $D_{ij}^{(2)}$ represents the forcing or noise component. Thus, a spatially dependent forcing will lead to a $D_{ij}^{(2)}$, that is a function of phase space location. Due to the lack of an analytical description of the stochastic model, we need to estimate the $D_i^{(1)}$ and $D_{ij}^{(2)}$ from the data by using their definition (e. g.

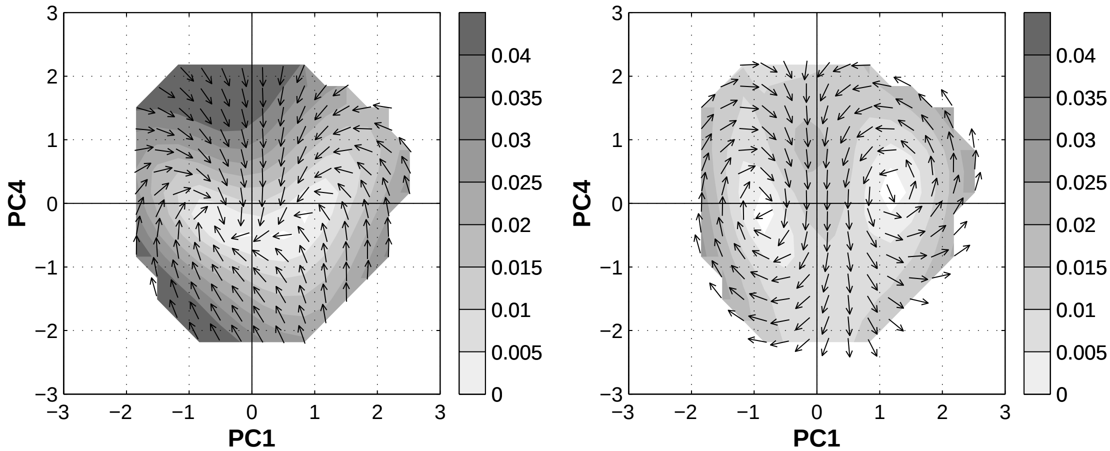

FIG. 2: a) Mean tendencies and b) nonlinear tendencies in the phase space spanned by EOF1 and EOF4 (sample size: 6×10^6). The nonlinear motion is obtained by removing the damping component of a linear fit from the mean tendencies. The shading indicates the speed in $\frac{1}{12\mathrm{h}}$.

Risken 1984):

$$
\begin{aligned}
D_i^{(1)}(\vec{x}) &= \frac{1}{n!} \lim_{\tau \to 0} \frac{1}{\tau} \langle [x_i(t+\tau) - x_i] \rangle \qquad (4) \\
D_{ij}^{(2)}(\vec{x}) &= \frac{1}{n!} \lim_{\tau \to 0} \frac{1}{\tau} \langle [x_i(t+\tau) - x_i] \rangle \langle [x_j(t+\tau) - x_j] \rangle ,
\end{aligned}
$$

where n denotes the sample size and τ the time lag. To our knowledge, though described in general terms by Siegert et al. 1998, the empirical estimation of these coefficients is a novel approach within the meteorological community, partly, because of demanding data requirements. The stationary solution of the FP-Eq. is found by computing the change of the distribution function with time given by Eq. 4 until a stationary PDF is reached.

a. Stationary PDF of the FP-Eq.

The drift coefficients are analytically and practically identical to the mean tendencies in Fig. 2a. The diffusion tensor in 2D has four elements. The diagonal elements represent the dominating diffusion processes, whereas the off-diagonal elements contain information about the much less important spatial correlation of the forcing. The estimated diffusion coefficients are spatially inhomogeneous (e. g. the first element D_{11} of the diffusion tensor is shown in Fig. 3). However, the diffusion coefficients for each element of the diffusion tensor have the same order of magnitude. The stationary PDF (not shown) given by the FP-Eq. and its MID (Fig. 4a) show remarkably good agreement with the GCM's PDF.

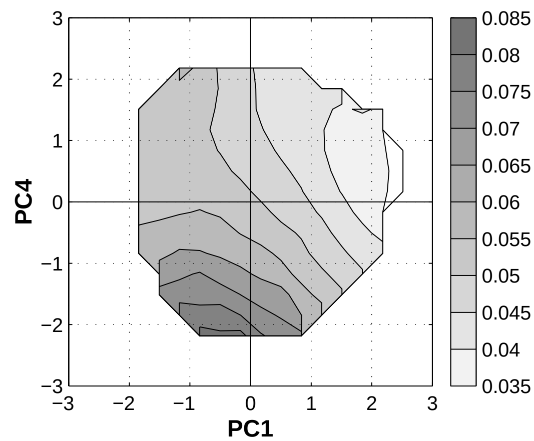

FIG. 3: The first diffusion coefficients $D_2^{(11)}$ estimated from the data by using Eq. 4. The contours indicate the values in $\frac{1}{12\mathrm{h}}$.

b. Stationary PDF of the FP-Eq. with spatially independent forcing

Since the diffusion coefficients are a function of phase space it is not clear whether the goodness-of-fit is a result of the spatially dependent forcing or the nonlinear deterministic drift. To investigate which component is responsible for the good agreement, we repeat the calculation above, but replace the forcing with a spatially independent forcing. This is done by spatially averaging each diffusion coefficient within the domain, where the underlying PDF has a value of at least 0.01. The MID of the stationary PDF of the FP-Eq. with constant diffusion and full nonlinear drift (Fig. 4b) still show very good agreement with the MID of the atmospheric GCM, indicating, that the goodness-of-fit is the result of the deterministic part and not of the spatial dependency in the forcing.

c. *Stationary PDF of the "linear" FP-Eq.*

To confirm this result we now repeat the calculation with the full spatially dependent forcing, but replace the drift coefficients with their least square linear fit (i. e. the difference between the mean tendencies and the nonlinear part shown in Fig. 2). The stationary PDF (not shown) of the FP-Eq. and its MID (Fig. 4c) now differs in essential ways from the distribution of the GCM.

5. Conclusion

We conclude that it is the drift and especially its nonlinear component that cause the remarkable agreement between the PDF of the stochastic model and the CCM, and not the spatially dependent and correlated forcing. The fact that the Fokker-Planck solution yields such good results, is an indication that the underlying process can be modeled as a Markov process. The distribution of geopotential height states in EOF phase space is remarkably well captured by a stochastically forced *nonlinear* model, but cannot be modeled well by a *linear* stochastic model.

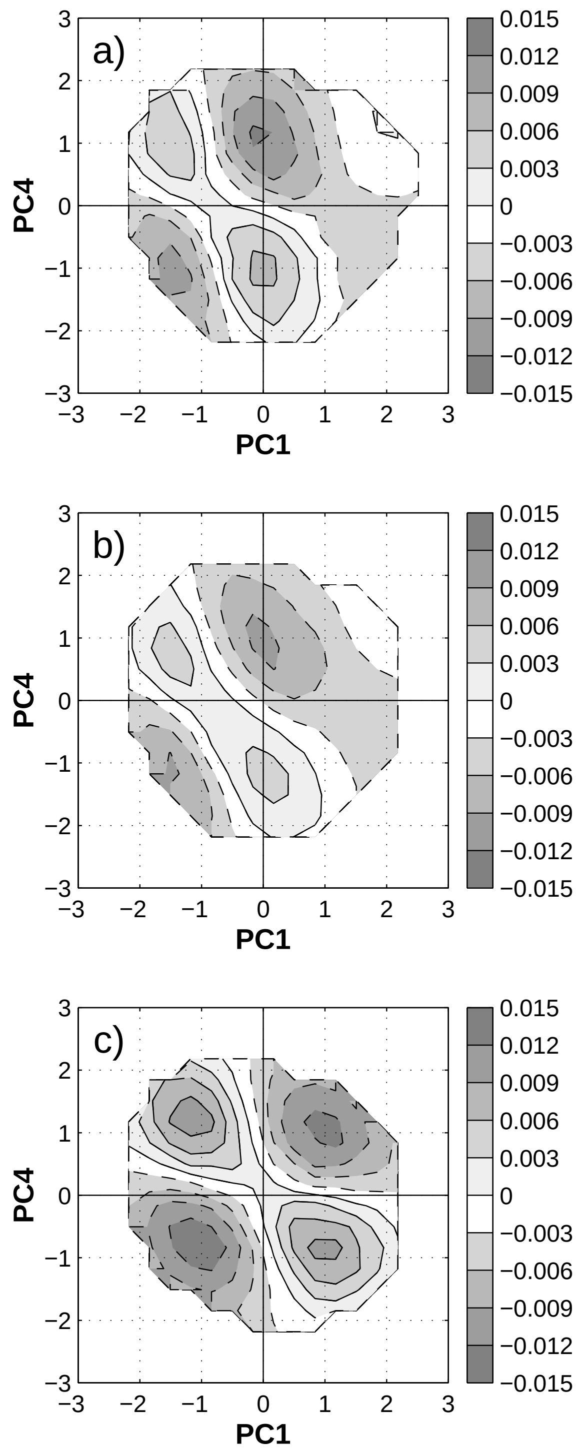

FIG. 4: a) Mutual information density (MID) of the stationary PDF of the Fokker-Planck-Eq.. b) MID of the PDF representing the stationary solution of the Fokker-Planck Eq., with spatially independent diffusion and full nonlinear drift. c) MID of the PDF representing the stationary solution of the Fokker-Planck Eq. , where the drift coefficients were replaced by linear tendencies estimated by a least square linear fit. The diffusion coefficients are the same as in the nonlinear case.

REFERENCES

Berner, J., 1999: *Weather Regimes and Transitions in a General Circulation Model.* Diplomarbeit, Meteorological Institute of the University of Bonn.

Risken, H., 1984: *The Fokker-Planck Equation. Methods of solution and applications.* Springer Verlag, Berlin Heidelberg New York.

Shannon, Claude E., and Warren Weaver, 1949: *The mathematical theory of communication.* University of Illnois Press.

Siegert, Silke, R. Friedrich, and J.Peinke, 1998: Analysis of data sets of stochastic systems. *Physics Letters A*, **243**, 275–280.