P3.4 A Theory for the Vertical Alignment of a Tilted Geophysical Vortex

David A. Schecter*

Advanced Study Program, National Center for Atmospheric Research,†Boulder, Colorado

Michael T. Montgomery and Paul D. Reasor‡

Department of Atmospheric Science, Colorado State University, Fort Collins, Colorado

1 Introduction

During its early stages of development, a geophysical vortex, such as a tropical cyclone or an ocean eddy, can be destroyed by episodes of external vertical shear. However, some vortices can survive, because they have a dominant tendency to stand upright (Polvani 1991; Sutyrin et al. 1998; Reasor and Montgomery 2001). In the following, we will show that this vertical alignment is often driven by the resonant damping of a "vortex Rossby mode." Furthermore, we will examine how the alignment rate varies with the vortex height and the atmospheric/oceanic stratification. A more detailed account of these results will appear in Schecter et al. (2001).

1.1 Quasi-Geostrophic Approximation

For simplicity, we assume that the alignment process is governed by quasi-geostrophic (QG) Boussinesq fluid dynamics. In QG theory, there is an approximate balance between the horizontal pressure gradient and the Coriolis force. That is, the dominant component of the velocity field is given by

$$\vec{v}_g \;=\; \hat{z} \times \nabla_h \psi. \qquad (1)$$

Here, ∇_h is the horizontal gradient operator, $\hat{z}$ is the unit vector pointing vertically upward, and ψ is the "geostrophic streamfunction." The geostrophic streamfunction is related to the pressure anomaly p, the unperturbed atmospheric density ρ_o, and the local value of the Coriolis parameter f, by the equation $\psi \equiv p/f\rho_o$.

The evolution of a QG flow is conveniently described by conservation of potential vorticity q:

$$\frac{\partial q}{\partial t} \;+\; \vec{v}_g \cdot \nabla_h q \;=\; 0, \qquad (2)$$

where

$$q \;=\; \nabla_h^2 \psi \;+\; \frac{f^2}{N^2}\frac{\partial^2 \psi}{\partial z^2}. \qquad (3)$$

Here, N is the (constant) basic-state buoyancy frequency, which measures the stratification of the atmosphere/ocean. The geostrophic streamfunction is obtained from Eq. (3), plus boundary conditions. In this paper, we assume that ψ vanishes at infinity in the horizontal plane. Motivated by tropical cyclone observations, we also assume that $\partial_z \psi = 0$ at $z = 0$ (the base of the vortex) and at $z = H$ (the height of the vortex).

1.2 Simulation of Vortex Alignment

Figure 1(a) shows a nonlinear numerical simulation of vortex alignment in an atmosphere/ocean with finite stratification (Reasor and Montgomery 2001). The shaded object represents the vortex core. The lateral boundary of the core is an isosurface of PV. A low level of PV (not shown) extends beyond the core, and decreases monotonically with radius. At $t = 0$ the vortex is given a simple tilt. In time, the tilt decays, and the vortex core becomes upright.

Figure 1(b) is a typical plot of tilt versus time for alignment simulations like that in Fig. 1(a). Notice that the tilt exhibits an early stage of exponential decay. Here, the tilt is measured by the magnitude of the $(1, 1)$ component of ψ, evaluated at the core radius r_v. That is, we let $\psi = \sum_{m,n} b_{m,n}(r,t)cos(m\pi z/H)e^{in\theta} + c.c.$, and define the tilt as $|b_{1,1}(r_v,t)|$.

It is surprising that the tilt decays exponentially with time. Time-asymptotic linear theory predicts that the streamfunction of a weak asymmetric perturbation on a monotonic vortex decays as a power-law ($t^{-\alpha}$). Apparently, time-asymptotic theory does not apply to the alignment process in Fig. 1. Nevertheless, the exponential decay can be explained: as we will see, it is due to the resonant damping of a vortex Rossby mode.

*Corresponding author address: David A. Schecter, NCAR/ASP, P.O. Box 3000, Boulder, CO 80307; e-mail: daspla@ucar.edu.

†NCAR is sponsored by the National Science Foundation.

‡Present affiliation: Hurricane Research Division AOML/NOAA, Miami, FL.

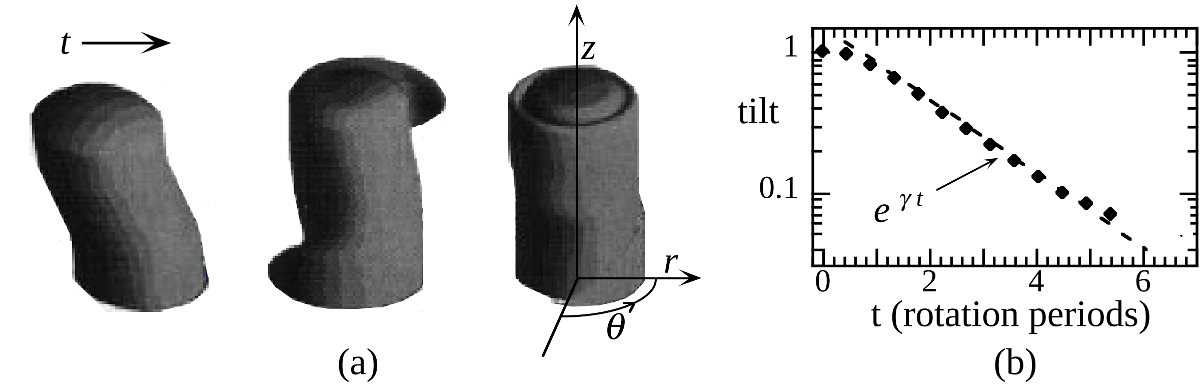

Figure 1: Vortex alignment. (a) Typical evolution of potential vorticity. (b) Typical plot of tilt versus time. The tilt (defined in Sec. 1.2) is normalized to its initial value. Time is measured in central rotation periods, $2\pi/\Omega_o(0)$.

2 Misalignment and Vortex Rossby Modes

We now show that a misalignment, as at $t = 0$ in Fig. 1(a), corresponds to exciting one or more vortex Rossby modes. To begin with, we decompose the PV distribution into an equilibrium component q_o and a perturbation Δq; that is,

$$q(r,\theta,z,t) = q_o(r) + \Delta q(r,\theta,z,t). \qquad (4)$$

Note that q_o has azimuthal and vertical symmetry. Unless otherwise stated, we further assume that q_o decreases monotonically with r until reaching zero. A typical q_o is sketched in Fig. 2.

A misalignment is created by shifting the equilibrium PV distribution horizontally to varying degrees at all vertical levels; i.e., $q \to q_o(|r\hat{r} - \vec{d}(z)|)$. To lowest order in the displacement amplitude $|\vec{d}|$, the corresponding PV perturbation is

$$\Delta q \ = \ - \, q_o'(r) \, |\vec{d}(z)| \, cos\left[\theta - \theta_d(z)\right]. \qquad (5)$$

Here, $q_o' \equiv dq_o/dr$, and θ_d is the orientation angle of the horizontal displacement vector $\vec{d}$.

A simple example of a "vortex Rossby mode" is an azimuthally propagating wave of the form

$$\Delta q = \frac{a}{2} \, \xi(r) \, cos(m\pi z/H) \, e^{i(n\theta - \omega t)} \ + \ c.c. \qquad (6)$$

Here, a is a complex mode amplitude, ω is the mode frequency, $m \in \{0, 1, 2, \ldots\}$, and $n \in \{1, 2, \ldots\}$. The radial part $\xi(r)$ of the mode depends implicitly on m, n and ω. Like Rossby waves on the β-plane, vortex Rossby modes are supported by a background PV gradient. On a monotonic vortex, they are retrograde; that is the phase-speed of a vortex Rossby mode is less than the angular velocity of the vortex in the region where the mode is peaked.

Of all the modes that are represented by Eq. (6), a misalignment [Eq. (5)] can project only onto those with azimuthal wave-number $n = 1$. When the stratification (N) is infinite, one can prove (Reasor and Montgomery 2001) that for each vertical wave-number m, there is an $n = 1$ vortex Rossby mode with $\omega = 0$ and

$$\xi(r) = c \, q_o'(r). \qquad (7)$$

Here, c is a normalization constant. When the stratification is finite, similar vortex Rossby modes exist.[1] In general, ω is nonzero, but Eq. (7) remains approximately valid. Upon comparing Eq. (7) to Eq. (5), it is clear that the $n = 1$ vortex Rossby modes have the same radial dependence as a misalignment. Hence, these $n = 1$ modes dominate a misalignment.

For future analysis, we define the critical radius of a mode. The critical radius r_* is where the fluid in the vortex co-rotates with the mode; that is,

$$\Omega_o(r_*) \equiv \omega/n. \qquad (8)$$

Here, $\Omega_o(r)$ is the unperturbed angular rotation frequency of the vortex. For monotonic vortices, $\Omega_o' < 0$ for all $r > 0$.

Finally, the streamfunction of a vortex Rossby mode is given by

$$\Delta\psi = \frac{a}{2}\Psi(r) \, cos(m\pi z/H) \, e^{i(n\theta - \omega t)} \ + \ c.c., \qquad (9)$$

where

$$\Psi(r) = \int_0^\infty dr' r' G_{mn}(r|r')\xi(r'). \qquad (10)$$

The Green function G_{mn} is determined by the invertibility relation [Eq. (3)] and the conditions that Ψ is regular at $r = 0$ and at $r = \infty$. This yields

$$G_{mn}(r|r') = -I_n(r_</l_R)K_n(r_>/l_R), \qquad (11)$$

[1] See Sec. 5 for exceptions.

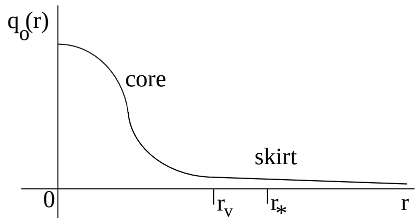

Figure 2: Core and skirt of the equilibrium PV profile.

where I_n and K_n are modified Bessel functions, $r_>$ ($r_<$) is the greater (smaller) of r and r', and

$$l_R \equiv NH/(m\pi f). \tag{12}$$

The length l_R is referred to as the "internal Rossby deformation radius." It increases with the level of stratification (N) and the height of the vortex (H).

3 Resonant Wave-Fluid Interaction

3.1 Vortex Core and Skirt

Figure 2 illustrates how a vortex can be decomposed into a core of radius r_v, and an outer skirt of relatively small PV. According to Eq. (7), the $n = 1$ vortex Rossby modes are concentrated in the core, where the equilibrium PV gradient is relatively large. Even if $n > 1$, the vortex Rossby modes are generally concentrated in the core.

The skirt may be viewed as an external object, with which a core vortex Rossby mode interacts. This interaction intensifies at r_*, where the mode resonates with the fluid rotation. If there is infinite stratification, or more generally if l_R is infinite, then $\omega = 0$ and r_* is infinite for $n = 1$. In this case, there is no resonance in the skirt. However, if l_R is finite, then in general $0 < \omega/n < \Omega_o(r_v)$, and r_* is somewhere in the skirt (see Fig. 2). As we will see, the resonance at r_* causes the mode amplitude $|a|$ to decay exponentially, provided that the PV gradient is nonzero and negative at r_*.

3.2 Conservation of Wave Activity

Conservation of wave activity $\mathcal{A}$ can be used to understand how a resonance in the skirt affects the amplitude of a vortex Rossby mode in the core. The wave activity is defined by the following:

$$\mathcal{A} \equiv \int d^3r \, \frac{r}{2\, q_o'} \, (\Delta q)^2, \tag{13}$$

where $q_o' \equiv dq_o/dr$ and the integral is over the entire volume of the vortex. It will prove useful to express conservation of wave activity as a balance between the rates of change of $\mathcal{A}$ in the core and in the skirt:

$$\frac{d}{dt}\int_c d^3r \, \frac{r}{2\, q_o'} \, (\Delta q)^2 = -\frac{d}{dt}\int_s d^3r \, \frac{r}{2\, q_o'} \, (\Delta q)^2. \tag{14}$$

Here, the subscripts 'c' and 's' denote integration over the core and skirt, respectively.

For concreteness, suppose that at $t = 0$ we excite the $(1, 1)$ vortex Rossby mode; that is, the mode with $m = 1$ and $n = 1$. This corresponds to tilting the vortex core, as at $t = 0$ in Fig. 1(a).

The streamfunction of the excited vortex Rossby mode extends into the skirt. As a result, the mode creates a PV perturbation there. Figure 3 illustrates the evolution of this PV perturbation near the critical radius r_*, in a reference frame that co-rotates with the mode. In Sec. 3.3, we will verify that the PV perturbation near r_* has the greatest effect on the evolution of the vortex Rossby mode, after a few rotation periods. For now, we simply assume that the skirt integral on the right side of Eq. (14) is dominated by the contribution near r_*.

For monotonic vortices, $q_o' < 0$ in the core and at the critical radius r_*. So, by conservation of wave activity [Eq. (14)], the growth of Δq^2 near r_* must cause Δq^2 in the core (i.e., the mode) to decay. On the other hand, if $q_o'(r_*)$ is positive, then the disturbance about r_* causes the mode, and the associated tilt, to grow in the core. If $q_o'(r_*) = 0$, then no PV perturbation develops at r_*, and the mode amplitude stays roughly constant.

We note that the resonant damping (or growth) of a vortex Rossby mode can also be explained using conservation of canonical angular momentum, $P \equiv \int d^3r \, r^2 q$ (Schecter et al. 2001).

3.3 Exponential Damping

We now transform the balance of wave activity [Eq. (14)] into an equation for the evolution of the mode amplitude $|a|$. To begin with, we assume that the perturbation in the vortex core is dominated by

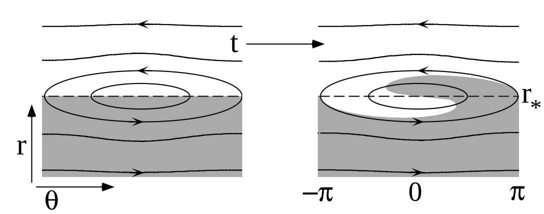

Figure 3: Streamlines and evolution of PV near r_* in the r-θ plane, at an arbitrary vertical level. The boundary between the shaded and unshaded regions is a line of constant PV.

a single vortex Rossby mode, with vertical and azimuthal wave-numbers m and n, respectively. The PV perturbation Δq of this mode is given by Eq. (6), with $\xi(r) = 0$ outside the core and $a \rightarrow a(t)$.

In time, the mode creates a PV perturbation in the skirt. This perturbation has the same wavenumbers as the vortex Rossby mode; that is, in the skirt, $\Delta q = \delta q(r,t) cos(m\pi z/H) e^{in\theta} + c.c.$ The linearized advection equation in the skirt is

$$\frac{\partial}{\partial t}\delta q + in\Omega_o(r)\delta q = \frac{inq_o'(r)}{r}\frac{a(t)}{2}\Psi(r)e^{-i\omega t}. \quad (15)$$

In deriving Eq. (15), we have used Eq. (9) for the streamfunction perturbation $\Delta\psi$. This assumes that the mode's contribution to $\Delta\psi$ dominates. Integrating Eq. (15), we obtain

$$\delta q = in\frac{q_o'\Psi}{2r}e^{-in\Omega_o t}\int_0^t dt' a(t')e^{i(n\Omega_o-\omega)t'}. \quad (16)$$

Here, we have assumed that $\delta q = 0$ at $t = 0$.

We may use Eqs. (6) and (16) to evaluate the left and right sides, respectively, of Eq. (14). This yields an equation for the mode amplitude of the form

$$\frac{d}{dt}|a| \; = \; \gamma\, |a|, \quad (17)$$

which is approximately valid for times in the range $\omega^{-1} \ll t \lesssim |\gamma|^{-1}$. Equation (17) implies that $|a(t)| \propto e^{\gamma t}$. The decay/growth rate γ depends crucially on the PV gradient at r_*, where the resonance occurs. Specifically,

$$\gamma \equiv -\frac{\pi n}{\int_c drr^2\xi\xi^*/q_o'}\left.\frac{q_o'\Psi\Psi^*}{|\Omega_o'|}\right|_{r_*}. \quad (18)$$

For monotonic vortices, $q_o' < 0$ in the vortex core, and the integral in Eq. (18) is negative. So, if $q_o'(r_*) < 0$, γ is also negative and the vortex Rossby mode decays exponentially. On the other hand, if $q_o'(r_*) > 0$, the mode grows, as explained in Sec. 3.2.

In general, γ must be computed numerically. A damped vortex Rossby mode is not associated with a complex eigenfrequency, since at late times ($t >> |\gamma|^{-1}$) the tilt decays non-exponentially. So, computing the eigenmodes of a vortex does not reveal the decay rate. Instead, one can first remove the skirt. The remaining core will have a discrete eigenmode which is undamped. The radial eigenfunctions (ξ and Ψ) and the critical radius r_* of this mode can be used to evaluate Eq. (18). A more accurate approach is to search for "Landau poles." This numerical method is explained by Spencer and Rasband (1997) and Schecter et al. (2000) for a related problem- mode damping in an ideal 2D vortex.

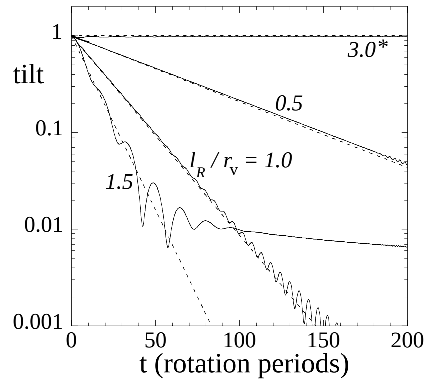

Figure 4: Tilt versus time for an RWS vortex.

In special cases, Eq. (18) is analytically tractable. For example, consider a Rankine-with-skirt (RWS) vortex:

$$\begin{array}{lll} q_o(r) & = Q_o & (r < r_v - \varepsilon), \\ & \ll Q_o & (r > r_v + \varepsilon), \end{array} \quad (19)$$

where Q_o is a constant and $\varepsilon << r_v$. In the transition layer, $|r - r_v| < \varepsilon$, the PV decreases monotonically with radius to a value that is much less than Q_o. It can be shown that the core Rossby modes of an RWS vortex have the following decay rates:

$$\gamma \simeq \pi n Q_o \frac{q_o'(r_*)G_{mn}^2(r_*|r_v)}{|\Omega_o'(r_*)|}, \quad (20)$$

where

$$r_* \simeq r_v\left[1 + 2G_{mn}(r_v|r_v)\right]^{-1/2}, \quad (21)$$

and $\Omega_o'(r_*) \simeq -Q_o r_v^2/r_*^3$. In the limit where ε is zero, the PV radial profiles of these modes are $\xi(r) = Q_o r_v \delta(r - r_v)$, where $\delta(r - r_v)$ is the Dirac delta-function centered at r_v.

4 Alignment Rate Versus l_R

We now show that the theory of resonant damping (Sec. 3) accurately predicts the alignment rates that are observed in numerical simulations. In addition, we show that the alignment rate can either increase or decrease with the internal Rossby deformation radius l_R, depending on the form of $q_o(r)$.

In all simulations, we give the vortex a simple tilt at $t = 0$, as in Fig. 1(a). By definition, a simple tilt is a misalignment [Eq. (5)] with $\vec{d} = D\, cos(\pi z/H)\, \hat{i}$. Here, D is a constant length and $\hat{i}$ is a constant horizontal unit vector. A simple tilt primarily excites the $(1,1)$ vortex Rossby mode.

We first consider the decay of a simple tilt on an RWS vortex [Eq. (19)]. The skirt of this particular RWS vortex extends to $r = 2.67r_v$, beyond which

$q_o(r) = 0$. Figure 4 shows the decay for several values of l_R in the range $0.5r_v \leq l_R \leq 3r_v$. The measures of tilt and time are the same as in Fig. 1(b). The solid curves are obtained from numerical integrations of the linearized QG equations. The dashed curves correspond to the theory of resonant damping, in which the tilt varies with time like $e^{\gamma t}$, and γ is given by Eq. (20) with $m = n = 1$. Early on, there is excellent agreement between the theory of resonant damping and the linear simulations. However, in accord with time-asymptotic linear theory, when $t >> |\gamma|^{-1}$, the streamfunction tends toward power-law decay.

Note that the vortex does not align when $l_R = 3r_v$. In this case, the critical radius ($r_* = 3.44r_v$) is beyond the skirt; i.e., $q_o'(r_*) = 0$. Therefore, there is no available mechanism for resonant damping.

We now consider the decay of a simple tilt on a Gaussian vortex, for which the equilibrium profile is $q_o(r) = Q_o e^{-9r^2/2r_v^2}$. Figure 5 shows the evolution of the tilt amplitude (measured at $r = r_v/3$) for $0.28r_v \leq l_R \leq 1.51r_v$. The solid curves are from numerical integrations of the linearized QG equations. The diamonds are from nonlinear simulations, taken directly from Reasor and Montgomery (2001). In these nonlinear simulations, $D = 0.3 \cdot min\,\{|Q_o/q_o'|\}$.

The dashed lines in Fig. 5 correspond to the theory of resonant damping. In each case, the theoretical value of γ was obtained numerically, by searching for a Landau pole (Spencer and Rasband 1997). Clearly, the theory of resonant damping accurately describes the alignment process in the Gaussian simulations.

Interestingly, Figs. 4 and 5 indicate that there is no universal dependence of the alignment rate on the internal Rossby deformation radius, at least in the range $.3r_v \lesssim l_R \lesssim 1.5r_v$. In this range, the alignment rate $|\gamma|$ for an RWS vortex increases with l_R. In contrast, for a Gaussian vortex, the alignment rate decreases as l_R increases.

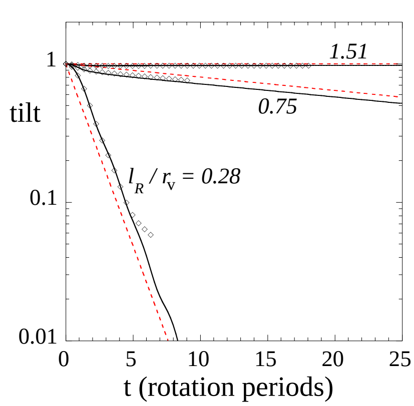

Figure 5: Tilt versus time for a Gaussian vortex.

5 Limitations

We have shown that geophysical vortex alignment can be driven by the resonant damping of a vortex Rossby mode. However, our analysis has limitations.

To begin with, we have only discussed the linear damping of a vortex Rossby mode. If the amplitude of the mode is initially large, nonlinear effects may rapidly dominate. For example, after a brief period of decay, the mode amplitude can oscillate and then equilibrate at a finite value; hence, nonlinear effects can inhibit vortex alignment. This possibility is anticipated from studies of mode damping in 2D vortices (e.g., Schecter et al. 2000).

Furthermore, we have found that there are no $n = 1$ vortex Rossby modes if l_R is less than a critical length scale l_c. The value of l_c depends on the particular form of $q_o(r)$. For an RWS vortex [Eq. (19)], l_c approaches zero as the width 2ε of the transition layer approaches zero. For a Gaussian vortex, we have found that $l_c \simeq r_v/4$. If $l_R \lesssim l_c$, the misalignment generally decays non-exponentially with time.

Finally, the QG equations of Sec. 1.1 assume that the vortex rotation frequency is small compared to the Coriolis parameter f. In addition, they assume that diabatic processes, such as moist convection, are negligible. Both of these assumptions are debatable for vortices of interest, such as incipient tropical cyclones. Future work will determine whether or not the resonant damping of vortex Rossby modes contributes significantly to the alignment of such complex vortices.

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