P3.15 THE TRADITIONAL APPROXIMATION AND EQUATORIAL INERTIAL INSTABILITY

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1 INTRODUCTION

The "traditional" hydrostatic approximation to the equations of fluid motion on a sphere includes certain geometrical approximations necessary for retaining conservation of energy and absolute angular momentum. In particular, the radial coordinate is replaced by the earth's mean radius (thereby suppressing the vertical variation in the planetary angular momentum), and the horizontal component of the earth's rotation vector is neglected. The neglected terms are most significant near the equator. White and Bromley (1995) and de Verdière and Schopp (1994) have suggested that neglecting these terms near the equator is not justified.

We consider the importance of making the hydrostatic approximation in the context of equatorial inertial instability. Inertial instability refers to a flow becoming unstable due to its distribution of angular momentum. The simplest case, axisymmetric circular flow, is unstable if the angular momentum decreases with distance from the axis of rotation. Adjustment to a stable state involves the formation of vertical rolls superposed on the circular flow (known as Taylor vortices). In the equatorial middle atmosphere, the approximately zonal mean flow can become inertially unstable. In the hydrostatic system, stability requires that the angular momentum be maximum at the equator and decrease with increasing absolute latitude. Dunkerton (1981) and others have shown the formation of Taylor vortices in the vertical-meridional plane when the stability condition are violated.

To focus attention on the essential dynamics, this study is restricted to the equatorial β-plane.

2 HYDROSTATIC CASE

2.1 Governing equations

The governing equations in pressure coordinates (x, y, p) for axisymmetric, adiabatic flow on the equatorial β-plane are

$$\begin{aligned} u_t &= -vu_y - \omega u_p + \beta yv && (1)\\ v_t &= -vv_y - \omega v_p - \beta yu + \Phi_y && (2)\\ \theta_t &= -v\theta_y - \omega\theta_p && (3)\\ v_y + \omega_p &= 0 && (4) \end{aligned}$$

where $u = Dx/Dt$, $v = Dy/Dt$ and $\omega = Dp/Dt$, θ is the potential temperature (equivalent to entropy), Φ is the geopotential, and $\beta \equiv 2\Omega/a$.

It follows from (1) and (2) that the equivalent of zonal absolute angular momentum,

$$m \equiv u - \tfrac{1}{2}\beta y^2 \qquad (5)$$

is a material invariant ($Dm/Dt = 0$).

The system (1)-(4) conserves the *Hamiltonian* functional

$$\begin{aligned} \mathcal{H}(m, v, \theta) &= \iint \Big\{ \tfrac{1}{2}v^2 + \tfrac{1}{2}\beta y^2 m \\ &\quad + \mathcal{E}(\rho, \theta) + \frac{p}{\rho^2} \Big\}\, dydp. \qquad (6) \end{aligned}$$

$\mathcal{E}(\rho, \theta)$ is the internal energy and ρ is the density. It also conserves the family of *Casimir invariants*

$$\mathcal{C}(m, \theta) = \iint C(m, \theta)\, dydp \qquad (7)$$

where C is an arbitrary function of m and θ.

2.2 Linear stability

An equilibrium X_0 of a set of partial differential equations is *linearly stable* if the equations linearized about X_0 have no exponentially growing modes.

It can be shown that X_0 is linearly stable if a conserved functional of the dynamics can be found that is minimized at X_0. The *Energy-Casimir* method (see Shepherd 1990) proceeds by constructing the combined invariant $\mathcal{H}+\mathcal{C}$, and finding conditions on the function $C(m, \theta)$ such that X_0 is a minimum.

Let X_0 be an equilibrium state with

$$m = M(y, p), \quad v = 0, \quad \theta = \Theta(y, p), \quad \rho = R(\Theta, p)$$

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and satisfying *thermal wind balance*

$$\beta y \frac{\partial M}{\partial p} = \frac{1}{R\Theta} \frac{\partial \Theta}{\partial y}. \tag{8}$$

Requiring X_0 to be a critical point of $\mathcal{H}+\mathcal{C}$ leads to the restrictions on $C(m, \theta)$:

$$C_m = -\frac{1}{2}\beta y^2, \quad C_\theta = -c_p \frac{T}{\Theta}, \tag{9}$$

where $T(\Theta, p)$ is the temperature, c_p is the specific heat capacity at constant pressure and the ideal gas law $p = \rho R_g T$ has been assumed. $\mathcal{H} + \mathcal{C}$ is convex (X_0 is a minimum) if

$$C_{mm} > 0 \tag{10}$$

$$C_{\theta\theta} > 0 \tag{11}$$

$$C_{mm} C_{\theta\theta} - C_{m\theta}^2 > 0. \tag{12}$$

In terms of the potential vorticity

$$Q = \frac{\partial M}{\partial y}\frac{\partial \Theta}{\partial p} - \frac{\partial M}{\partial p}\frac{\partial \Theta}{\partial y}, \tag{13}$$

(10)-(12) can be more instructively written as

$$-\frac{\beta y}{Q}\frac{\partial \Theta}{\partial p} > 0 \tag{14}$$

$$-\frac{1}{R\Theta Q}\frac{\partial M}{\partial y} > 0 \tag{15}$$

$$\frac{\beta y}{R\Theta Q} > 0. \tag{16}$$

(16) indicates that Q must have the same sign as y. (14) expresses static stability (entropy increases with height) and (15) inertial stability (angular momentum decreases with distance from the equator).

2.3 Nonlinear stability

An equilibrium satisfying the conditions (14)-(16) can be shown to be *nonlinearly stable* (that is, stable to finite amplitude disturbances of the dynamical fields). Nonlinear stability of a state X_0 with respect to a norm $||\cdot||$ means that for a nonequilibrium state $X(t)$, and for all $t > 0$,

$$||X(0) - X_0|| < \varepsilon \Rightarrow ||X(t) - X_0|| < \delta(\varepsilon) \tag{17}$$

We demonstrate nonlinear stability by defining the *pseudoenergy*

$$\mathcal{A} = \mathcal{H}(X(t)) + \mathcal{C}(X(t)) - \mathcal{H}(X_0) - \mathcal{C}(X_0), \tag{18}$$

and bounding it from below and above by disturbance norms for all t. Since $\mathcal{A}$ is conserved, it follows that disturbances do not grow without bound.

The nonlinear stability analysis provides a method for deriving rigorous upper bounds on the growth of instabilities. Unstable equilibria can be considered finite amplitude disturbances to nonlinearly stable equilibria. See Mu et al. (1996) in which the method is applied to a related problem.

3 NONHYDROSTATIC CASE

If the hydrostatic approximation is relaxed, then the symmetric β-plane equations include extra Coriolis force terms in the zonal and vertical momentum equations, and the materially conserved equivalent of zonal angular momentum is

$$m \equiv u - \tfrac{1}{2}\beta y^2 + \gamma z \tag{19}$$

Similar conditions to (14)-(16) can be derived. Hua et al. (1997) observed that inertial instability in the oceans occurs when the nonhydrostatic stability conditions are violated.

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