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THE CONSTRUCTION OF THE NUMERICAL SCHEMES IN THE HERMITIAN FINITE ELEMENTS SPACES

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1. INTRODUCTION

A number of different methods have been developed for numerical modeling of initial-boundary value problems of hydrodynamics. The most popular ones are: the finite differences, spectral, pseudo-spectral and collocation methods. The finite element methods have been known for over thirty years. Their theoretical discussion was presented by Zienkiewicz (1971) and Strang and Fix (1973).

Cullen (1974) was the first to use the finite elements in modeling of atmospheric processes. His example was followed by Staniforth and Daley (1977) and Staniforth and Mitchell (1977). In their considerations they mainly applied Lagrange's approximations, which for any variable v can be written in the form $\Sigma v_i c_i$, where c_i, basis functions, usually linear or low-order polynomials, v_i – the value of v at the node i.

On the basis of the theory of approximation Strang and Fix (1973) shortly described a second approach to the problem of solving partial differential equations using finite elements. They defined new finite elements spaces, called Hermitian spaces, and suggested that unknown solutions could be approximated in the form:

$$v_h(x,t) = \sum [v_i(t)\varphi_i(x) + v_i^{'}(t)\psi_i(x)] \qquad (*)$$

Ten years later Rymarz and Winnicki (1984) analyzed more complicated finite element approximations, which were written in the general form:

$$v_h(x,t) = \sum [v_i(t)\varphi_i(x) + v_i^{'}(t)\psi_i(x) + v_i^{''}(t)\gamma_i(x)] \qquad (**)$$

where $v_h(x,t)$ is a discretization of the exact solution $v(x,t)$; $v_i^{'}, v_i^{''}$ - the first and the second derivative of the solution.

2. THE FINITE ELEMENT SPACES

We will concentrate on the finite element approximation for initial-boundary value problem:

$$\begin{cases} \mathbf{f}_h = \sum [\mathbf{f}_i(t)\varphi_i(x) + \mathbf{f}_i^{'}(t)\psi_i(x)], & t > 0 \\ \mathbf{f}_{0,h} = \sum [\mathbf{f}_{0,i}(t)\varphi_i(x) + \mathbf{f}_{0,i}^{'}(t)\psi_i(x)], & t = 0 \end{cases} \qquad (1)$$

where $\mathbf{f}_i$ vector of values of sought-for functions $\mathbf{f}_h$ at the node i; h - mesh spacing; $\mathbf{f}_i^{'}$ - values of the first derivatives of the solution $\mathbf{f}$; $\varphi_i(x), \psi_i(x)$ - Hermitian space basis functions. It is important to notice that each basis function has the compact support and this support is *small*. It is given by a few elements of the space. These functions vanish at the nodes outside the element and take the value 1 at the node i. For (*) the assumptions hold piece-wise cubic functions.

Now we can define Hermitian space $V_h^{(3)}$:

$$\begin{aligned} V_h^{(3)} = \{v : v \in C^1[0,l], v|_K \in P_3^{(i)}(K), i = 1,\ldots,N_h, \\ v = 0,\ Dv = 0,\ x = 0,\ x = l\} \end{aligned} \qquad (2)$$

The basis of $V_h^{(3)}$ is a pair of cubic piece-wise polynomials. They are of C^1 type. The space $V_h^{(3)}$ is a Hilbert space, $V_h^{(3)} \subset \tilde{H}_0^2(0,l)$, endowed by the scalar product:

$$(w,v)_{\tilde{H}_0^2(\Omega)} = \int_\Omega w(x)v(x)dx \qquad (3)$$

3. PROBLEM FORMULATION FOR DIFFUSION EQUATION

Let us consider the linear one-dimensional diffusion equation with irregular initial condition:

$$\frac{\partial u}{\partial t} - a^2 \frac{\partial^2 u}{\partial x^2} = 0, \quad u(x,0) = u_0(x) \qquad (4)$$

After applying the finite element method's technique in the Hermitian space $V_h^{(3)}$ for $u_h = v_h(x,t)$ (HFEM):

$$\begin{cases} (\partial u/\partial t - a^2\partial^2 u/\partial x^2, \varphi_k) = 0 \\ (\partial u/\partial t - a^2\partial^2 u/\partial x^2, \psi_k) = 0 \\ (u(x,0), \varphi_k) = (u_0(x), \varphi_k) \\ (u(x,0), \psi_k) = (u_0(x), \psi_k) \end{cases} \qquad (5)$$

we obtain the system of implicit discrete equations:

$$\begin{cases} H_i^{n+1} + \beta^2 L_i^{n+1} = H_i^n \\ H_i^{'n+1} + \beta^2 L_i^{'n+1} = H_i^{'n} \\ H_i^0 = (420/h)\int_{x_{i-1}}^{x_{i+1}} u_0(x)\varphi_i(x)dx \\ H_i^{'0} = (420/h^2)\int_{x_{i-1}}^{x_{i+1}} u_0(x)\psi_i(x)dx \end{cases} \qquad (6)$$

$$\begin{aligned} H_i &= 54(u_{i-1} + u_{i+1}) + 312u_i + 13h(u_{i-1}^{'} - u_{i+1}^{'}) \\ H_i^{'} &= -3h(u_{i-1}^{'} + u_{i+1}^{'}) + 8hu_i^{'} - 13(u_{i-1} - u_{i+1}) \\ L_i &= -504(u_{i-1} - 2u_i + u_{i+1}) - 42h(u_{i-1}^{'} - u_{i+1}^{'}) \\ L_i^{'} &= 42(u_{i-1} - u_{i+1}) - 14h(u_{i-1}^{'} + u_{i+1}^{'}) + 112hu_i^{'} \end{aligned} \qquad (7)$$

where $\beta = a^2\tau/h^2$; τ, h - time integrating and spatial steps; a^2 – diffusion coefficient; $a^2 = \text{const} > 0$. The values u_i, $u_i^{'}$ are usually called *the nodal parameters* of the Hermitian space $V_h^{(3)}$.

System Eqns. (6) is equivalent to the system of linear algebraic equations. We rewrite it in the form:

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$$-\mathbf{A}\mathbf{U}_{i-1}^{n+1}+\mathbf{B}\mathbf{U}_{i}^{n+1}-\mathbf{C}\mathbf{U}_{i+1}^{n+1}=\mathbf{F}_{i}^{n} \qquad (8)$$

so that the vectorial variant of passage method could be used for solving it; where: $\mathbf{U}$ - vector of solutions, $\mathbf{U}=(u,u')^T$ for (*) and $\mathbf{U}=(u,u',u'')^T$ for (**); $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ - matrices of the coefficients built for Eqn. (7),

$$\mathbf{A}=\begin{bmatrix}-54+504\beta & -13h+42h\beta\\ 13-42\beta & 3h+14h\beta\end{bmatrix}$$

$$\mathbf{B}=\begin{bmatrix}312+1008\beta & 0\\ 0 & 8h+112h\beta\end{bmatrix} \qquad (9)$$

$$\mathbf{C}=\begin{bmatrix}-54+504\beta & 13h-42h\beta\\ -13+42\beta & 3h+14h\beta\end{bmatrix}$$

for (*) and

$$\mathbf{A}=\begin{bmatrix}-6000+79200\beta & -1812h+11880h\beta & -181h^2+660h^2\beta\\ 1812-11880\beta & 532h+792h\beta & 52h^2+264h^2\beta\\ -181+660\beta & -52h-264h\beta & -5h^2-44h^2\beta\end{bmatrix}$$

$$\mathbf{B}=\begin{bmatrix}43440+158400\beta & 0 & 562h^2+1320h^2\beta\\ 0 & 1664h+25344h\beta & 0\\ 562+1320\beta & 0 & 12h^2+176h^2\beta\end{bmatrix}$$

$$\mathbf{C}=\begin{bmatrix}-6000+79200\beta & 1812h-11880h\beta & -181h^2+660h^2\beta\\ -1812+11880\beta & 532h+792h\beta & -52h^2-264h^2\beta\\ -181+660\beta & 52h+264h\beta & -5h^2-44h^2\beta\end{bmatrix}$$

for (**). It is well known that the finite element methods lead to the difference schemes, which are similar to the finite differences schemes. In the Hermitian spaces we obtain not only the solution, but also the values of its first or first and second derivatives.

To the analysis of the diffusion process let us apply more completed finite element approximation, which includes function and its first and second derivatives. This problem is graphically illustrated in Fig. 1.

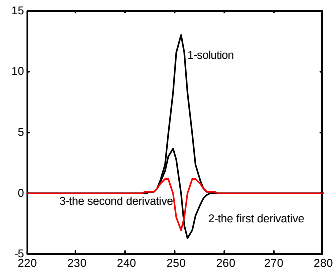

Fig. 1. Solutions of the diffusion equation.

Figure 1 presents the diffusion equation (4) solutions with the initial condition in the Dirac's delta function form: $u_0=\begin{cases}1, & x=x_0\\ 0, & x\neq x_0\end{cases}$. The curve 1 is the graph of the solution u, curve 2 its first derivative u', and the curve 3 its second derivative u''.

4. CONCLUSION

Applying the Hermitian idea of constructing difference schemes we can describe the irregular problems in a much more complete and correct manner. Considering additional nodal parameters in the approximated solutions furnishes us with new information about the analyzed process and makes possible not only accurate localization of any discontinuity or irregularity of the initial condition, but also more accurate plotting of that irregularity in consecutive time levels, Winnicki (1988) and Winnicki and Jasinski (2000). Regardless of the number of nodal parameters the Hermitian method leads to three-point difference schemes on the supports $(i-1,i,i+1)$ with additional degrees of freedom at every point. This feature is not true for Lagrangian schemes, which include $2p+1$ points, p – order of polynomial; and their supports, for $p=2$, are $(i-2,i-1,i,i+1,i+2)$.

The accuracy of the system (8) is the fourth order in space for (*) and the sixth order for (**). It can be proved that their order of accuracy is equal to h^{p+1} (in (**) the basis functions are fifth order polynomials). The obtained schemes are unconditionally stable (they are always implicit ones). They are not dispersive, but they are lightly dissipative. The schemes can be the second order accurate in time if for spatial operator the Crank-Nicholson approximation is applied.

The results suggest that more attention should be given to the finite element methods in Hermitian spaces investigations. It seems that the nodal parameter in the first derivative form can be a very good *dissipative mesh* of the difference schemes for advection equation (Winnicki, 2000). It is still unknown the role of the second derivative of the solution.

5. REFERENCES

Cullen, M. J. P., 1974, Integration of the primitive equations on a sphere using the finite element method., *Quart. J. R. Met. Soc.*, **100**, 555-562.

Rymarz, Cz., and Winnicki, I., 1984, Some effective numerical methods for solving irregular boundary-value problems of diffusion and hydrodynamics, *J. Tech. Phys.*, Polish Academy of Science, (in English), **25**, 351-365.

Staniforth, A. N., and Daley, R., 1977, A finite element formulation for the vertical discretisation of σ-coordinate primitive equation models., *Mon. Weath. Rev.*, **105**, 1108-1118.

Staniforth, A. N., and Mitchell, H. L., 1977, A semi-implicit finite element barotropic model., *Ibid.*, **105**, 154-169.

Strang, G., and Fix, G. J., 1973, *An Analysis of the Finite Element Method*, Prentice-Hall, Englewood Cliffs.

Winnicki, I., 1988, Analysis of Some Selected Numerical Methods from the Point of View of Irregular Initial Conditions, *J. Tech. Phys.*, Polish Academy of Sciences, (in English), **29**, 297-309.

Winnicki, I., and Jasinski, J., 2000, Finite element method in Hermitian spaces in the analysis of irregular atmospheric processes., *Proc.* 16*th International Conference on Interactive Information and Processing Systems for Meteorology, Oceanography and Hydrology*., Long Beach, USA, 09-14.01.2000.

Zienkiewicz, O. C., 1971, *The finite element method in engineering science*., McGraw-Hill.