1. INTRODUCTION

A number of different methods have been developed for numerical modeling of initial-boundary value problems of hydrodynamics. The most popular ones are: the finite differences, spectral, pseudo-spectral and collocation methods. The finite element methods have been known for over thirty years. Their theoretical discussion was presented by Zienkiewicz (1971) and Strang and Fix (1973).

Cullen (1974) was the first to use the finite elements in modeling of atmospheric processes. His example was followed by Staniforth and Daley (1977) and Staniforth and Mitchell (1977). In their considerations they mainly applied Lagrange's approximations, which for any variable \( v \) can be written in the form \( \sum \chi_i v \), where \( \chi_i \) basis functions, usually linear or low-order polynomials, \( v \) - the value of \( v \) at the node \( i \).

On the basis of the theory of approximation Strang and Fix (1973) shortly described a second approach to the problem of solving partial differential equations using finite elements. They defined new finite elements spaces, called Hermitian spaces, and suggested that unknown solutions could be approximated in the form:

\[ v_h(x,t) = \sum [v_j(t)\psi_j(x) + v'_j(t)\psi'_j(x)] \]  

Ten years later Rymarz and Winnicki (1984) analyzed more complicated finite element approximations, which were written in the general form:

\[ v_h(x,t) = \sum [v_j(t)\psi_j(x) + v'_j(t)\psi'_j(x) + v''_j(t)\psi''_j(x)] \]  

where \( v_h(x,t) \) is a discretization of the exact solution \( v(x,t) \); \( v, v' \) - the first and the second derivative of the function; \( \psi, \psi' \) - Hermitean space basis functions. It is important to notice that each basis function has the compact support and this support is small. It is given by a few elements of the space. These functions vanish at the nodes outside the element and take the value 1 at the node \( i \). For (*) the assumptions hold piece-wise cubic functions.

Now we can define Hermitian space \( V_h^{(3)} \):

\[ V_h^{(3)} = \{ v : v \in C^1[0,1], v_{ix} \in P_{i}^{(3)}(K), i = 1, \ldots , N_h \}, \]

\[ v = 0, Dv = 0, x = 0, x = 1 \]  

The basis of \( V_h^{(3)} \) is a pair of cubic piece-wise polynomials. They are of \( C^1 \) type. The space \( V_h^{(3)} \) is a Hilbert space, \( V_h^{(3)} \subset H^1(0,1) \), endowed by the scalar product:

\[ (w,v)_{H^1(0,1)} = \int_0^1 w(x)v(x)dx \]  

3. PROBLEM FORMULATION FOR DIFFUSION EQUATION

Let us consider the linear one-dimensional diffusion equation with irregular initial condition:

\[ \frac{d\phi}{dt} - \alpha^2 \frac{d^2\phi}{dx^2} = 0, \quad \phi(x,0) = \phi_0(x) \]  

After applying the finite element method's technique in the Hermitian space \( V_h^{(3)} \) for \( \phi_h(x,t) \) (HFEM):

\[ (d\phi/dt - \alpha^2 d^2\phi/dx^2, \psi_{h_{k+1}}) = 0 \]

\[ (d\phi/dt - \alpha^2 d^2\phi/dx^2, \psi_{h_k}) = 0 \]

\[ (\phi(x,0), \psi_{h_k}) = (\phi_0(x), \psi_{h_k}) \]

we obtain the system of implicit discrete equations:

\[ H_{h_{k+1}}{\phi}_{h_{k+1}} + \beta^2 L_{h_{k+1}}{\phi}_{h_{k+1}} = H_{h_k}{\phi}_{h_k} \]

\[ H_{h_{k+1}} = (420/h^2) \sum_{i=1}^{N_h} \psi_i(x)\psi_i(x)dx \]

\[ L_{h_{k+1}} = (420/h^2) \sum_{i=1}^{N_h} \psi_i(x)\psi'_i(x)dx \]

\[ H_{h_0} = 54(u_{i+1} + u_{i+1}) + 312 u_i + 13h(u_{i+1} - u_{i+1}) \]

\[ H_{h_1} = 3h(u_{i+1} + u_{i+1}) + 8hu_i - 13h(u_{i-1} - u_{i+1}) \]

\[ L_{h_0} = -504(u_{i-1} - 2u_i + u_{i+1}) - 42h(u_{i-1} - u_{i+1}) \]

\[ L_{h_1} = 42h(u_{i-1} - u_{i+1}) - 14h(u_{i+1} + u_{i+1}) + 112 hu_i \]

where \( \beta = \alpha^2 \tau/h^2 \), \( \tau \) - time integrating and spatial steps; \( \alpha^2 \) - diffusion coefficient; \( \alpha^2 \) - const > 0. The values \( u_i \), \( u_i \) are usually called the nodal parameters of the Hermitian space \( V_h^{(3)} \).

System Eqns. (6) is equivalent to the system of linear algebraic equations. We rewrite it in the form:
finite differences schemes. In the Hermitian spaces we obtain not only the solution, but also the values of its approximation, which includes function.

4. CONCLUSION

The accuracy of the system (8) is the fourth order in space for (*) and the sixth order for (**). It can be proved that their order of accuracy is equal to $h^{p+1}$ (in (** the basis functions are fifth order polynomials). The obtained schemes are unconditionally stable (they are always implicit ones). They are not dissipative, but they are lightly dissipative. The schemes can be the second order accurate in time if for spatial operator the Crank-Nicholson approximation is applied. The results suggest that more attention should be given to the finite element methods in Hermitian spaces investigations. It seems that the nodal parameter in the first derivative form can be a very good dissipative mesh of the difference schemes for advection equation (Winnicki, 2000). It is still unknown the role of the second derivative of the solution.

5. REFERENCES


