P6.19 THE ROLE OF THE MOMENTUM DIVERGENCE EQUATION ELLIPTICITY IN THE NUMERICAL SOLUTIONS

Ireneusz A. Winnicki*, Krzysztof Kroszczynski*
Military University of Technology (MUT), Warsaw, Poland

1. INTRODUCTION

This paper can be treated as a short lecture addressed to the beginners at applied meteorology, especially at numerical weather prediction. Its aim is to discuss the influence of the ellipticity of the momentum divergence equation (so-called balance equation) on the solution of non-divergent equation of the solenoidal or geostrophic models in the barotropic approaches. The equation considered is a nonlinear differential equation of Monge-Ampere type (MA) used for streamfunction ψ field determining if the geopotential field is known. Transforming the governing equations and assuming that the divergence of the wind field is equal to zero it can be obtained, $\mathrm{div}\mathbf{V} = 0$. The nonlinear balance equation is a part of the general system equations describing quasi-solenoidal or quasi-geostrophic models.

Making use of the real aerological measurement data furnished by the international GRID network the Monge-Ampere equation has been solved numerically together with the solenoidal model in barotropic approximation.

2. PROBLEM FORMULATION

The simplest barotropic model is a solenoidal model, in which the velocity vector can be introduced as a hypothetical solenoidal wind described by a streamfunction ψ and a geostrophic model, where the geostrophic wind relations define the velocity field. In both cases the most accurate results are obtained by performing the analysis on the non-divergent layer of the atmosphere ($\mathrm{div}\mathbf{V} = 0$) at the height between 3 and 5 km. The models just mentioned are in meteorology known as non-divergent models.

The numerical weather forecasting process based upon the non-divergent models in barotropic approaches can be divided into three following stages:

1. computing the horizontal components of the wind velocity (the geostrophic model) or the streamfunction field (the solenoidal model) treating the geopotential as known data - solution of the balance equation;
2. predicting of the geopotential (geostrophic mo-del) or the streamfunction (the solenoidal model) for 12 hours, for instance, and
3. solving the balance equation for geopotential. In the geostrophic model, on the basis of the prognostic geopotential field, we obtain a prognostic wind field. In the solenoidal model, on the basis of prognostic streamfunction we obtain a prognostic geopotential field.

For the realization of the first stage of the weather forecasting process it is necessary to analyze a nonlinear balance equation for streamfunction y, a particular case of equation of Monge-Ampere type: (1)

$$\frac{\partial^2\psi}{\partial x^2}\cdot\frac{\partial^2\psi}{\partial y^2}-\left(\frac{\partial^2\psi}{\partial x\partial y}\right)^2=-\frac{f}{2}\nabla^2\psi+\frac{1}{2}\nabla^2\Phi-\frac{1}{2}\left(\frac{\partial\psi}{\partial x}\cdot\frac{\partial f}{\partial x}+\frac{\partial\psi}{\partial y}\cdot\frac{\partial f}{\partial y}\right)$$

f – Coriolis parameter. The MA equation is a second order nonlinear partial differential equation, written in general form:

$$r\cdot t-s^2=a\cdot r+2b\cdot s+c\cdot t+\vartheta,$$
$$r=\frac{\partial^2\psi}{\partial x^2},\quad s=\frac{\partial^2\psi}{\partial x\partial y},\quad t=\frac{\partial^2\psi}{\partial y^2}. \tag{2}$$

Its coefficients a, b, c and the term ϑ depend on x, y, the sought-for function and its first derivatives

$$p=\frac{\partial\psi}{\partial x},\quad q=\frac{\partial\psi}{\partial y} \tag{3}$$

The basic problem is the geopotential field geometry influence on the physically acceptable solution of MA equation. The criterion for classifying Eqn. (2) mainly depends on $\nabla^2\Phi$ and has the form:

$$\Delta=\vartheta+ac-b^2 \tag{4}$$

The MA equation is classified as elliptic if $\Delta>0$. In the opposite case it is hyperbolic, and if $\Delta=0$ this equation is parabolic. The solution of the Eqn. (2) is physically acceptable only when it is of elliptic type. Leading Eqn. (1) into Eqn. (2) form we obtain: (5)

$$m^4\frac{\partial^2\psi}{\partial x^2}\cdot\frac{\partial^2\psi}{\partial y^2}-m^4\left(\frac{\partial^2\psi}{\partial x\partial y}\right)^2=a\frac{\partial^2\psi}{\partial x^2}+2b\frac{\partial^2\psi}{\partial x\partial y}+c\frac{\partial^2\psi}{\partial x\partial y}+\vartheta$$

$$a=c=-\frac{fm^2}{2};b=0;\vartheta=\frac{m^2}{2}\nabla^2\Phi-\frac{m}{2}\left(\frac{\partial\psi}{\partial x}\cdot\frac{\partial f}{\partial x}+\frac{\partial\psi}{\partial y}\cdot\frac{\partial f}{\partial y}\right)$$

Where m^2 - the map scale factor in the polar stereographic projection. The term can be estimated as it follows: $\frac{m}{2}\left(\frac{\partial\psi}{\partial x}\frac{\partial f}{\partial x}+\frac{\partial\psi}{\partial y}\frac{\partial f}{\partial y}\right)<<\frac{m^2}{2}\nabla^2\Phi$. The type of the differential Eqn. (5) now can be determined by checking the expression sign:

$$\Delta=\vartheta+ac-b^2\approx\frac{f^2}{m^2}+2\cdot\nabla^2\Phi>0 \tag{6}$$

The Eqn. (5) is a diagnostic equation. Because of the nonlinearity, an iterative solution procedure must be used. A number of iterative methods may be applied to solve this problem (Gauss elimination, Fourier series expansion, relaxation). For example, Paegle and

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* *Corresponding authors address*: Prof. Ireneusz Winnicki, Ph.D.; Krzysztof Kroszczynski, Ph.D.: Mil. Univ. of Techn., Dept. of Civil Engineering, Chemistry and Applied Physics, Institute of Meteorology, 00-908 Warsaw, Poland, ul Kaliski 2, +48 22 6839475, irekwin@ack.wat.waw.pl
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Tomlinson (1975) described the Fourier transform and Gauss elimination algorithms applied to the balance equation in a spherical coordinate system. The same equation Gent et al. (1993) solved in cylindrical coordinates system. Winnicki (1995) presented the algorithm of solution of this problem written in the polar stereographic projection form.

The main point is that if the iterative methods lead to convergent sequences of solutions, the equation to be solved satisfies the general ellipticity condition of the MA equation. That condition depends mostly on the geometry of the geopotential field. It means the curvature of this field must be elliptic. We rewrite the ellipticity criterion (6) in the discrete form:

$$\delta_{i,j} = 2 \cdot \left(\Phi_{i,j+1} + \Phi_{i,j-1} + \Phi_{i+1,j} + \Phi_{i-1,j} - 4 \cdot \Phi_{i,j}\right) + S_{i,j} \geq 0$$
$$S_{i,j} = \frac{d^2}{m_{i,j}^2} \cdot f_{i,j}^2 \qquad (7)$$

where d is horizontal grid size.

We have to analyze the equations of MA type not only at the numerical solution of balance equation. This problem appears also in frontogenesis description and omega equation in the terms of the **Q**-vector.

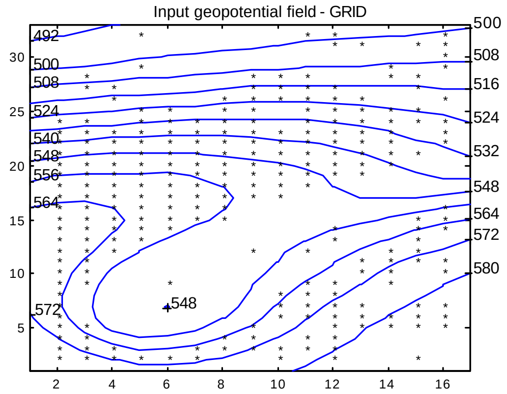

Fig. 1. The geopotential field. The stars indicate the point the ellipticity condition is not satisfied.

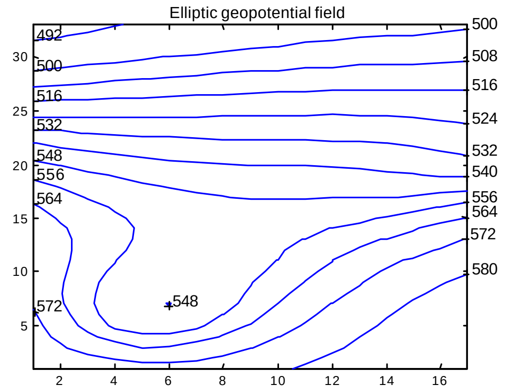

Fig. 2. The elliptic geopotential field.

3. AN EXAMPLE

Let us consider a geopotential field taken from the international GRID data (Fig. 1). The middle of the Figure 1 corresponds with the latitude of 10^{o}N (the equator zone). The number of non-elliptic nodes is equal to 235. The input field after modification is presented in Fig. 2.

Figure 3 graphically presents the solution of the momentum divergence equation (5) (MA equation) after geopotential field modification. For the Eqn. (5) solving the explicit finite differences method was applied.

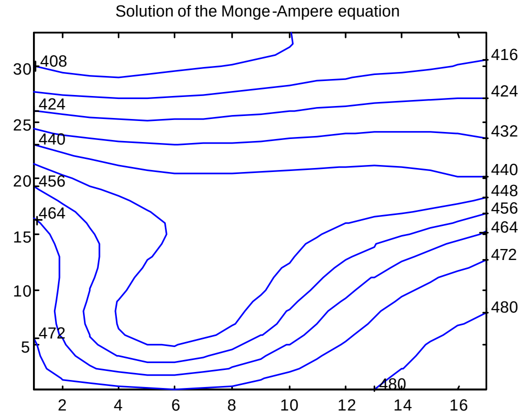

Fig. 3. The solution of the Monge-Ampere equation.

On the basis of the ellipticity criterion (7) we see that the condition strongly depends on the latitude. The same geopotential field situated on 25^{o}N indicates 155 non-elliptic nodes, on 45^{o}N – 61 nodes, and on 60^{o}N – 5 nodes. On the higher latitudes the geopotential fields are usually elliptic and therefore they do not require any modification. As an effect of the Eqn. (5) conversion we obtained the Poisson equation that was solved using the FFT technique (NDP FORTRAN compiler + IMSL library).

4. CONCLUSION

By analyzing geopotential fields we can determine regions, which can influence on the ellipticity of those fields, and then the ellipticity of the balance equation. They are mainly regions of high pressure (anticyclones and ridges) situated in the low latitudes. In the Fig. 2 we can see that the ridge distinctly weakens – an effect of using numerical subroutine. The value of f^2 in the low latitudes is positive and nearly equal to zero. It means the ellipticity condition $f^2/m^2 + 2 \cdot \nabla^2\Phi > 0$ may not be satisfied because of the negative values of the Laplacian of the geopotential, $\nabla^2\Phi$, over the high-pressure area. It is the main cause of the lack or slow convergence of the iteration process of balance equation solving.

5. REFERENCES

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