18th Conference on Weather Analysis and Forecasting/14th Conference on Numerical Weather Prediction, 30 July – 2 August, 2001, Fort Lauderdale, Florida

JP1.16 AGEOSTROPHIC FORCING IN A HEIGHT TENDENCY EQUATION: TWO CASE STUDIES

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1. INTRODUCTION

Ageostrophic motions, and ageostrophic (AG) vorticities, have been traditionally viewed as a response to dynamic and thermodynamic forcing processes, that when applied, perturb the balanced state of the atmosphere. These motions have typically been analyzed in the context of an atmosphere that is initially in geostrophic balance. AG motions or AG winds have been considered to be a component of the observed wind, or the difference between the observed wind and the geostrophic wind (e.g., Trenberth and Chen, 1988, *hereafter TC88*). The AG wind is proportional to and to the left (right) of the wind acceleration vector in Northern (Southern) Hemisphere (e.g., Bluestein, 1992). An atmosphere initially in geostrophic balance will respond to some imposed forcing mechanism that generates AG motions and induces vertical motions. Thus, vertical motions, and the 'secondary' circulations they are part of, are also considered an important component in examining AG circulations (e.g. Uccellini et al., 1984; Keyser et al., 1989; Loughe et al., 1995).

Formulations for the AG winds have included calculating this quantity as a residual, which is dependent upon how geostrophy itself is defined (Phillips, 1963; Blackburn, 1985). Alternative formulations for AG motions relating AG circulations to geostrophy have also been used in the examination of the three-dimensional circulations associated with upper tropospheric jets streaks and fronts (TC88; Xu, 1992; Xu and Keyser, 1993; Loughe et al., 1995). For the calculation of AG vorticity tendencies in a height tendency equation, such as the Zwack-Okossi (Z-O) equation (Zwack and Okossi, 1986; Lupo et al., 1992), it would be ideal to find a relationship that yields an instantaneous estimate of the AG vorticity tendency that is a diagnostic quantity.

The Z-O equation has been used as the diagnostic framework in the study of many atmospheric phenomena (e.g., Lupo et al., 1992; King et al., 1995; Lupo and Smith, 1995, *hereafter LS95*; Lupo and Bosart, 1999; Zhang et al., 1999). Of these papers, only Zwack and Okossi (1986) and Lupo et al. (1992) discuss the physical processes represented by the forcing terms in a detailed manner. Lupo et al. (1992) addresses the role of AG vorticity tendencies using the generalized form of the Z-O equation. They characterize this term as a “correction term that accounts for the possibility that the unbalanced atmosphere may attempt to reach a balanced state other than geostrophic”. Thus, AG vorticity tendencies, which appear explicitly in the Z-O equation, have not been viewed solely as either a forcing mechanism or a response to unbalanced forcing in a strict sense within this framework. Therefore, the exact role of AG vorticities in this equation has not been satisfactorily addressed.

An expression is developed here that can be substituted for the AG vorticity tendency in the Z-O equation. This relationship is derived using the TC88 methodology, which describes the AG wind. In deriving this relationship, it will be shown that the AG vorticity tendency term in the Z-O equation can be interpreted in the traditional sense and as a forcing mechanism. Also, it will be shown that the AG vorticity tendency can be partially partitioned among the other terms in the Z-O equation. Finally, it will be shown that the derived relationship produces a better estimate of the total Z-O height tendencies than simply calculating the AG vorticity tendency in the equation as a residual.

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2. THE EQUATION

The generalized Z-O equation, derived in Lupo et al., 1992 or LS95, is given by (1). Note that all equations are provided on the last page in order to save space. Also please refer to the referenced publications for the definitions of all symbols in these expressions, however, the symbols each have their usual meteorological meaning. The Z-O equation for pressure levels aloft (LS95) in which the diagnostic quantity includes the near-surface geostrophic vorticity tendency as a forcing process gives (2). In Eq. (1), the diabatic heating and other quantities that must be parameterized are treated in more detail by Lupo et al., (1992) and LS95. The vorticity tendencies are then relaxed to get height tendencies using sequential overrelaxation.

One advantage to using the Z-O equation is that observed quantities can be used in (1) instead of a geostrophic estimation (Zwack and Okossi, 1986). Since the observed winds, and thus vorticity tendencies, can be partitioned mathematically into both geostrophic and AG winds, this gives rise to an AG vorticity tendency term in Eq. (1) (Lupo et al., 1992). Lupo et al. (1992) describe the manner in which all the terms in Eq. (1) physically force vorticity and height changes. With the exception of the AG vorticity tendency term, those arguments are appropriate to the extent that the atmosphere attempts to achieve some geostrophically balanced state. If the atmosphere does not reach a new geostrophic equilibrium within the time-scale defined by both the data set used and the phenomenon studied; then not all of the vorticity forcing is being utilized to produce geostrophic vorticity changes. Rather, a portion is being utilized to produce AG vorticity changes. Thus, the AG vorticity tendency term provides an “adjustment” which corrects for this situation. Using the above argument, this term could be considered as a response to atmospheric forcing. However, the appearance of this term on the right-hand-side of (1) suggests that it can be expressed in terms of forcing mechanisms. The following sections will demonstrate the validity of this interpretation.

3. AG VORTICITY TENDENCIES

The AG vorticity tendency term represents an adjustment process in which the term corrects for the fact that the atmosphere may tend toward some new geostrophically balanced state or a balanced state that is other than geostrophic (Lupo et al., 1992). In deriving a relationship for this term, the methodology of TC88 is used. TC88 argue that since the geostrophic wind relationship is a diagnostic relationship, the geostrophic wind could be inverted in order to define the geostrophic geopotential field. This field is defined by TC88 as the geopotential field which, given the wind field, satisfies the geostrophic relation. Thus, the AG geopotential would be defined as the difference between the observed geopotential field and the geostrophic geopotential. They argue that in order to *properly* evaluate AG processes *both* the AG components of the wind and the geopotential should be considered (see TC88 for a more detailed rationale).

Blackburn (1985) argues that the term "ageostrophic" is itself ambiguous since the geostrophic relationship could be calculated, for example, using an f-plane assumption (e.g. Phillips, 1963) which he calls geostropy – 0 or “theoretician’s geostrophy”. Geostrophy can be defined such that f varies (called geostrophy-1 by Blackburn (1985) or “synoptician’s geostrophy"). Each relationship can produce very different AG wind components if solved as a residual since geostrophy-1 contains divergence. Even so, in the use of geostrophy-1, the AG component does not account for all the divergent component of the wind (e.g., TC88). TC88 also show that the AG component of the wind contains a rotational component that, along with the divergent geopotential, can be substantial in the entrance and exit regions of jet maxima. Thus, they demonstrate that it is useful to partition both the *wind and geopotential height fields* into divergent and rotational components in order to describe the AG wind.

The following is a brief derivation of the TC88 approach to calculating the AG wind component. For a more rigorous derivation, the reader should consult TC88. In addition to partitioning the wind into geostrophic and AG components, the wind can also be partitioned using Helmholtz theorem (3a-d). V_g (V_{ag}) is the geostrophic (AG) wind (geostrophy –1 used here), V_r (V_d) the rotational (divergent) component of the wind, and ψ (χ) is the geostrophic streamfunction (velocity potential).

A scalar field such as the geopotential, can be partitioned into rotational (ϕ_r) and divergent (ϕ_d) geopotentials using a balance relationship. The TC88 expressions (their 8 and 9) and methods are used here for consistency with that study since the partition of Φ is not unique. TC88 also solve for the rotational geopotential using their (8) then solve for ϕ_d by subtracting ϕ_r from the total geopotential field.

Using (3), the AG wind component is (4), and then applying the del operator to the rotational component of the wind gives (5). This relationship for the AG wind now accounts for all the divergent wind, the divergent geopotential, and the rotational component that cannot balance exactly since coriolis parameter (f) varies. The latter two terms come from the geostrophic component of the wind and are not accounted for by calculating V_{ag} as a residual. It should be noted here that the third term on the right-hand-side of (5) would vanish if geostrophy-0 were used.

Then χ can be calculated, following Hurrell and Vincent (1991), in (5) by using (6). In (6), ω is vertical motion and *Li* is the inverse Laplacian. Since vertical motions (ω) appear explicitly in the Z-O relationship, this variable must be estimated quantitatively. A generalized form of the omega equation similar to that derived by Krishnamurti (1968) can be expressed as (7) here, where (σ) is the static stability parameter. This form of the omega equation was chosen because of its compatibility with (1), i.e. each of the forcing processes that appear in (1) have a complement in (7). The effect of each term in (1) involves the creation of divergence/convergence fields as the result of AG motions imparted on balanced flow. As the mass field in a column of atmosphere adjusts to these new divergence/convergence fields in attempting to establish a new balanced state, the vertical motion profile must also adjust to the new atmospheric state. Then, a similar equation for AG vorticity tendencies can be derived.

From Eq. (7), the vertical motions can be determined using sequential overrelaxation as subject to a suitable boundary condition, in this case the Dirichlet condition. Thus, (7) can be represented in symbolic form as (8), where the subscripts correspond to each forcing mechanism. These are; horizontal vorticity advection, friction, vorticity tilting, vertical vorticity advection, horizontal temperature advection, and diabatic processes, respectively. Thus, (6b) and (8) can be substituted into the velocity potential term in (5) to obtain (9). Thus, (9) becomes an expression for the AG wind containing forcing mechanisms that are compatible with those found in (1) and (7), and can be used to calculate the ageostrophic vorticity tendency term in (1). Partitioning the first term in (9) among the forcing processes in (1) demonstrates mathematically the argument for the AG vorticity tendency term as a “correction term” (Lupo et al., 1992). Using the horizontal vorticity advection term as an example gives (10).

In section 2, it was argued that the AG vorticity tendency could be considered a response to unbalanced forcing, provided that the atmosphere was initially in, or very close to, geostrophic balance. If the atmosphere is considered to be in an unbalanced or transitional state to begin with, then the time rate of change of that forcing mechanism can either modulate or enhance the "final" instantaneous geostrophic vorticity tendency by removing the AG component as in (9) or (10). Thus, the formulation also 'eliminates' the expression for the AG vorticity tendency from the right-hand-side side of the equation, as many of the terms from the partitioned tendencies can be paired with the appropriate terms in (1).

The AG motions produced by unbalanced forcing are proportional to the acceleration of the wind (e.g., Bluestein, 1992). Typically, the AG vorticity tendencies are proportional to the magnitude of the forcing, e.g., large forcing will result in larger geostrophic and AG vorticity tendencies (and typically of the same sign, aside from serendipitous horizontal and/or vertical distributions of the forcing functions (Eq. 13, LS95). In (9) and (10), the AG tendency term, which is a correction term, counters the AG component and the divergent part of the forcing functions in the Z-O equation, which themselves could be partitioned in a similar manner to the development described in this section. Also, the time tendency of the rotational and divergent height tendencies act as "correction" terms and demonstrate that the geostrophic height tendency depends not only on the shape of the height tendency field, but its rate of change with time as well. Thus, as a "correction term", the terms arising from the AG vorticity tendency can dampen the total forcing in (9), and as the forcing shrinks, the AG tendency term shrinks (and consequently AG forcing as well) until the system re-establishes geostrophic balance. This re-establishment of geostrophic balance likely represents a new balanced state rather than a return to the old state.

4. TWO CASE STUDIES

A comparison between the height tendencies calculated using (1) and (2), and then by substituting expression (9) for the AG component of the wind, was performed by examining two case studies. The first case is that of a blocking anticyclone lifecycle studied in LS95, and a full description of the event can be found there. This event was a moderate-to-strong 5-day event that occurred over the middle and western Atlantic during late Oct. and early Nov. 1985. The second case examines a rapidly developing cyclone event that occurred over the mid-western United States in a 24-h period between 0000 UTC 10 and 11 Nov. 1998. This cyclone can be classified as a "land bomb" (central pressure fell 24 hPa in 24 h) as it developed very rapidly during the 0000 UTC to 1200 UTC time period. A brief synoptic description is given here.

At 1200 UTC 09 November (not shown), a weak surface disturbance was over southeastern Colorado, and strong surface fronts were analyzed across the southern great plains (warm front) and southwest (cold front). At 850 hPa, warm moist southerly flow was bringing in copious amounts of water vapor from over the Gulf of Mexico. The 500-hPa map showed that a strong trough was moving over the Rocky Mountains during this time, and weak ridging could be found over the eastern United States. At 300 hPa a strong jet streak was digging into the trough over the western United States. By 0000 UTC 10 Nov., all the ingredients for a rapidly developing storm had moved into place, and the storm was under the poleward exit region of a jet streak. Strong convection was developing rapidly over east-central Kansas south into Oklahoma just prior to 0000 UTC 10 Nov. as indicated from satellite and NEXRAD WSR-88D imagery (not shown). Thus, latent heat release played a key role in surface cyclone development as calculated using (1) for 0000 UTC 10 November. Latent heat release was the third largest contributor to the total height tendencies (height falls).

a. data

In the blocking event, the data used were obtained from the NASA/Goddard Laboratory for Atmospheres (GLA; Schubert et al., 1993). These fields include the standard set of upper air parameters u and v (horizontal wind vectors in m s^{-1}), z (height in m), T (absolute temperature), RH (relative humidity), and q (mixing ratio in g kg^{-1}), which were interpolated linearly in ln(p) to 50-hPa isobaric levels. Also, a suite of surface variables was provided with this data set and the list of variables can be found in the reference cited above. The analyzed fields were provided on a 2.0^o x 2.5^o lat./lon. grid at 14 mandatory pressure levels from 1000 hPa to 20 hPa for 6-h intervals.

To study the 10 - 11 Nov. 1998 mid-western cyclone, the data set used was the National Center for Environmental Prediction (NCEP) gridded re-analyses, which are described in more detail by Kalnay et al., (1996). These analyses are archived at the National Center of Atmospheric Research (NCAR) and are available through their mass-store facilities in Boulder, CO. The upper air analyses (u, v, z, T, RH, and vertical motion (ω)) interpolated quadratic in ln[p] onto 50 hPa isobaric levels. The horizontal resolution is 2.5^o x 2.5^o lat./lon. and variables are available at 6-h intervals. The re-analyses are available on 17 mandatory levels from 1000 hPa to 10 hPa, and include a set of various surface fields, and tropopause information. Lastly, since vertical motions are included in the NCEP re-analyses and not the GEOS-1 analyses, the vertical motions for both cases were calculated using (7) for experimental consistency.

b. results

The resulting height tendencies were compared using the TC88 AG winds (5) and a computation of the AG wind as a residual to calculate the AG tendency term in (1). The height tendencies for the blocking event were located within a moving 40^o x 60^o lat./lon. box centered on the blocking event itself. For the cyclone event, the same size box was centered on North America. Only 3 time periods were used in the case of the rapidly developing cyclone; chosen to capture the cyclone development environment (1200 UTC 9 to 1200 UTC 10 Nov.). The height tendencies in each case were compared to height tendencies derived from the 24 h observed geostrophic vorticity tendencies.

Tables 1 and 2 show the comparative statistics for the blocking event and the cyclone at both 500 hPa and at the surface. Note that there was a similar degree of improvement using TC88 calculations between the surface field correlations, even though the cyclone case correlations were lower than those for the blocking event. At 500 hPa there was little difference between the correlations for the calculation using the residual and the TC88 method. As with the surface, the correlations are better for the blocking height tendencies. Higher correlations for calculated height tendencies over a marine environment were also noted in Lupo et al. (1992). The higher correlations for the blocking surface tendencies could partially be due to the elevation of the underlying surface. The observed tendencies for both cases were on the 1000 hPa surface, while the calculated height tendencies were calculated on the first 50 hPa level above the surface. Thus in locations with elevated terrain, lower tropospheric calculated height tendencies were being compared to observed height tendencies at the surface.

Both calculated AG methods overestimated the observed height tendencies, especially for the cyclone case. This overestimation was greater at the surface than aloft and for the cyclone rather than the blocking event. The overestimation in the cyclone case is not as severe if the comparison region used focuses more closely on the cyclone (20^o x 40^o lat./lon. box) and the overestimation for both the surface and 500 hPa height tendencies using either method was by 50% or less (not shown). There was also an improvement (about 0.05 - 0.1) in the correlations when the smaller region was used, the improvement being larger for the surface fields. Since better pattern similarity would be an overriding concern for a diagnostic study, the TC88 method of calculating the AG vorticity tendencies would be a superior technique. This improvement in pattern similarity occurs despite the fact that far more calculations were needed to get the same quantity, and thus, there was more chance for the introduction of numerical (e.g. truncation) error.

5. SUMMARY AND CONCLUSIONS

The role of the AG vorticity tendencies in the Z-O equation were examined in the context that this term acts as a correction term within the vorticity or height tendency calculations. This term is shown to act as both a response to initial forcing and as a forcing mechanism itself in modulating the initial forcing. A partition of the winds using the TC 88 methodology demonstrated that the AG winds contain a divergent wind component as well as components that resulted from partitioning the geostrophic wind (the geopotential field) into divergent and rotational parts. This partition demonstrates that an equation can be derived for the AG vorticity tendency that is complementary to the Z-O and Omega equations. Each forcing mechanism in the Z-O equation has an AG component that can be "corrected" for by subtracting out the AG forcing which itself can be partitioned. The strength of the AG tendencies are proportional to the time rate of change of the forcing itself, thus demonstrating the non-linear nature of height changes as forced by dynamic or thermodynamic forcing mechanisms.

Two case studies were then examined to determine whether this method improved the final calculated Z-O height tendencies as compared to observed height tendencies. These case studies examined an Atlantic region blocking event and a rapidly developing North American cyclone. Even though two different phenomena were studied using two different data sets in two adjacent regions of the world, the conclusions were the same. The height tendencies using the derived method for calculating ageostrophic vorticity tendencies and the residual method both resulted in a comparable overestimation of the 500 hPa and surface height tendencies. While the 500 hPa correlations to observed height tendencies were similar for both cases, the comparison of surface height tendencies resulted in improved pattern similarity using the TC88 derived AG winds. These improvements occur in spite of the larger number of calculations needed to compute the derived AG winds. With today's computing power, however, the time taken to calculate each was similar.

6. ACKNOWLEDGEMENTS

The author would like to thank Dr. James T. Moore, Dr. Phillip J. Smith, Dr. Patrick Market, and Dr. Peter Zwack for their helpful comments and insight during completion of this work.

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Table 1. Comparative statistics for calculating the total Z-O height tendencies using the ageostrophic winds calculated as a residual (R) and the TC88 approach vs. the observed finite difference height tendencies (OBS) for the LS95 blocking event. Displayed are correlation coefficients (CC), mean absolute values (MAV) and the ratio of the calculated and observed height tendencies displayed as a percentage (% of OBS). Mean absolute values are given in units of 10^{-3} m s^{-1}.

Entire Case (top)
0600 UTC 30 October 1985 (bottom)

	CCs	MAVs	% of OBS
Surface (R)	0.76	1.25	150
	0.69	1.29	153
(TC88)	0.83	1.79	215
	0.78	2.07	244
500 hPa (R)	0.79	0.87	85
	0.78	0.99	75
(TC88)	0.78	1.29	126
	0.80	1.77	134

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Table 2. As in Table 1 except for the 10 - 11 November 1998 cyclone case.

Entire Case (top)
0000 UTC 10 November 1998 (bottom)

	CCs	MAVs	% of OBS
Surface (R)	0.55	0.93	151
	0.55	0.91	154
(TC88)	0.59	1.40	226
	0.57	1.36	231
500 hPa (R)	0.76	1.09	130
	0.76	1.11	139
(TC88)	0.74	1.52	181
	0.74	1.52	190

$$\frac{\partial \zeta_{gL}}{\partial t} = \frac{1}{(p_L - p_t)} \int_{p_t}^{p_L} \left[(-\vec{V} \cdot \nabla \zeta_a - \omega \frac{\partial \zeta_a}{\partial p} + \zeta_a \frac{\partial \omega}{\partial p} - \hat{k}(\nabla \omega \times \frac{\partial \vec{V}}{\partial p}) + \hat{k} \cdot (\nabla \times \vec{F}) - \frac{\partial \zeta_{ag}}{\partial t}) \right] dp$$

$$- \frac{1}{(p_L - p_t)} \frac{R}{f} \int_{p_t}^{pL} \int_{p}^{pL} \nabla^2 \left[-\vec{V} \cdot \nabla T + S\omega + \frac{\dot{Q}}{c_p} \right] \frac{dp}{p} dp \quad (1)$$

$$\left. \frac{\partial \zeta_g}{\partial t} \right)_{pi} = \left. \frac{\partial \zeta_g}{\partial t} \right)_{pL} + \frac{R}{f} \int_{pi}^{pL} \nabla^2 \left(-V \cdot \nabla T + S\omega + \frac{\dot{Q}}{c_p} \right) \frac{dp}{p} \quad (2)$$

$$\vec{V} = \vec{V}_g + \vec{V}_{ag} \quad (3a)$$

$$\vec{V} = \vec{V}_r + \vec{V}_d \quad (3b)$$

$$\vec{V}_r = \hat{k} \times \nabla \psi \quad (3c)$$

$$\vec{V}_d = \nabla \chi \quad (3d)$$

$$\vec{V}_{ag} = \hat{k} \times \nabla \psi + \nabla \chi - \left(\frac{\hat{k}}{f} \times \nabla \Phi_r + \frac{\hat{k}}{f} \times \nabla \Phi_d \right) \quad (4)$$

$$\vec{V}_{ag} = \nabla \chi - \left(\frac{\hat{k}}{f} \times \nabla \Phi_d \right) - \left(\frac{\beta \Phi_r}{f^2} \right) \quad (5)$$

$$\nabla^2 \chi = \nabla \cdot \vec{V} \quad (6a)$$

$$\chi = L_i \left(-\frac{\partial \omega}{\partial p} \right) \quad (6b)$$

$$\omega = \omega_{hv} + \omega_f + \omega_{ti} + \omega_{ag} + \omega_{vv} + \omega_{di} + \omega_{ta} + \omega_{diab} \quad (8)$$

$$\left(\nabla^2 \sigma + f \zeta_a \frac{\partial^2}{\partial p^2} \right) \omega = -f \frac{\partial}{\partial p} \left(-\vec{V} \cdot \nabla \zeta_a + \hat{k} \cdot \nabla \times F - (\frac{\partial \vec{V}}{\partial p} \times \nabla \omega) - \frac{\partial \zeta_{ag}}{\partial t} \right) + f \omega \frac{\partial^2 \zeta}{\partial p^2} - \frac{R}{p} \nabla^2 \left(-\vec{V} \cdot \nabla T + \frac{\dot{Q}}{c_p} \right) \quad (7)$$

$$\vec{V}_{ag} = \nabla \left(L_i \frac{\partial}{\partial p} \left(\omega_{hv} + \omega_f + \omega_{ti} + \omega_{ag} + \omega_{di} + \omega_{ta} + \omega_{diab} \right) \right) - \frac{\hat{k}}{f} \times \nabla \Phi_d - \frac{\beta \Phi_r}{f^2} \quad (9)$$

$$\left. \frac{\partial \zeta_{gL}}{\partial t} \right|_{hv} = -\frac{1}{p_L - p_t} \int_{pt}^{pL} -\vec{V} \cdot \nabla \zeta_a - \left. \frac{\partial \zeta_{ag}}{\partial t} \right|_{hv} \quad (10)$$