P3.3 AN UPPER BOUNDARY CONDITION FOR NONHYDROSTATIC MODELS ABSORBING BOTH GRAVITY AND ACOUSTIC WAVES

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1. INTRODUCTION

We present a formulation of an upper boundary condition for non-hydrostatic models that, by employing a second-order recursive filter in time to record a 'memory' of the recent changes to the presssure perturbation and vertical velocity, enables an appropriate linear combination of these variables to be constrained so as to minimize the spurious reflection of gravity waves at a range of horizontal and vertical wavelengths and to at least partially absorb vertically-propagating acoustic waves. The method can be thought of as generalizing the upper boundary condition of Klemp and Durran (1983) and Bougeault (1983).

2. PERTURBATION EQUATIONS FOR AN ISOTHERMAL BASIC STATE

We shall assume a constant basic state temperature, T_0, constant gravitational acceleration, g, and we ignore rotation. From the usual specific heats for air, we define $\gamma = C_p/C_v = 7/5$ and $\kappa = (\gamma - 1)/\gamma = R/C_p = 2/7$. For this basic state, the sound speed is $c = \sqrt{\gamma R T_0}$, the atmospheric scale height is $H = RT_0/g$ and the Brunt-Vaisala frequency is $N = \sqrt{\kappa/\gamma}c/H$.

In a two-dimensional vertical slice, we let u and w be the horizontal and vertical wind components. T is the temperature, P is the pressure, $\rho = P/RT$ is the density. For convenience, we define the Exner function and potential temperature:

$$\pi = C_p(P/P_{00})^\kappa, \tag{1}$$

$$\theta = C_p T/\pi. \tag{2}$$

2.1 *The basic state*

We use a zero suffix to denote the basic state variables and find that hydrostatic balance,

$$\theta_0 \frac{\partial \pi_0}{\partial z} = -g \tag{3}$$

implies the exponential vertcial profiles of the state variables:

$$\pi_0 = C_p \exp\left(\frac{-gz}{C_p T_0}\right) \equiv C_p \left[\exp\left(\frac{-z}{H}\right)\right]^\kappa, \tag{4a}$$

$$\rho_0 = \frac{P_{00}}{RT_0} \exp\left(\frac{-z}{H}\right), \tag{4b}$$

$$P_0 = P_{00} \exp\left(\frac{-z}{H}\right), \tag{4c}$$

$$\theta_0 = T_0 \left[\exp\left(\frac{z}{H}\right)\right]^\kappa. \tag{4d}$$

2.2 *Perturbation equations*

About this basic state, the equations for perturbations are:

$$\frac{\partial u}{\partial t} = -\theta_0 \frac{\partial \pi}{\partial x}, \tag{5a}$$

$$\frac{\partial w}{\partial t} = -\theta_0 \frac{\partial \pi}{\partial z} + g\frac{\theta}{\theta_0}, \tag{5b}$$

$$\frac{\partial \pi}{\partial t} = -(\gamma - 1)\pi_0 \left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right) + g\frac{w}{\theta_0}, \tag{5c}$$

$$\frac{\partial \theta}{\partial t} = -wN^2 \frac{\theta_0}{g} \tag{5d}$$

But if we adopt the rescaling:

$$u \leftarrow \rho_0^{1/2} u, \tag{6a}$$

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$$w \leftarrow \rho_0^{1/2} w, \tag{6b}$$

$$\pi \leftarrow \left(\frac{\rho_0^{1/2} \theta_0}{c} \right) \pi, \tag{6c}$$

$$\theta \leftarrow \left(\frac{g \rho_0^{1/2}}{\theta_0 N} \right) \theta. \tag{6d}$$

then the perturbation equations become more symmetrical:

$$\frac{\partial u}{\partial t} = -c \frac{\partial \pi}{\partial x}, \tag{7a}$$

$$\frac{\partial w}{\partial t} = -c \left(\frac{\partial}{\partial z} + \frac{1}{L} \right) \pi + N\theta, \tag{7b}$$

$$\frac{\partial \pi}{\partial t} = -c \frac{\partial u}{\partial x} - c \left(\frac{\partial}{\partial z} - \frac{1}{L} \right) w, \tag{7c}$$

$$\frac{\partial \theta}{\partial t} = -Nw, \tag{7d}$$

where $L = 14H/3 \equiv 14RT_0/(3g)$ is the "Lamb-height" (e-folding distance for the Lamb wave perturbation, in units for which the energy density scales as the square of the perturbation). The energy density becomes half the integrated sum of squares of these new perturbation variables.

3. DISPERSION ANALYSIS AND WAVE IMPEDANCE

Assume $\partial/\partial x \equiv ik$ for a horizontal Fourier harmonic and that all the dependent variables scale like:

$$\psi = \tilde{\psi} e^{st}, \tag{8}$$

for $s > 0$. Our original equations then become:

$$s\tilde{u} = -ick\tilde{\pi}, \tag{9a}$$

$$s\tilde{w} = -c\mathcal{D}\tilde{\pi} + N\tilde{\theta}, \tag{9b}$$

$$s\tilde{\pi} = -ick\tilde{u} - c\mathcal{D}^*\tilde{w}, \tag{9c}$$

$$s\tilde{\theta} = -N\tilde{w}. \tag{9d}$$

where,

$$\mathcal{D} \equiv \left(\frac{d}{dz} + \frac{1}{L} \right), \tag{10a}$$

$$\mathcal{D}^* \equiv \left(\frac{d}{dz} - \frac{1}{L} \right). \tag{10b}$$

Applying substitutions,

$$\tilde{u} = \frac{-ick}{s} \tilde{\pi}, \tag{11a}$$

$$\tilde{\theta} = \frac{-N}{s} \tilde{w}, \tag{11b}$$

we obtain:

$$\left(\frac{N^2}{s} + s \right) \tilde{w} = -c\mathcal{D}\tilde{\pi}, \tag{12a}$$

$$\left(\frac{c^2 k^2}{s} + s \right) \tilde{\pi} = -c\mathcal{D}^*\tilde{w}, \tag{12b}$$

or the single equation for $\tilde{\pi}$:

$$\tilde{\pi} = K^2 \mathcal{D}^* \mathcal{D} \tilde{\pi}, \tag{13}$$

where

$$K^2 = \frac{s^2 c^2}{(N^2 + s^2)(c^2 k^2 + s^2)}, \tag{14}$$

Note that,

$$\mathcal{D}\mathcal{D}^* \equiv \left(\frac{d^2}{dz^2} - \frac{1}{L^2} \right), \tag{15}$$

so the only solution corresponding to forcing from the lower boundary is

$$\tilde{\pi} = \tilde{\pi}^+ e^{-\mu z}, \tag{16}$$

where

$$\mu^2 = \left(\frac{1}{L^2} + \frac{1}{K^2} \right), \tag{17}$$

and $\mu \geq 0$. For later algebraic convenience, we note that,

$$s^2 c^2 \mu^2 = N^2 c^2 k^2 + s^2 c^2 \left(k^2 + \frac{1}{4H^2} \right) + s^4. \tag{18}$$

In this case of forcing from below,

$$\tilde{\pi} = \hat{Z} \tilde{w}, \tag{19}$$

where,

$$\hat{Z} = \frac{sc}{c^2 k^2 + s^2} \left(\mu + \frac{1}{L} \right), \tag{20}$$

is the "impedance" of the disturbance at this s and k.

Suppose the upper boundary condition is written:

$$\pi(t) = \int_0^\infty Z(t') w(t - t') dt', \tag{21}$$

that is, a superposition of "effect" π depending on preceding "causes" w through the impedance kernel $Z(t')$, with $Z(t') = 0$ for $t' < 0$. Use Laplace transforms to express:

$$\pi(t) = \int_0^\infty \tilde{\pi} e^{st} ds, \tag{22a}$$

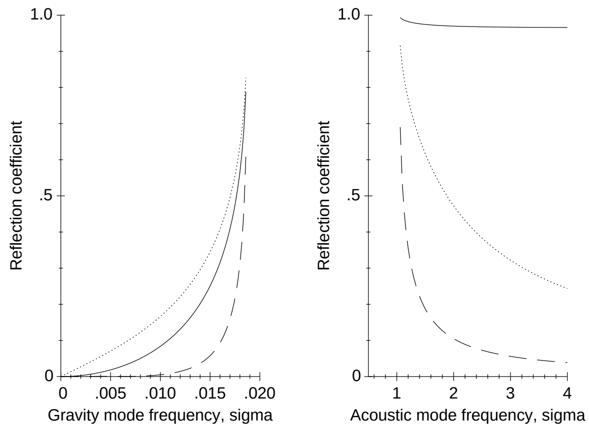

Figure 1. Reflection coefficients. Solid curves for $Z = Z_g$; dotted for $Z = Z_1$ with $r = 2s^{-1}$; dashed for $Z = Z_2$ with $b = r/\sqrt{2}$.

$$w(t) = \int_0^\infty \tilde{w} e^{st} ds, \qquad (22b)$$

so that,

$$\tilde{\pi}(s) = \int_0^\infty Z(t')e^{-st'} dt' \, \tilde{w}(s). \qquad (23)$$

Hence, inverting the relation,

$$\hat{Z}(s) = \int_0^\infty Z(t')e^{-st'} dt'. \qquad (24)$$

provides the "perfect" radiative boundary condition for these linearized waves. Unfortunately, the kernel, $Z(t)$ is highly structured in the t domain and computationally practical approximations to $Z(t)$ that can be expressed with only a small storage burden inevitably make significant errors in at least some range of complex parameter s. Good radiative boundary conditions seek to keep this error small in the portion of s that matters most for meteorology, while maintaining overall numerical stability.

In the absence of any special boundary condition, for example, when the vertical velocity is set to zero at the model top all incident waves (acoustic and gravity) are fully reflected. A vast improvement is obtained by the choice of an impedance set equal to the asymptotic limit:

$$Z_0(t) = \lim_{s \to 0} \hat{Z}(s) = Z_g = \frac{N}{c|k|}, \qquad (25)$$

given by the Klemp and Durran (1983) theory, which is reasonably simple to apply in any model whose horizontal domain is of a shape that facilitates double Fourier transformation.

We can insert $s = i\sigma$ and use (20) to express the *complex* impedance $\hat{Z}$ as a function of frequency σ. The two branches of complex μ, which we may denote μ^+ and μ^- correspond to upward and downward propagating waves of the common frequency, σ. In general, both branches contribute to the solution satisfying a given boundary impedance condition. Thus, if

$$\tilde{w} = \tilde{w}^+ + \tilde{w}^-, \qquad (26a)$$

and

$$\tilde{\pi} = \tilde{\pi}^+ + \tilde{\pi}^- = \hat{Z}^+\tilde{w}^+ + \hat{Z}^-\tilde{w}^-, \qquad (26b)$$

then the boundary impedance condition, $\hat{Z} = \hat{Z}_b$, is only satisfied when

$$\tilde{w}^- = \left(\frac{\hat{Z}^+ - \hat{Z}_b}{\hat{Z}_b - \hat{Z}^-} \right) \tilde{w}^+. \qquad (27)$$

The magnitude of this ratio between the amplitudes of downward and upward waves will be referred to as the reflection coefficient (its square is a measure of the fractional wave *power* reflected):

$$R_b(\sigma) = \left| \frac{\hat{Z}^+ - \hat{Z}_b}{\hat{Z}_b - \hat{Z}^-} \right|. \qquad (28)$$

For the case, $\hat{Z}_0(\sigma) = Z_g$, and for a wave of horizontal wavelength, $\lambda = 2\pi/k = 2000m$ we obtain the reflection profiles for the gravity and acoustic modes shown in Fig. 1 by the solid curves. While this method is reasonably good for gravity waves, especially in the hydrostatic and high vertical wavenumber limit, $s \to 0$, where the reflection coefficient tends to zero, it is clear that this boundary condition does very little to absorb the acoustic modes (for which it was never originally designed). The reason is that the impedance of purely vertically propagating acoustic waves in our model is, in our particular choice of scaling units,

$$\lim_{s = \infty} \hat{Z}(s) \equiv Z_a = 1, \qquad (29)$$

which is many times larger than the magnitudes of impedances typical of atmospheric gravity waves, and the impedance of obliquely-propagating acoustic waves is even greater.

A partial solution to radiating both gravity and acoustic waves is to employ a recursive time filter to retain a 'memory' of recent vertical velocity at the top:

$$\bar{w}(t) = \int_0^\infty re^{-rt'} w(t - t') \, dt', \qquad (30)$$

and then to use this for the upper boundary condition:

$$\pi(t) = Z_a w(t) - (Z_a - Z_g)\bar{w}(t). \qquad (31)$$

The filter whose effect is the smoothing integral (30) can be obtained as the recursive expression of the differential equation:

$$\frac{d\bar{w}}{dt} = -r(\bar{w} - w). \qquad (32)$$

The impedance implied by this new boundary condition is

$$\hat{Z}_1 = Z_a - \frac{(Z_a - Z_g)r}{r + s}. \qquad (33)$$

The choice of the smoothing rate, r, can be used to trade off the quality of the radiation condition for gravity modes against the quality of treatment for the acoustic modes. For an intermediate value, $r = 2.s^{-1}$ (a time scale of about half a second), the dotted curves of Fig. 1 show the reflection coefficient obtained with horizontal wave length $\lambda = 2000m$. For acoustic waves propagating in the strictly vertical direction the new boundary condition is now efficiently absorbing but, compared with the Klemp and Durran condition, the treatment of gravity waves is now made rather worse. Evidently, the first-order filtering approach does not constitute a completely satisfactory solution.

We consider refining the definition of $\bar{w}$ in (31) by the substitution of a second-order recursive filter. For example:

$$\hat{Z}_2 = Z_a - \frac{(Z_a - Z_g)(2bs + r^2)}{s^2 + 2bs + r^2}. \qquad (34)$$

Such a filter is provided by recursively solving the numerical representation of:

$$\frac{d^2\bar{w}}{dt^2} = -2b\left(\frac{d\bar{w}}{dt} - \frac{dw}{dt}\right) - r^2(\bar{w} - w). \qquad (35)$$

The impedance $\hat{Z}_2$ retains the desirable asymptotic limits,

$$\lim_{s\to 0} \hat{Z}_2 = Z_g, \qquad (36a)$$

$$\lim_{s\to\infty} \hat{Z}_2 = Z_a, \qquad (36b)$$

but we can select coefficients b and r^2 to obtain a better impedance match near $s = 0$. Since

$$\left.\frac{d^2\hat{Z}_2}{ds^2}\right|_{s=0} = \frac{2(Z_a - Z_g)}{r^2} \qquad (37)$$

and, for the actual upward-propagating gravity wave,

$$\left.\frac{d^2\hat{Z}^+}{ds^2}\right|_{s=0} = \frac{c^2[k^2 + 1/(4H^2)] - 2N^2}{N|ck|^3}, \qquad (38)$$

we need to choose,

$$r^2 = \frac{2(Z_a - Z_g)N|ck|^3}{c^2[k^2 + 1/(4H^2)] - 2N^2}. \qquad (39)$$

The choice of coefficient b remains undetermined, although it must be nonnegative if the filter is to remain stable. One possible choice is $b = r/\sqrt{2}$, the consequence of which is shown in the dashed curves of Fig. 1 for the same horizontal wave length selected for the other curves.

4. DISCUSSION

This preliminary study of radiation boundary conditions for a compressible nonhydrostatic atmosphere suggests that it is indeed possible to extend the method of Bougeault (1983) and Klemp and Durran (1983) to include partial absorption of acoustic modes while improving the handling of those gravity modes significantly modified by nonhydrostatic effects. The method involves the application of a temporal recursive filter at the model top. However, first-order recursive filtering is found not to be sufficent — satisfactory results require the application of a filter of at least second order. Like the conditions of Bougeault, Klemp and Durran, the new method requires a horizontal spectral decomposition at the model top. This raises the question whether the resulting boundary condition can then accommodate horizontal variations in the Brunt-Vaisala frequency (assumed uniform in this limited study). This question needs to be addressed in a more complete model before we can give an authoritative answer.

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