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A SEMI-LAGRANGIAN DYNAMICAL CORE FOR THE NON-HYDROSTATIC WRF MODEL

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1. INTRODUCTION

A non-hydrostatic semi-Lagrangian dynamical core is under development as part of the multi-institution Weather Research and Forecasting (WRF) initiative (for example, Michalakes et al. 2000, or the web-site, http://wrf-model.org). This version is intended to complement an Eulerian dynamical core, developed at NCAR (Klemp et al. 2001), that uses relatively simple spatial numerics and a split-explicit vertically-implicit time integration scheme to maintain stability of the acoustic modes.

2. SPATIAL ELEMENTS OF THE SEMI-LAGRANGIAN MODEL

Our semi-Lagrangian option offers high-order 'compact' operators for all spatial differencing and integration, together with high-order grid interpolations for the semi-Lagrangian advection itself. This emphasis on high formal accuracy in the spatial discretization recognizes the importance to operational mesoscale prediction of achieving correct timing of mobile meteorological features. The additional computational cost incurred by high-order operators can be at least partially mitigated by the longer time steps that a semi-Lagrangian model allows (Robert 1981), provided the stability of the 'fast' modes, (acoustic, but possibly also the deepest of the gravity modes) can be reliably maintained by using semi-implicit time discretization. Since high-order spatial operators allow the use of unstaggered variables without much loss of accuracy (Purser and Leslie 1988), the semi-Lagrangian model can be formulated with just a single family of trajectories each time step. The interpolations associated with these displaced trajectories are handled using the efficient 'cascade' interpolation method of Purser and Leslie (1991) adapted to accommodate forward trajectories (Purser and Leslie 1994) and mass conservation (Leslie and Purser 1995).

Indirectly, the adoption of adaptive hybrid sigma/theta vertical coordinates (Bleck and Benjamin 1993, Konor and Arakawa 1997) can also mitigate the computational burden. The model levels tend to concentrate within layers of especially stable air where important vertical structural detail is often found. Thus, the hybrid coodinate model resolves features which would require significantly more levels using a conventional vertical grid (Johnson et al. 1993). The additional freedom obtained in the model by allowing a more general prescription for the vertical coordinate is something we plan to incorporate in a later version of this model.

The efficient parallelization of the spatial operators of this model has been the subject of particular study. For the case of the compact operators Fujita and Purser (2001) supply a more detailed review. Initially, the use of global ('all-to-all') data transposes is being used, but more efficient methods may be used in a later version.

3. TEMPORAL DISCRETIZATION

One consequence of selecting a forward-trajectory semi-Lagrangian advection scheme is

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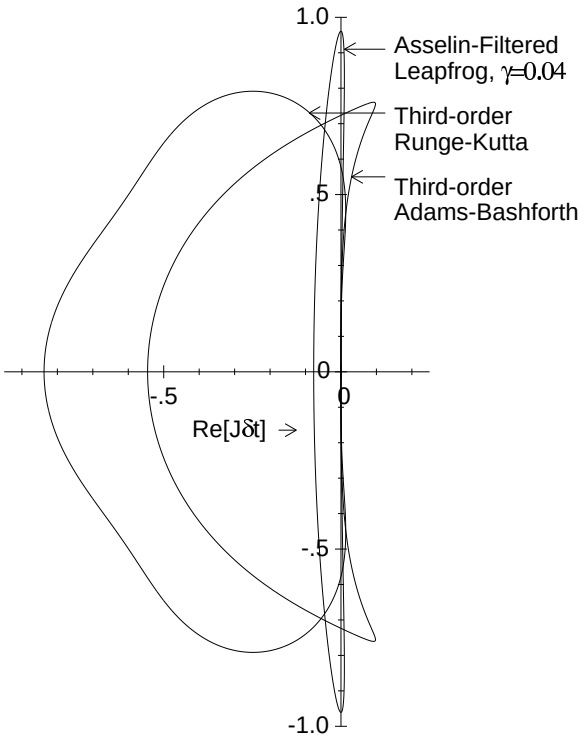

Figure 1. Stability regions in the complex $J\delta t$ plane of the filtered leapfrog and the third-order Adams-Bashforth and Runge-Kutta schemes, with time steps adjusted for equivalent rate of tendency function evaluations.

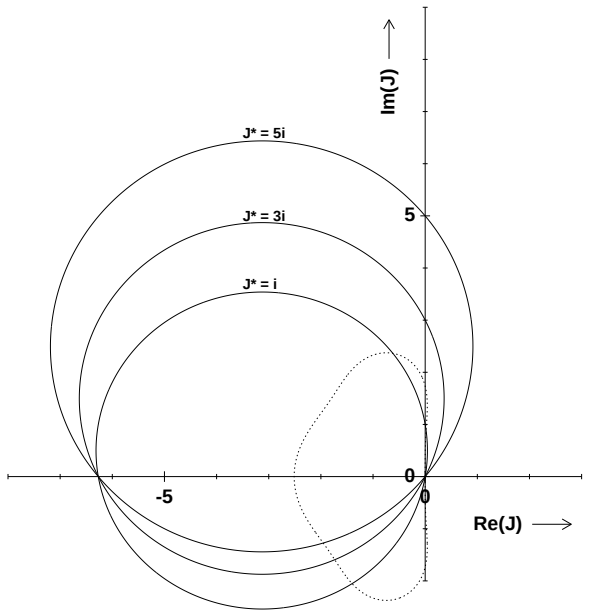

Figure 2. Stability regions (interiors of closed curves) in the complex plane of $J\delta t$ for the explicit third-order Runge-Kutta scheme (the dotted curve that is symmetric about the real-axis) together with the stability curves for the semi-implicit modification of Williamson's scheme at assumed frequencies, $J^* = i, 3i, 5i$ for the parameter choice, $a_1 = a_2 = a_3 = b = 0$. J is the true complex frequency of the mode. The curves for the implicit modified scheme all pass precisely through the respective complex values of J^*, implying that the fast oscillatory modes are exactly neutrally stable when $J^* = J$, but are unstable for the smallest underestimation, $J^* < J$, which implies a lack of numerical robustness.

that it makes available a wider range of numerical time integration schemes — essentially any scheme for ordinary differential equations that can be adapted for a semi-implicit treatment of the 'fast' components. In an Eulerian model, the advective Courant-Friedrichs-Lewy condition imposes a strong inhibition against using anything other than the filtered leap-frog method, because of this method's exceptionally lenient explicit stability properties for advected structures. This is visible in Fig. 1 which shows the stability regions in the complex frequency plane $J\delta t$ for the equation,

$$\frac{d\psi}{dt} = J\psi, \tag{1}$$

for a time-filtered leap-frog, together with alternatives in the form of third-order Adams-Bashforth and Runge-Kutta methods. What is relevant to linear advection is the fact that the leap-frog's stability region encloses a larger extent of the imaginary axis, and can therefore get by with longer time steps in Eulerian advection.

However, semi-Lagrangian methods are freed from this particular restriction and the choice of integration method can be better judged by other criteria. For example, the leap-frog is second-order accurate only in its pure form, but a practical implementation involves measures to control the computational mode, usually through a filter (Robert 1966, Asselin 1972) which reduces formal accuracy to first-order and requires additional storage. Of the more accurate third- and fourth-order methods exhibiting stability for oscillatory modes, Adams-Bashforth methods are unfortunately even more demanding on storage. However, certain Runge-Kutta methods (Williamson 1980, Gill 1951) combine storage economy with formal accuracy and are unburdened by any compuational modes. Since other characteristics of the semi-Lagrangian model are described elsewhere, we devote the bulk of this paper to an outline examination of one of these low-storage Runge-Kutta schemes adapted to include semi-implicit adjustments.

4. SEMI-IMPLICIT ADAPTATION OF WILLIAMSON'S SCHEME

One of these methods is a third-order Runge-Kutta scheme advocated by Williamson (1980). For

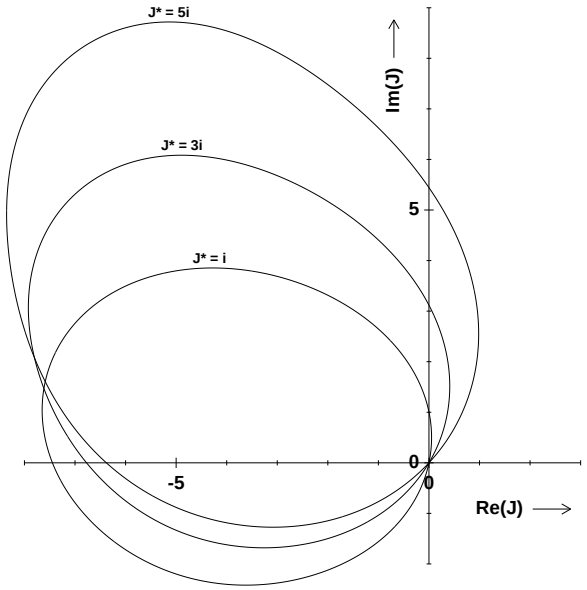

Figure 3. Stability regions in the J plane for the semi-implicit modified Williamson scheme for $J^* = i, 3i, 5i$ when $a_1 = a_2 = a_3 = 0$. but with the second-order de-centering parameter, $b = 1$. For the higher frequencies (larger loops), a degree of robustness is obtained, since the curve of neutral stability misses the point $J = J^*$ by a significant marging, but this margin is not significant at smaller frequencies (e.g., for the curve for $J^* = i$).

the case of a multi-component system more general than (1),

$$\frac{d\psi}{dt} = \mathcal{F}(\psi) \tag{2}$$

the method advances by a cycle of duration δt in stages. In Williamson's scheme, the cycle is subdivided into shorter intervals in the ratio, $\{3 : 5 : 4\}$. Each cycle begins with $\psi^0 \equiv \psi(t)$, and $E^0 \equiv 0$, and finishes with the complete update, $\psi(t + \delta t) \equiv \psi^3$. The three intermediate stages, $i = 1, 2, 3$, within each cycle advance the state ψ according to:

$$E^{i-1} = R_{i-1}\mathcal{F}(\psi^{i-1})\delta t + Q_{i-1}E^{i-1}, \tag{3a}$$

$$\psi^i = \psi^{i-1} + E^{i-1} + \hat{\psi}^i, \tag{3b}$$

where constants R and Q are defined:

$$(R_0, R_1, R_2) = (1/3, 15/16, 8/15), \tag{4a}$$

$$(Q_0, Q_1, Q_2) = (0, -25/16, -17/15), \tag{4b}$$

and the terms $\hat{\psi}$ denote the semi-implicit adjustment terms, which can be ignored in the explicit algorithm. In order to evaluate the implicit terms for the fast modes we assume that the Jabobian operator J encoding the sensitivity of the fast-mode forcing term $F \equiv \mathcal{F}\delta t$ to changes in the state ψ:

$$dF = Jd\psi, \tag{5}$$

can be approximated by $J^* \approx J$ where J^* is the linearization about a sufficiently simple basic state. It can be shown that there is a family of consistent semi-implicit adjustment formulae for the three stages of this Runge-Kutta scheme:

$$\hat{\psi}^1 = (1 - W_1^1 J^*)^{-1} W_0^1 F^0 - E^0, \tag{6a}$$

$$\hat{\psi}^2 = (1 - W_2^2 J^*)^{-1} (W_E E^1 + W_1^2 F^1) - E^1, \tag{6b}$$

$$\hat{\psi}^3 = (1 - W_3^3 J^*)^{-1} W_2^3 F^2 - E^2, \tag{6c}$$

where the weights are defined in terms of three 'first-order de-centering parameters' a_1, a_2, a_3 associated with the three stages and a 'second-order de-centering parameter', b associated with the middle stage, according to:

$$[W_0^1, W_1^2, W_2^3] = \left[\frac{1}{3}, \frac{5}{6}\left(\frac{1}{2} + \frac{b}{9}\right), \frac{1}{4}\right], \tag{7a}$$

$$[W_1^1, W_2^2, W_3^3] = \left[\frac{1+a_1}{6}, \frac{5}{6}\left(\frac{1+a_2}{4} + \frac{b}{9}\right), \frac{1+a_3}{8}\right], \tag{7b}$$

$$W_E = -\frac{2b}{9}, \tag{7c}$$

In the context of a complete model, the Jacobian operator J^* becomes a matrix of spatial partial derivative operators structured so that, with further manipulations, each adjustment equation involves the solution of a Helmholtz equation. However, it is illuminating to see the effect of the semi-implicit

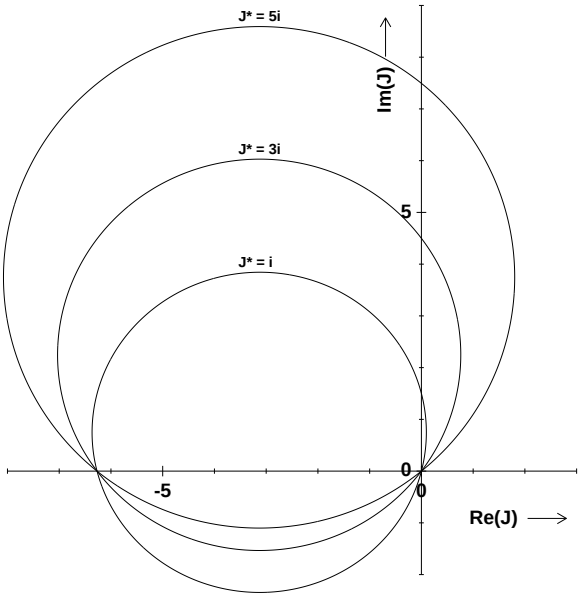

Figure 4. Modified Williamson scheme, as in Fig. 3, but with $a_1 = a_2 = a_3 = .5$ and $b = 0$.. All implicit modes are now robustly stable.

scheme we have described when it is applied to a much simpler system, (1), where the Jacobian J reduces to a complex number.

For the semi-implicit case, J^* represents an assumed complex frequency for the given mode. As in Fig. 1, it becomes possible to examine for the region of J for which stability occurs, but now for each assumed J^*. Formally, when all the de-centering parameters vanish, each stage's adjustments become equivalent to the trapezoidal rule and the stability regions for non-dimensional $J^* = i, 3i, 5i$ are given in Fig. 2. When $J = J^*$, these schemes are indeed stable, but only the slightest underestimation of the frequencies of the oscillatory modes $|J^*| < |J|$ induces a numerical instability that would render this scheme impractical. For schemes with vanishing first-order parameters, a, but $b > 0$ the implicit modes remain formally second-order accurate. Fig. 3 shows how such a scheme, with $b = 1$, provides a small margin of 'robustness' in the sense that a small error in the approximating frequency J^* does not destabilize the scheme. However, this safety margin becomes very thin for the slowest of the implicit modes. Finally, by using $a_1 = a_2 = a_3 = .5$ (Fig. 4) substantial robust stability is achieved for all oscillatory modes and, while they now become formally only first-order accurate, this is deemed an acceptable price to pay given that the implicit modes are specifically those *not* of meteorolgical significance.

The robustness we achieve by de-centering the implicit calculations makes it significantly easier and computationally less costly to stabilize the fast modes of the model. Numerically, the task involves solving a Helmholtz equation for the acoustic terms; robustness allows the coefficients of this equation to be simplified without risk of destabilizing the integration.

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