Assimilation in Land Surface Hydrology : A General Theory

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1 INTRODUCTION

There has been enormous developments in the theory of data assimilation ever since estimation theory was introduced. The majority of the applications have been directed towards oceanic and atmospheric research. Ocean and atmosphere prognostic equations are well established along with the proper (and formal) parameterization schemes. However, land surface hydrology suffers from a lack of formalization that makes such an application difficult. There have been numerous developments in land data assimilation (McLaughlin, 1995). In the recent past simple methods for surface temperature assimilation have been developed (Lakshmi, 2000) which take into account the sensitivity nature of hydrological variables (Lakshmi and Susskind, 2001) in order to adjust alll the hydrological variables associated with the water and energy budget. In this paper, we attempt to examine the basis for the development of data assimilation in hydrology, i.e. a close examination of the prognostic equations as well as a few examples of data assimilation for land surface variables. Finally, an interesting example of understanding and underscoring the importance of dependence of soil hydraulic properties on soil moisture is presented.

2 THEORY

The land surface hydrology can be represented by a model having in general n layers, The water balance for the model can be written as (along the lines of Mahrt and Pan, 1984)

$$\begin{aligned}
z_1\frac{\partial\theta_1}{\partial t} &= P_n - E - R - \alpha_1 T - q_{1,2} - Q_{b1}\\
z_2\frac{\partial\theta_2}{\partial t} &= q_{1,2} - q_{2,3} - \alpha_2 T - Q_{b2}\\
z_3\frac{\partial\theta_3}{\partial t} &= q_{2,3} - q_{3,4} - \alpha_3 T - Q_{b3}\\
z_i\frac{\partial\theta_i}{\partial t} &= q_{i-1,i} - q_{i,i+1} - \alpha_i T - Q_{bi}\\
z_{n-1}\frac{\partial\theta_{n-1}}{\partial t} &= q_{n-2,n-1} - q_{n-1,n} - \alpha_{n-1} T - Q_{bn-1}\\
z_n\frac{\partial\theta_n}{\partial t} &= q_{n-1,n} - q_{n,WT} - \alpha_n T - Q_{bn}
\end{aligned} \tag{1}$$

where θ_1, θ_2, θ_i,...,θ_n are the volumetric soil moistures of layer 1 (with thickness z_1), layer 2 (with thickness z_2), etc., P_n is the net precipitation (actual precipitation P minus the canopy interception), E is the bare soil evaporation, R is the surface runoff, T is the transpiration, $q_{i,i+1}$ is the moisture flow from layer i to layer $i+1$ and Q_{bi} is the base flow layer i. In this model, the bare soil evaporation is assumed to take place from the top layer only and the vegetation transpiration from the bottom layer only. The moisture flow from layer i to layer $i+1$ ($q_{i,i+1}$) is modeled using the Richards equation accounting for the gravity advection and the moisture gradient.

$$q_{i,i+1} = D(max[\theta_i,\theta_{i+1}])\frac{\theta_i - \theta_{i+1}}{0.5(z_i + z_{i+1})} + K(max[\theta_i,\theta_{i+1}]) \tag{2}$$

The bare soil evaporation and the vegetation transpiration are estimated using the supply and demand principle, i.e. if there is enough moisture to satisfy the potential value, the evaporation and transpiration occur at the potential rate, else they occur at a rate limited by the amount of available soil moisture. Potential evapotranspiration is defined as the evapotranspiration occuring in the absence of any restrictions in the supply of moisture or energy to the land surface. This is a important variable as the actual evapotranspiration is computed as being less than (in the case adequate amount of moisture or energy to satisfy the potential is not available) or equal to the potential evapotranspiration. In this paper we compute the potential evapotranspiration by solving the water and energy budgets of the land surface. The total transpiration is partitioned into contribution from the various layers based on the amount of roots in each of the layers, i.e. α_i is a function of the proportional to the fraction of roots of the vegetation present in layer i relative to layers 1 through n. The baseflow is a combination of baseflow from each of the layers, i.e.

$$Q_b = \sum_{i=1}^{i=n} Q_{bi} \tag{3}$$

where the baseflow from each individual layer Q_{bi} is given by

$$Q_{bi} = Q_b^* \frac{\theta_i}{\theta_b^*}; \theta_i \leq \theta_b^*$$

$$Q_{bi} = Q_b^* \frac{\theta_i}{\theta_b^*} + (Q_b^{max} - \frac{Q_b^*}{\theta_b^*/\theta_s}\left(\frac{\theta_i - \theta_b^*}{\theta_s - \theta_b^*}\right)^2 ; \theta_i \geq \theta_b^* \tag{4}$$

The energy balance equation for the land surface can be written as

$$\begin{aligned} R_n &= H + LE + G \\ R_n &= R_{sd}(1-\alpha) + R_{ld} - \epsilon\sigma T_s^4 \\ LE &= \frac{\rho C_p}{\gamma(r_{av}+r_c)}(e_s(T_s) - e_a) \\ H &= \frac{\rho C_p}{r_{ah}}(T_s - T_a) \\ G &= \frac{\kappa}{D}(T_s - T_d) \end{aligned} \tag{5}$$

where R_{sd}, R_{ld} are the incoming shortwave and longwave radiation respectively. α and ϵ and σ are the albedo, emissivity and the Stefan-Boltzmann's constant respectively. ET is the latent heat of evapotranspiration equals sum of bare soil evaporation and transpiration: $ET = E + T$; and T_s, T_a and T_d are the surface temperature, air temperature and the deep soil ($50cm$) temperature respectively. $e_s(T_s)$ and e_a are the saturated vapor pressure at surface temperature T_s and actual vapor pressure of the air respectively. ρ, C_p and γ are the density, specific heat and psychrometric constant of air; r_{av} and r_{ah} are the aerodynamic resistances to vapor and heat and r_c is the canopy resistance. κ and D are the thermal conductivity and the diurnal damping depth of the soil.

The aerodynamic resistances to vapor (r_{av})and heat (r_{ah}) are taken as equal to each other and are evaluated as (Brutsaert, 1982),

$$r_{av} = r_{ah} = \frac{1}{k^2 u_2}(\ln(\frac{z-d}{z_0}))^2 \tag{6}$$

where k is the Von Karman constant (0.4), u_2 is the $2m$ wind speed, z is $2.0m$ (the reference height), z_0 is the roughness length and d is the zero plane displacement. The canopy resistance is given by (Feyen et al., 1980),

$$r_c = \frac{r_{min}^{st}}{\mathcal{L}} \tag{7}$$

r_{min}^{st} is the minimum stomatal resistance and $\mathcal{L}$ is the leaf area index. LE_1, H_1 and G_1 are variables that depend on surface resistance (LE_1 and H_1) and thermal capacity of the ground (G_1) such that $LE_1(e_s(T_s) - e_a)$ equals the evapotranspiration flux LE, $H_1(T_s - T_a)$ equals the sensible heat flux H and $G_1(T_s - T_d)$ equals the ground heat flux. The latent heat coefficient LE_1 and the sensible heat coefficient H_1 are a function of the wind speed through the dependence of the aerodynamic resistances r_{av} and r_{ah} on wind speed (Figure 2). The heat storage is not included in Eqn. (2) as we have a thin upper layer (1.0cm) and a short time step (1 hour) in our computations. Therefore, the heat storage term is negligible. The latent heat coefficient is defined as the heat in Wm^{-2} per unit vapor pressure difference between the saturated surface vapor pressure and the ambient air vapor pressure in mb. The sensible heat coefficient is defined as the heat flux in Wm^{-2} per unit temperature difference between the surface and the air in K. The latent heat coefficient depends on the wind speed, roughness length (bare soil: z_0 =0.001m, vegetation: $z_0 = 0.07m$), zero plane displacement (bare soil: d=0.0m, vegetation: d=0.25m), leaf area index and the minimum stomatal resistance ($r^{st}_{min} = 100sm^{-1}$).

3 ASSIMILATION

The various variables that will be altered by assimilation will be surface temperature, soil moisture and streamflow.

3.1 Surface Temperature

The surface temperature computed by the mode T_s^m and the surface temperature observed using a hand-held, in-situ or remote-sensor on aircraft or satellite T_s^o can be merged according to their individual error characteristics, i.e.

$$T_s = \frac{w_o T_s^o + w_m T_s^m}{w_o + w_m} \tag{8}$$

where the weights w_o and w_m reflect the confidence we have in our estimates of the observation and the model output and the sum of the weights equals unity, i.e. $w_o + w_m = 1$. These can be envisioned as the inverse of the standard deviation of the difference between the quantity and the "truth". Using the energy balance expression from above, we have

$$\delta LE = -(4\epsilon\sigma T_s^3 + H_1 + G_1)\delta T_s \tag{9}$$

where $H_1 = \frac{\rho C_p}{r_{ah}}$ and $G_1 = \frac{\kappa}{D}$ and $\delta T_s = T_s - T_s^m$. Let us weight the corrections for bare soil evaporation and transpiration by their relative magnitude in the combination to get $ET = E + T$, viz.,

$$\begin{aligned} \delta E &= w_E \delta LE \\ \delta T &= w_T \delta LE \end{aligned} \tag{10}$$

such that $w_E + w_T = 1$ as water needs to be conserved in the system.

If $\delta E \neq 0$ then

$$\begin{aligned} \delta\theta_1 &= (\delta E + \delta T_1)\frac{\delta t}{\rho_w L z_1} \\ \delta\theta_i &= \delta T_i \frac{\delta t}{\rho_w L z_i} \end{aligned} \tag{11}$$

The "adjusted" soil moisture will be given by the sum of the initial state and the adjustment, i.e.,

$$\theta_i = \theta_i^m + \delta\theta_i \tag{12}$$

3.1.1 Convergence

In the above assimilation equations, we have two free parameters, viz., w_o, and w_E. We can state this formally as,

$$T_s = T_s(w_o, w_E) \tag{13}$$

Therefore it follows that

$$\delta\theta_i = \delta\theta_i(w_o, w_E) \tag{14}$$

The loss function to be minimized is given by

$$L = \left(\frac{1}{N} \sum_{j=1}^{N} (\theta_{ij}^m(w_o, w_E) - \theta_{ij}^o)^2 \right)^{1/2} \tag{15}$$

where the model propogation of soil moisture through time j depends on the increment corrections $\delta\theta_i$ implemented at the previous times prior to the present. Therefore, using the criterion for minima we obtain

$$\begin{aligned} \frac{\partial L}{\partial w_o} &= 0 \\ \frac{\partial L}{\partial w_E} &= 0 \end{aligned} \tag{16}$$

Minimization can proceed numerically

3.2 Soil Moisture

If we observe the profile soil moisture $\theta_{i,j}^o$, we have an optimal soil moisture $\theta_{i,j}$ using the observation and the model (superscript m) for layer i and time j

$$\theta_{i,j} = \frac{w_i^o \theta_{i,j}^o + w_i^m \theta_{i,j}^m}{w_i^o + w_i^m} \tag{17}$$

Now, we need to assimilate the differences, $\theta_{i,j} - \theta^m_{i,j}$ so as to make better predictions of the fluxes. Now $\theta_{i,j}$ will be used at $j+1$ for further propogation instead of $\theta^m_{i,j}$

3.3 Streamflow

Streamflow is given by:

$$Q(t) = R(t) + Q_b(t) + I(t) \tag{18}$$

where $R(t)$ is the overland runoff, $Q_b(t)$ is the sum og all baseflow components (from each of the n layers) and $I(t)$ is the inflow to the stream from the previous stream segment. The observations of the streamflow is given by

$$Q^o(t) = R^o(t) + Q^o_b(t) + I(t) \tag{19}$$

where, we need to ensure that the model computed streamflow equals the observed streamflow. Therefore we need to fulfill the following relationship

$$Q^o(t) = R^{'}(t) + Q^{'}_b(t) + I(t) \tag{20}$$

where the overland runoff and the baseflow have been changed to reflect the adjustment of the model streamflow to the observed streamflow. Therefore, we have the change (between model and adjusted components) as

$$\delta Q(t) = \delta R(t) + \delta Q_b(t) \tag{21}$$

Let us distribute the change $\delta Q(t)$ between the surface runoff and the baseflow as the ratio of their magnitudes, i.e.,

$$\begin{aligned} \delta R(t) &= \delta Q(t) \frac{R(t)}{R(t) + Q_b(t)} \\ \delta Q_b(t) &= \delta Q(t) \frac{Q_b(t)}{R(t) + Q_b(t)} \end{aligned} \tag{22}$$

where in the case of baseflow the individual layers are corrected in proportion to their soil moisture content, i.e.,

$$\delta Q_{bi}(t) = \delta Q_b(t) \frac{\theta_i(t)}{\sum_{j=1}^n \theta_j(t)} \tag{23}$$

4 RESULTS AND ANALYSIS

4.1 Soil Moisture and Hydraulic Properties of the Soil

Hydraulic properties of the soil are a major determinant of the soil wetness, temperature and the transfer of heat and moisture fluxes. It is therefore imperative to quantify the effect of "incorrect" soil wetness on the "inferred" soil hydraulic proerties. The soil hydraulic properties are : The variation of soil hydraulic conductivity with soil moisture is given by the Brooks-Corey relations as (Brooks and Corey, 1964) given by

$$\begin{aligned} K(\psi) &= K_s \left(\frac{\psi_c)}{\psi}\right)^{2+3m} & \psi > \psi_c \\ \theta(\psi) &= \theta_r + (\theta_s - \theta_r)\left(\frac{\psi_c}{\psi}\right)^m & \psi > \psi_c \\ K(\psi) &= K_s \quad \theta(\psi) = \theta_s & \psi \le \psi_c \end{aligned} \tag{24}$$

where θ_s is the saturated soil moisture content, θ_r is the residual soil moisture content and m is the pore distribution index (Brooks-Corey parameter), ψ_c is the air-entry suction head and K_s is the saturated hydraulic conductivity. The parameters for the above set of relations is obtained from Rawls et. al., 1982 for various soil types. The exchange fluxes are given by: $q_{1,2}$ is the drainage from layer 1 to layer 2 (if > 0, flux is from layer 1 to layer 2, else vice-versa) and is given by (Mahrt and Pan, 1984)

$$q_{1,2} = K(max(\theta_1, \theta_2)) + D(max(\theta_1, \theta_2))\frac{\partial \theta}{\partial z} \tag{25}$$

where $max(\theta_1, \theta_2)$ is the larger of θ_1 and θ_2. The usage of $max(\theta_1, \theta_2)$ follows Mahrt et. al., (1984) to reduce the truncation errors caused due to asymmetric finite differencing between layer 1 and 2. This formulation follows the moisture movement from "upstream" where the movement originates and helps in reducing truncation errors. q_2 is the drainage flux from layer 2,

$$q_2 = K(\theta_2) \tag{26}$$

In case the soil moisture is biased by 1, 2, 5 or 10%, we will have a difference in the computed ver-

Texture	θ_s	θ_r	ψ_c (cm)	m	K_s (cm/hr)
S	0.44	0.02	7.26	0.59	21.0
SL	0.45	0.04	14.66	0.322	2.59
SiL	0.50	0.02	20.76	0.21	0.68
SCL	0.47	0.04	32.56	0.15	0.15
C	0.48	0.09	37.30	0.13	0.06

Table 1: Parameters for Soil Texture Classes; θ_s is the volumetric soil moisture content at saturation; θ_r is the volumeteric soil moisture at residual; ψ_c is the bubbl;ing pressure; m is the pore size distribution index; and K_s is the saturated hydraulic conductivity S = Sand, SL = Sandy Loam, SiL = Silty Loam, SCL = Sandy Clay Loam and C = Clay

sus "true" hydraulic quantity. These can be simply quantified using the popular Brooks-Corey relationship (Brooks and Corey, 1964) and the Rawls, Brakensiek and Saxton soil properties (Rawls et. al. 1982). The size of the percentage difference would suggest that the accurate representation of soil moisture is particularly important in order to correctly compute the vertical flow between soil layers.

The following Figure 1 is based on results for $\delta\theta$ = 0.3 (volumetric) and $(z_i + z_{i+1})/2 = 15cm$ (assuming a $10cm$ upper layer and a $20cm$ bottom layer thickness). It can be seen that a small (10%) change in soil moisture causes up to a 60% change in hydraulic conductivity and a 40% change in hydraulic diffusivity and a 20% change in vertical flow for a volumetric soil moisture of 0.4.

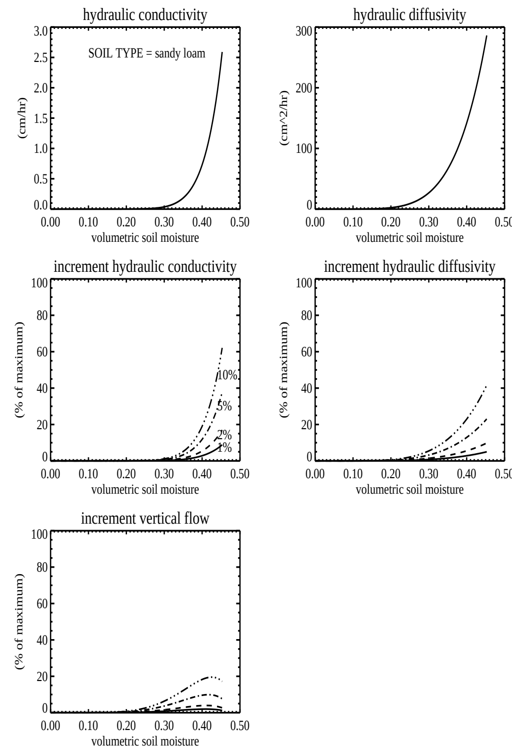

Figure 1: Variation in soil hydraulic properties corresponding to a percentage change in soil moisture versus soil moisture

5 CONCLUSIONS AND DISCUSSION

It can be seen that there is an enormous amount of research that needs to be undertaken before we can use formalistic state space representation for land surface data assimilation. We have demonstrated using a simple framework that we can achieve a high degree of accuracy by simply using an adjustment technique with surface temperature, soil moisture and streamflow.

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